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ON THE CONVERGENCE OF THE ZETA FUNCTION FOR CERTAIN PREHOMOGENEOUS VECTOR SPACES

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Introduction

Let (G, V) be an irreducible prehomogeneous vector space defined over a number field k, $P \in k[V]$ a relative invariant polynomial, and χ a rational character of G such that $P(gx) = \chi(g)P(x)$. Let $V_k^{ss} = \{x \in V_k \mid P(x) \neq 0\}$. For $x \in V_k^{ss}$, let G_x be the stabilizer of x, and G_x^0 the connected component of 1 of G_x. We define L_0 to be the set of $x \in V_k^{ss}$ such that G_x^0 does not have a non-trivial rational character. Then we define the zeta function for (G, Y) by the following integral

$$Z(\Phi, s) = \int_{G_{\mathbb{A}}/G_k} |\chi(g)|^s \sum_{x \in L_0} \Phi(gx)\, dg,$$

where Φ is a Schwartz-Bruhat function, s is a complex variable, and dg is an invariant measure.

Shintani showed the convergence of $Z(\Phi, s)$ for $\mathrm{Re}(s) \gg 0$ for the spaces $\mathrm{Sym}^2 k^n$ and $\mathrm{Sym}^3 k^2$ (see [4], [5]). F. Sato showed the convergence of $Z(\Phi, s)$ when $G_x \cap \mathrm{Ker}(\chi)$ is connected semi-simple (which implies that $L_0 = V_k^{ss}$) (see [1]). Note that his assumptions in [1] were later proved by other people. Also he considered prehomogeneous vector spaces over $\mathbb{Q}$, but if (G, V) is a prehomogeneous vector space over k, we can consider (G, V) as a prehomogeneous vector space over $\mathbb{Q}$. Then the zeta function of (G, V) over k and the zeta function of (G, V) over $\mathbb{Q}$ are the same. So his result implies the convergence of the zeta function for prehomogeneous vector spaces as above over an arbitrary number field k. In [8], we showed the convergence of $Z(\Phi, s)$ when $\dim G = \dim V$ (in this case $L_0 = V_k^{ss}$ also). These cover 23 types of irreducible reduced prehomogeneous vector spaces. Ying recently showed the convergence of $Z(\Phi, s)$ for a few cases when

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$L_0 \neq V_k^{ss}$. In this paper, we prove the convergence of $Z(\Phi, s)$ for prehomogeneous vector spaces of the form $(G/\tilde{T}, V)$, where G, V are as follows:

(1) $G = \mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$, $V = k^2 \otimes k^2 \otimes k^2$,

(2) $G = \mathrm{GL}(3) \times \mathrm{GL}(3) \times \mathrm{GL}(2)$, $V = k^3 \otimes k^3 \otimes k^2$,

(3) $G = \mathrm{GL}(4) \times \mathrm{GL}(2)$, $V = \wedge^2 k^4 \otimes k^2$,

(4) $G = \mathrm{GL}(6) \times \mathrm{GL}(2)$, $V = \wedge^2 k^6 \otimes k^2$,

and $\tilde{T} = \mathrm{Ker}(G \to \mathrm{GL}(V))$ for all the cases. These are the D_4, E_6, D_5, E_7 cases in [6].

Note that since $L_0 = V_k^{ss}$ for the case (4), the result of M. Sato and Shintani (see [3]) on the meromorphic continuation and the functional equation of the local zeta function at an infinite place implies the meromorphic continuation of $Z(\Phi, s)$ and the functional equation of the form

$$Z(\Phi, s) = Z(\hat{\Phi}, N - s),$$

where $\hat{\Phi}$ is an appropriate Fourier transform and N is a number which can easily be figured out depending on the normalization. (see §0.3 of [8]). For the cases (1)–(3), the meromorphic continuation of $Z(\Phi, s)$ is unknown.

In [7], Ying considered three types of prehomogeneous vector spaces, one of which is the case where $G = \mathrm{GSpin}(Q) \times \mathrm{GL}(2)$ for a non-degenerate quadratic form Q in $n \geq 4$ variables, and V is the tensor product of the standard representations. When $\mathrm{GSpin}(Q)$ is split, the case $n = 4$ (resp. $n = 6$) is the case (1) (resp. case (3)) of this paper. So cases (1) and (3) of this paper are covered by Ying. However, our method is totally different from Ying's method. For example, his method is based on the consideration of Tamagawa numbers as in F. Sato's paper [1] and does not prove that the incomplete theta series $\sum_{x \in L_0} \Phi(gx)$ satisfies the assumption of Shintani's lemma (see §3.4 of [8]). Our method is to estimate the incomplete theta series on a Siegel set. Therefore, we can show that $\sum_{x \in L_0} \Phi(gx)$ satisfies the assumption of Shintani's lemma.

We handle the cases (1), (2) in §2, and the cases (3), (4) in §3.

§1. Preliminaries

We basically follow the notations of [8], but we recall the most basic ones. For a finite set X, $\# X$ is its cardinality. If f, g are functions on a set X (not necessarily finite), $f \ll g$ means that there exists a constant C such that $f(x) \leq Cg(x)$ for all $x \in X$. We also use the classical notation $x \ll y$ when y is a much larger number than x. We hope the meaning of this notation will be clear from the context. The ring of adeles (resp. the group of ideles) over k is denoted by $\mathbf{A}$ (resp. $\mathbf{A}^\times$). For a vector space V over k, $V_\mathbf{A}$ is the adelization and $\mathcal{S}(V_\mathbf{A})$ is the space of Schwartz-Bruhat functions. We define $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x > 0\}$. For $\lambda \in \mathbf{R}_+$, $\underline{\lambda}$ is the idele whose component at any infinite place is $\lambda^{\frac{1}{[k:\mathbf{Q}]}}$ and whose component at any finite place is 1. Let $|x|$ be the adelic absolute value of $x \in \mathbf{A}$. Then $|\underline{\lambda}| = \lambda$. Let $a_n(t_1, \cdots, t_n)$ be the n-dimensional diagonal matrix whose (i, i)-entry is t_i for all i. We define $\mathrm{GL}(n)^0_\mathbf{A} = \{g \in \mathrm{GL}(n)_\mathbf{A} \mid |\det g| = 1\}$.

For all the four cases in this paper, G is of the form $G = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_f)$. ($f$ is either 2 or 3). Let $G_i = \mathrm{GL}(n_i)$ for all i. Let $T_i \subset G_i$ be the set of diagonal matrices, and $T = T_1 \times \cdots \times T_f$. Let $\varepsilon > 0$ be a sufficiently small constant. We define

$$G^0_{i\mathbf{A}} = \mathrm{GL}(n_i)^0_\mathbf{A},$$

$$T^0_{i+} = \{a_{n_i}(\underline{\lambda}_{i1}, \cdots, \underline{\lambda}_{in_i}) \mid \lambda_{i1}, \cdots, \lambda_{in_i} \in \mathbf{R}_+, \lambda_{i1} \cdots \lambda_{in_i} = 1\},$$

$$T^0_{i\varepsilon} = \left\{ a_{n_i}(\underline{\lambda}_{i1}, \cdots, \underline{\lambda}_{in_i}) \;\middle|\; \begin{array}{c} \lambda_{i1}, \cdots, \lambda_{in_i} \in \mathbf{R}_+, \lambda_{i1} \cdots \lambda_{in_i} = 1 \\ \lambda_{i1}\lambda_{i2}^{-2}, \cdots, \lambda_{in_i-1}\lambda_{in_i}^{-1} \geq \varepsilon \end{array} \right\},$$

$$\mathfrak{t}^*_i = \{y_i = (y_{i1}, \cdots, y_{in_i}) \in \mathbf{R}^{n_i} \mid y_{i1} + \cdots + y_{in_i} = 0\}.$$

For $c_i = (c_{i1}, \cdots, c_{in_i-1}) \in \mathbf{R}^{n_i-1}$, we define

$$w_i(c_i) = c_{i1}(1, -1, 0, \cdots, 0) + c_{i2}(0, 1, -1, 0, \cdots, 0) + \cdots + c_{in_i-1}(0, \cdots, 0, 1, -1).$$

Let $\mathfrak{t}^*_{i,\mathrm{pc}}$ be the cone generated by positive weights, i.e.

$$\mathfrak{t}^*_{i,\mathrm{pc}} = \{w_i(c_i) \in \mathfrak{t}^*_i \mid c_{i1}, \cdots, c_{in_i-1} \geq 0\}.$$

Apparently, the set of interior points of $\mathfrak{t}^*_{i,\mathrm{pc}}$ consists of points of the form $w_i(c_i)$ where $c_{i1}, \cdots, c_{in_i-1} > 0$. For $c = (c_1, \cdots, c_f)$, we define

$$w(c) = (w_1(c_1), \cdots, w_f(c_f)).$$

Let

$$G^0_\mathbf{A} = G^0_{1\mathbf{A}} \times \cdots \times G^0_{f\mathbf{A}},$$
$$T^0_+ = T^0_{1+} \times \cdots \times T^0_{f+},$$
$$T^0_\varepsilon = T^0_{1\varepsilon} \times \cdots \times T^0_{f\varepsilon},$$
$$\mathfrak{t}^* = \mathfrak{t}^*_1 \times \cdots \times \mathfrak{t}^*_f,$$
$$\mathfrak{t}^*_{\mathrm{pc}} = \mathfrak{t}^*_{1,\mathrm{pc}} \times \cdots \times \mathfrak{t}^*_{f,\mathrm{pc}}.$$

For $t \in T^0_+$ and $y = (y_1, \cdots, y_f) \in \mathfrak{t}^*$, we can define $t^y \in \mathbf{R}_+$ in the usual manner. Let $\rho \in \mathfrak{t}^*$ be half the sum of positive weights. This means that $t^\rho = \prod_i \prod_{j<k} (\lambda_{ij}\lambda_{ik}^{-1})$ for

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$$t=(a_{n_1}(\underline{\lambda}_{11},\cdots,\underline{\lambda}_{1n_1}),\cdots,a_{n_f}(\underline{\lambda}_{11},\cdots,\underline{\lambda}_{fn_f})).$$

The Weyl group W of G is the product of the Weyl groups of $\mathrm{GL}(n_1),\cdots,\mathrm{GL}(n_f)$ and we identify the Weyl group of $\mathrm{GL}(n_i)$ as the set of permutation matrices for all i. The group W acts on T_+^0 from the left by $t\to gtg^{-1}$ for $g\in W$, $t\in T_+^0$. We define the left action of W on $\mathfrak{t}^*$ by $t^{gy}=(g^{-1}tg)^y$ for $g\in W$, $y\in\mathfrak{t}^*$, $t\in T_+^0$.

For the cases in this paper, up to a constant, $Z(\Phi,s)$ coincides with the following integral

$$\int_{\mathbf{R}_+\times G_{\mathbf{A}}^0/G_k}\lambda^s\sum_{x\in L_0}\Phi(\underline{\lambda}g^0x)d^\times\lambda dg^0,$$

where the action of $\underline{\lambda}$ is the usual multiplication by $\underline{\lambda}$, $d^\times\lambda=\lambda^{-1}d\lambda$, and dg^0 is an invariant measure on $G_{\mathbf{A}}^0$. We define

$$Z_+(\Phi,s)=\int_{[1,\infty)\times G_{\mathbf{A}}^0/G_k}\lambda^s\sum_{x\in L_0}\Phi(\underline{\lambda}g^0x)d^\times\lambda dg^0.$$

It is well known that there exists a compact set $\hat{\Omega}\subset G_{\mathbf{A}}^0$ such that $\hat{\Omega}T_\varepsilon^0$ surjects to $G_{\mathbf{A}}^0/G_k$. Therefore, by Proposition (1.2.3) [8], there exists $0\le\Psi\in\mathcal{S}(V_{\mathbf{A}})$ such that $Z(\Phi,s)$, $Z_+(\Phi,s)$ are bounded by constant multiples of the following integrals

$$(1.1)\qquad \int_{\mathbf{R}_+\times T_\varepsilon^0}\lambda^{\mathrm{Re}(s)}\sum_{x\in L_0}\Psi(\underline{\lambda}tx)t^{-2\rho}d^\times\lambda d^\times t,$$

$$\int_{[1,\infty)\times T_\varepsilon^0}\lambda^{\mathrm{Re}(s)}\sum_{x\in L_0}\Psi(\underline{\lambda}tx)t^{-2\rho}d^\times\lambda d^\times t$$

respectively, where $d^\times t$ is an invariant measure on T_+^0.

In the following sections, we choose a coordinate system $x=(x,\cdots,x_N)$ of V for each case so that there exists $\gamma_i\in\mathfrak{t}^*$ for $i=1,\cdots,N$ and $tx=(\underline{t}^{\gamma_i}x_i)$ for $t\in T_+^0$, $x\in V_{\mathbf{A}}$. The element γ_i is called the weight of the coordinate x_i. For $x=(x_1,\cdots,x_N)\in V_k$, we define $I_x=\{1\le i\le N\mid x_i\ne0\}$. Let Conv_x be the convex hull of the set $\{\gamma_i\mid i\in I_x\}$.

Definition (1.2). *A point $x\in V_k$ is k-stable if for all $g\in G_k$, the convex hull Conv_{gx} contains a neighborhood of the origin of* $\mathfrak{t}^*$.

We showed in Proposition (3.1.4) [8] that if L_0 coincides with the set of k-stable points, $Z(\Phi,s)$ converges absolutely for $\mathrm{Re}(s)\gg0$ and $Z_+(\Phi,s)$ is an entire function.

We need the following lemma in §2 to show that L_0 coincides with the set of k-stable points for the cases (1), (2).

LEMMA (1.3). *Suppose that $L \subset V_k^{ss}$ is a G_k-invariant subset such that* Conv_x *contains an interior point of* $\mathfrak{t}_{pc}^*$ *for any $x \in L$. Then x is k-stable for all $x \in L$.*

Proof. Suppose $x \in L$. Let $g \in W$, $t \in T_+^0$. We define

$$e_i = (\overbrace{0, \cdots, 0, 1}^{i}, 0, \cdots, 0) \in V_k$$

for $i = 1, \cdots, N$. Then $te_i = \underline{t}^{\gamma_i} e_i$. So

$$tge_i = gg^{-1}tge_i = g\underline{(g^{-1}tg)}^{\gamma_i} e_i = \underline{t}^{g\gamma_i} ge_i.$$

Therefore, $\mathrm{Conv} I_{gx} = g\mathrm{Conv}_x$. Since L_0 is G_k-invariant, $gx \in L_0$. This implies that $g\mathrm{Conv}_x$ contains an interior point of $\mathfrak{t}_{pc}^*$. So Conv_x contains an interior point of $g^{-1}\mathfrak{t}_{pc}^*$. Note that this statement is true for all $g \in W$.

Suppose that Conv_x does not contain a neighborhood of the origin of $\mathfrak{t}^*$. Since Conv_x is a finite convex polytope, this implies that Conv_x is contained in a half space containing the origin, say $\{y \in \mathfrak{t}^* \mid l(y) \leq 0\}$ where $l(y)$ is a non-zero linear form on $\mathfrak{t}^*$. There exists an element $g \in W$ such that $l(g^{-1}y)$ is of the form

$$l(g^{-1}y) = l_1(y_1) + \cdots + l_f(y_f),$$
$$l_i(y_i) = a_{i1}y_{i1} + \cdots + a_{in_i}y_{in_i}$$

for $y = (y_1, \cdots, y_f) \in \mathfrak{t}^*$ where $a_{i1} \geq \cdots \geq a_{in_i}$ are constants for $i = 1, \cdots, f$.

Since $y_{i1} + \cdots + y_{in_i} = 0$ for all i, we may assume that $a_{ij} > 0$ for all i, j. Also since the linear form l is not identically zero, we may assume that there exist i_0, j_0 such that $a_{i_0j_0} > a_{i_0j_0+1}$.

We showed that there exists an interior point $w(c) = (w_1(c_1), \cdots, w_f(c_f))$ of $\mathfrak{t}_{pc}^*$ such that Conv_x contains the point $g^{-1}w(c)$. Then

$$l(g^{-1}w(c)) = \sum_{i=1}^{f} \sum_{j=1}^{n_i-1} (a_{ij} - a_{ij+1})c_{ij}.$$

By assumption, all the terms are non-negative and at least one term is positive. Therefore, $l(g^{-1}w(c)) > 0$. This is a contradiction. So we can conclude that Conv_x contains a neighborhood of the origin. Q.E.D.

§2. D_4, E_6 cases

We consider the cases (1), (2) in the introduction in this section. We consider these prehomogeneous vector spaces as $\mathrm{M}(2,2) \otimes k^2$ or $\mathrm{M}(3,3) \otimes k^2$, i.e. the space of 2×2 or 3×3 matrices whose entries are linear forms in two variables $v = (v_1, v_2)$. We express a general element of V as $M_x(v) = v_1 x_1 + v_2 x_2$ where $x_1 = (x_{1,ij})$, $x_2 = (x_{2,ij})$ are 2×2 or 3×3 matrices. We choose $x = (x_1, x_2)$ as the coordinate system of V. If $g = (g_1, g_2, g_3)$ is an element of $\mathrm{GL}(2) \times \mathrm{GL}(2) \times \mathrm{GL}(2)$ or $\mathrm{GL}(3) \times \mathrm{GL}(3) \times \mathrm{GL}(3)$, the action of g is defined by

$$gM_x(v) = g_1 M_x(vg_3)\,{}^t g_2.$$

We define $F_x(v) = \det M_x(v)$. Then F_x is a binary quadratic or cubic form. It was proved in [2] that V_k^{ss} is the set of x such that F_x has distinct factors over the closure $\bar{k}$ of k. We showed in [6] that L_0 is the set of x such that F_x is irreducible.

THEOREM (2.1). *The set L_0 coincides with the set of k-stable points. Therefore, $Z(\Phi, s)$ converges absolutely for $\mathrm{Re}(s) \gg 0$ and $Z_+(\Phi, s)$ is an entire function.*

Proof. Suppose that F_x is irreducible. Then for any $v \in k^2 \setminus \{(0,0)\}$, $F_x(v) \neq 0$, i.e. $M_x(v)$ is a non-singular matrix. In particular x_1, x_2 are non-singular matrices. Let $t = (t_1, t_2, t_3) \in T_+^0$, where

$$\begin{cases} t_1 = a_2(\underline{\lambda}_{11}, \underline{\lambda}_{11}^{-1}),\ t_3 = a_2(\underline{\lambda}_{21}, \underline{\lambda}_{21}^{-1}),\ t_3 = a_2(\underline{\lambda}_{31}, \underline{\lambda}_{31}^{-1}) & \text{case (1)}, \\ t_1 = a_3(\underline{\lambda}_{11}, \underline{\lambda}_{12}, \underline{\lambda}_{13}),\ t_2 = a_3(\underline{\lambda}_{21}, \underline{\lambda}_{22}, \underline{\lambda}_{23}),\ t_3 = a_2(\underline{\lambda}_{31}, \underline{\lambda}_{31}^{-1}) & \text{case (2)}, \end{cases}$$

and $\lambda_{11}\lambda_{12}\lambda_{13} = \lambda_{21}\lambda_{22}\lambda_{23} = 1$.

The set L_0 is clearly G_k-invariant. So by Lemma (1.3), we only have to show that for any $x \in L_0$, Conv_x contains an interior point of $\mathfrak{t}_{\mathrm{pc}}^*$. Let $\gamma_{i,jk}$ be the weight of the coordinate $x_{i,jk}$ for all i, j, k. The element $\gamma_{i,jk}$ can be expressed in the form $\gamma_{i,jk} = w(d_{i,jk})$ ($d_{i,jk}$ may not be in $\mathfrak{t}_{\mathrm{pc}}^*$).

We first consider the case (1). The following lemma is easy to verify and the proof is left to the reader.

LEMMA (2.2). (1) $d_{1,11} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

(2) $d_{1,12} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

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(3) $d_{1,21}=\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.

Note that $d_{2,jk}$ can be obtained by replacing the last $\frac{1}{2}$ in $d_{1,jk}$ by $-\frac{1}{2}$.

Suppose $x\in L_0$. If $x_{1,11}\neq 0$, then $\gamma_{1,11}\in \mathrm{Conv}_x$ and $\gamma_{1,11}$ is an interior point of $\mathfrak{t}_{pc}^*$ by the above lemma.

Suppose $x_{1,11}=0$. Then since x_1 is non-singular. $x_{1,12}, x_{1,21}\neq 0$. Moreover if $x_{2,11}=0$, we can choose $v\in k^2\setminus\{0\}$ so that $M_x(v)$ is singular. This contradicts to the assumption $x\in L_0$. So we may assume that $x_{2,11}\neq 0$. Therefore, $\gamma_{1,12}, \gamma_{1,21}, \gamma_{2,11}\in \mathrm{Conv}_x$. This implies that $\gamma_{1,12}+\gamma_{1,21}+\gamma_{2,11}\in \mathrm{Conv}_x$ also and

$$d_{1,12}+d_{1,21}+d_{2,11}=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right).$$

So $\gamma_{1,12}+\gamma_{1,21}+\gamma_{2,11}$ is an interior point of Conv_x. This completes the proof of Theorem (2.1) for the case (1).

Next, we consider the case (2). The following lemma is easy to verify and the proof is left to the reader.

LEMMA (2.3). (1) $d_{1,11}=\left(\left(\frac{2}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{1}{3}\right),\frac{1}{2}\right)$.

(2) $d_{1,12}=\left(\left(\frac{2}{3},\frac{1}{3}\right),\left(-\frac{1}{3},\frac{1}{3}\right),\frac{1}{2}\right)$.

(3) $d_{1,13}=\left(\left(\frac{2}{3},\frac{1}{3}\right),\left(-\frac{1}{3},-\frac{2}{3}\right),\frac{1}{2}\right)$.

(4) $d_{1,21}=\left(\left(-\frac{1}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{1}{3}\right),\frac{1}{2}\right)$.

(5) $d_{1,22}=\left(\left(-\frac{1}{3},\frac{1}{3}\right),\left(-\frac{1}{3},\frac{1}{3}\right),\frac{1}{2}\right)$.

(6) $d_{1,31}=\left(\left(-\frac{1}{3},-\frac{2}{3}\right),\left(\frac{2}{3},\frac{1}{3}\right),\frac{1}{2}\right)$.

Note that $d_{2,jk}$ can be obtained by replacing the last $\frac{1}{2}$ in $d_{1,jk}$ by $-\frac{1}{2}$.

Suppose $x\in L_0$. If $x_{1,11}\neq 0$, then $\gamma_{1,11}\in \mathrm{Conv}_x$ and $\gamma_{1,11}$ is an interior point of $\mathfrak{t}_{pc}^*$.

Suppose $x_{1,11}=0$, $x_{1,12}, x_{1,21}\neq 0$. Then $\gamma_{1,12}, \gamma_{1,21}\in \mathrm{Conv}_x$. So $\gamma_{1,12}+\gamma_{1,21}\in \mathrm{Conv}_x$ also and

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$$d_{1,12}+d_{1,21}=\left(\left(\frac{1}{3},\frac{2}{3}\right),\left(\frac{1}{3},\frac{2}{3}\right),1\right).$$

So $\gamma_{1,12}+\gamma_{1,21}$ is an interior point of $\mathfrak{t}_{pc}^*$.

Consider the following two cases:

(1) $x_{1,11}=0$, $x_{1,12}=0$, and $x_{1,21}\neq 0$,

(2) $x_{1,11}=0$, $x_{1,21}=0$, and $x_{1,12}\neq 0$.

Since these cases are similar, we only consider the case (1). Since x_1 is a non-singular matrix, $x_{1,13}\neq 0$. If $x_{2,11}=x_{2,12}=0$, we can choose $v\in k^2\setminus\{0\}$ so that $v_1x_{1,13}+v_2x_{2,13}=0$. This contradicts to the assumption $x\in L_0$. So we may assume that either $x_{2,11}\neq 0$ or $x_{2,12}\neq 0$. Since

$$d_{2,11}=d_{2,12}+((0,0),(1,0),0),$$

we only consider the case $x_{2,12}\neq 0$.

With these assumptions, $\gamma_{1,21},\gamma_{1,13},\gamma_{2,12}\in \mathrm{Conv}_x$. Then

$$3\gamma_{1,21}+2\gamma_{1,13}+2\gamma_{2,12}\in \mathrm{Conv}_x$$

also and

$$3d_{1,12}+2d_{1,13}+2d_{2,12}=\left(\left(\frac{5}{3},\frac{7}{3}\right),\left(\frac{2}{3},\frac{1}{3}\right),\frac{3}{2}\right).$$

So $3\gamma_{1,21}+2\gamma_{1,13}+2\gamma_{2,12}$ is an interior point of $\mathfrak{t}_{pc}^*$.

Suppose $x_{1,11}=x_{1,12}=x_{1,21}=0$. Then since x_1 is a non-singular matrix, $x_{1,13}$, $x_{1,22}$, $x_{1,31}\neq 0$. Suppose $x_{2,11}\neq 0$. Then

$$\gamma_{1,13}+\gamma_{1,22}+\gamma_{1,31}+\gamma_{2,11}\in \mathrm{Conv}_x$$

and

$$d_{1,13}+d_{1,22}+d_{1,31}+d_{2,11}=\left(\left(\frac{2}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{1}{3}\right),1\right).$$

So $\gamma_{1,13}+\gamma_{1,22}+\gamma_{1,31}+\gamma_{2,11}$ is an interior point of $\mathfrak{t}_{pc}^*$.

Suppose $x_{1,11}=x_{1,12}=x_{1,21}=x_{2,11}=0$. Then if either $x_{2,12}=0$ or $x_{2,21}=0$, we can choose $v\in k^2\setminus\{0\}$ so that $v_1x_1+v_2x_2$ is singular, which is a contradiction. So $x_{2,12}$, $x_{2,21}\neq 0$. By assumption, $x_{1,13}$, $x_{1,22}$, $x_{1,31}\neq 0$ also. Then

$$\gamma_{1,13}+\gamma_{1,22}+\gamma_{1,31}+\gamma_{2,12}+\gamma_{2,21}\in \mathrm{Conv}_x$$

and

$$d_{1,13} + d_{1,22} + d_{1,31} + d_{2,12} + d_{2,21} = \left(\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right), \frac{1}{2}\right).$$

So $\gamma_{1,13} + \gamma_{1,22} + \gamma_{1,31} + \gamma_{2,12} + \gamma_{2,21}$ is an interior point of $\mathfrak{t}^*_{\mathrm{pc}}$. This completes the proof of Theorem (2.1) for the case (2). Q.E.D.

§3. D_5, E_7 cases

We consider the cases (3), (4) in the introduction in this section. We consider these cases as the space of 4×4 or 6×6 alternating matrices whose entries are linear forms in two variables $v = (v_1, v_2)$. We express a general element of V as $M_x(v) = v_1 x_1 + v_2 x_2$ where $x_1 = (x_{1,ij})$, $x_2 = (x_{2,ij})$ are 4×4 or 6×6 alternating matrices. We choose $x = (x_1, x_2)$ as the coordinate system of V (we only consider $x_{i,jk}$ such that $j > k$). If $g = (g_1, g_2)$ is an element of $\mathrm{GL}(4) \times \mathrm{GL}(2)$ or $\mathrm{GL}(6) \times \mathrm{GL}(2)$, the action of g is defined by

$$gM_x(v) = g_1 M_x(v g_2)^t g_1.$$

Since $M_x(v)$ is an alternating matrix, there exists a binary quadratic or cubic form $F_x(v)$ such that $\det M_x(v) = F_x(v)^2$ ($F_x(v)$ is the Pfaffian of $M_x(v)$). It was proved in [2] that V_k^{ss} is the set of x such that $F_x(v)$ has distinct factors. We showed in [6] that L_0 is the set of x such that F_x is irreducible for the case (3) and that $L_0 = V_k^{\mathrm{ss}}$ for the case (4).

THEOREM (3.1). *The integral* $Z(\Phi, s)$ *converges absolutely and locally uniformly for* $\mathrm{Re}(s) \gg 0$ *and* $Z_+(\Phi, s)$ *is an entire function.*

Proof. Unlike the cases (1), (2), there are no k-stable points, so we have to be a little more subtle for these cases. Let Ψ be as in §1. For $L \subset V_k$, we define

$$\Theta_L(\Psi, \underline{\lambda} t) = \sum_{x \in L} \Psi(\underline{\lambda} t x) \tag{3.2}$$

for $\lambda \in \mathbf{R}_+$, $t \in T^0_\varepsilon$.

We estimate $\Theta_{L_0}(\Psi, \underline{\lambda} t)$. Note that if $y \in \mathfrak{t}^*$, the integral $\int_{T^0_\varepsilon} t^{y-2\rho} d^\times t$ converges absolutely if $-(y - 2\rho)$ is an interior point of $\mathfrak{t}^*_{\mathrm{pc}}$.

Let $t = (t_1, t_2)$ where

$$\begin{cases} t_1 = a_4(\underline{\lambda}_{11}, \underline{\lambda}_{12}, \underline{\lambda}_{13}, \underline{\lambda}_{14}),\ t_2 = a_2(\underline{\lambda}_{21}, \underline{\lambda}_{21}^{-1}) & \text{case (3)}, \\ t_1 = a_6(\underline{\lambda}_{11}, \underline{\lambda}_{12}, \underline{\lambda}_{13}, \underline{\lambda}_{14}, \underline{\lambda}_{15}, \underline{\lambda}_{16}),\ t_2 = a_2(\underline{\lambda}_{21}, \underline{\lambda}_{21}^{-1}) & \text{case (4)}. \end{cases}$$

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Let $\gamma_{i,jk}$ be the weight of the coordinate $x_{i,jk}$ for all i, j, k $(j > k)$. The element $\gamma_{i,jk}$ can be expressed in the form $\gamma_{i,jk} = w(d_{i,jk})$, where $d_{i,jk} \in \mathbf{R}^4$ or $\mathbf{R}^6$.

Let $\sigma = \mathrm{Re}(s)$. We will prove that the function $\lambda^{\sigma}\Theta_{L_0}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T^0_{\varepsilon}$ for $\sigma \gg 0$. What we are going to do is to divide L_0 into a union of finite number of (not necessarily G_k-stable) subsets L_i and to estimate $\Theta_{L_i}(\Psi, \underline{\lambda t})t^{-2\rho}$ by a finite number of functions of the form $\lambda^{p_N}t^{w(c_N)}$ where $p_N \in \mathbf{R}$, $c_N \in \mathbf{R}^4$ or $\mathbf{R}^6$ depend on a finite number of positive numbers N. These numbers should have the property that if we choose N appropriately, $p_N \ll 0$ and all the entries of c_N are negative.

If $\lambda \geq 1$, for any $\sigma \in \mathbf{R}$, we can choose N depending on σ so that $\sigma + p_N < 0$ and all the entries of c_N are negative. This implies that the function $\lambda^{\sigma}\Theta_{L_i}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $[1, \infty) \times T^0_{\varepsilon}$. If $\lambda \leq 1$, we fix N so that all the entries of c_N are negative. Then if $\sigma + p_N > 0$, the function $\lambda^{\sigma}\Theta_{L_i}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $(0, 1] \times T^0_{\varepsilon}$. Since σ is arbitrary for the convergence of the integral on $[1, \infty) \times T^0_{\varepsilon}$, this proves the convergence of $Z(\Phi, s)$ for $\mathrm{Re}(s) \gg 0$ and $Z_+(\Phi, s)$ for all s.

Let

$$I_0 = \begin{cases} \{(i, j, k) \mid i = 1,2,\ 1 \leq k < j \leq 4\} & \text{case (3)}, \\ \{(i, j, k) \mid i = 1,2,\ 1 \leq k < j \leq 6\} & \text{case (4)}. \end{cases} \tag{3.3}$$

For $I \subset I_0$, we define

$$h_I(\lambda, t) = \prod_{(i,j,k)\in I} \sup(1, \lambda^{-1}t^{-\gamma_{i,jk}}) \tag{3.4}$$

for $\lambda \in \mathbf{R}_+$, $t \in T^0_{\varepsilon}$.

Functions of the form $h_I(\lambda, t)$ often appear in estimates of various incomplete theta series because our main tool is Lemma (1.2.6) [8]. So we first consider the function $h_I(\lambda, t)$. We start with the following two observations whose proofs are easy and are left to the reader.

LEMMA (3.5).

(1) *If* $I_1 \subset I_2 \subset I_0$, *then* $h_{I_1}(\lambda, t) \leq h_{I_2}(\lambda, t)$.

(2) *If* $I = I_1 \amalg I_2 \subset I_0$, *then* $h_I(\lambda, t) = h_{I_1}(\lambda, t)h_{I_2}(\lambda, t)$.

LEMMA (3.6).

$$h_I(\lambda, t) = \sup_{I' \subset I} \prod_{(i,j,k)\in I'} (\lambda^{-1}t^{-\gamma_{i,jk}}).$$

Next, to simplify the situation, we estimate $h_I(\lambda, t)$ by functions of the form

$\lambda^p t^{w(c)}$

Let

$$d_{i,jk} = \begin{cases} ((d_{i,jk,1}, \cdots, d_{i,jk,3}), d_{i,jk,4}) & \text{case (3)}, \\ ((d_{i,jk,1}, \cdots, d_{i,jk,5}), d_{i,jk,6}) & \text{case (4)}. \end{cases}$$

We define

$$c_{I,l} = \sum_{\substack{(i,j,k) \in I \\ d_{i,jk,l} < 0}} d_{i,jk,l},$$

for all l and put

$$c_I = \begin{cases} ((c_{I,1}, \cdots, c_{I,3}), c_{I,4}) & \text{case (3)}, \\ ((c_{I,1}, \cdots, c_{I,5}), c_{I,6}) & \text{case (4)}. \end{cases}$$

LEMMA (3.7). $h_I(\lambda, k) \ll \sup(1, \lambda^{-\#I}) t^{-w(c_I)}$ on $\mathbf{R}_+ \times T_\varepsilon^0$.

Proof. Note that

$$\prod_{(i,j,k) \in I'} (\lambda^{-1} t^{-\gamma_{i,jk}}) = \lambda^{-\#I'} \prod_{(i,j,k) \in I'} t^{-w(d_{i,jk})}$$

and $\lambda^{-\#I'} \le \sup(1, \lambda^{-\#I})$.

Let $\bar{d}_{i,jk} \in \mathbf{R}^4$ or $\mathbf{R}^6$ be the element obtained by replacing positive entries of $d_{i,jk}$ by 0. Then

$$\prod_{(i,j,k) \in I'} t^{-w(d_{i,jk})} \ll \prod_{(i,j,k) \in I'} t^{-w(\bar{d}_{i,jk})} = t^{-\sum_{(i,j,k) \in I'} w(\bar{d}_{i,jk})}.$$

However, since all the entries of $\bar{d}_{i,jk}$ are non-positive for all i, j, k.

$$t^{-\sum_{(i,j,k) \in I'} w(\bar{d}_{i,jk})} \ll t^{-\sum_{(i,j,k) \in I} w(\bar{d}_{i,jk})} = t^{-w(c_I)}.$$

This proves the lemma. Q.E.D.

For the rest of this section, $\lambda \in \mathbf{R}_+$, $t \in T_\varepsilon^0$. So in inequalities like Lemma (3.7), we will not mention that it is uniform with respect to $\lambda \in \mathbf{R}_+$, $t \in T_\varepsilon^0$.

We first consider the case (3). The following lemma is easy to verify and the proof is left to the reader.

LEMMA (3.8). (1) $d_{1,21} = \left(\left(\frac{1}{2}, 1, \frac{1}{2}\right), \frac{1}{2}\right)$.

(2) $d_{1,31} = \left(\left(\frac{1}{2}, 0, \frac{1}{2}\right), \frac{1}{2}\right)$.

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(3) $d_{1,41}=\left(\left(\frac{1}{2}, 0, -\frac{1}{2}\right), \frac{1}{2}\right)$.

(4) $d_{1,32}=\left(\left(-\frac{1}{2}, 0, \frac{1}{2}\right), \frac{1}{2}\right)$.

(5) $d_{1,42}=\left(\left(-\frac{1}{2}, 0, -\frac{1}{2}\right), \frac{1}{2}\right)$.

(6) $d_{1,43}=\left(\left(-\frac{1}{2}, -1, -\frac{1}{2}\right), \frac{1}{2}\right)$.

Note that $d_{2,jk}$ can be obtained by replacing the last $\frac{1}{2}$ in $d_{1,jk}$ by $-\frac{1}{2}$. Also if we put $d_0=((-3, -4, -3), -1)$, then $-2\rho=w(d_0)$. By Lemma (3.7), $h_{I_0}(\lambda, t) \ll \sup(1, \lambda^{-12})t^{-w(c_0)}$, where

(3.9) $$c_0=c_{I_0}=-((3, 2, 3), 3).$$

Definition (3.10). (1) $L_1=\{x \in L_0 \mid x_{1,21} \neq 0\}$.
(2) $L_2=\{x \in L_0 \mid x_{1,21}=0, x_{1,31} \neq 0\}$.
(3) $L_3=\{x \in L_0 \mid x_{1,21}=0, x_{1,31}=0\}$.

Apparently, $L_0=L_1 \cup L_2 \cup L_3$. So we estimate $\Theta_{L_i}(\Psi, \underline{\lambda t})t^{-2\rho}$ for $i=1,2,3$.

(1) Consider L_1.

Let $I=I_0 \setminus \{(1,2,1)\}$. By Lemma (1.2.6) [8], for any $N \geq 1$,

$$\Theta_{L_1}(\Psi, \underline{\lambda t})t^{-2\rho} \ll \lambda^{-N}t^{-N\gamma_{1,21}}h_I(\lambda, t)t^{-2\rho}.$$

By Lemma (3.7),

$$h_I(\lambda, t) \ll \sup(1, \lambda^{-11})t^{w(((3,2,3),3))},$$
$$h_I(\lambda, t)t^{-2\rho} \ll \sup(1, \lambda^{-11})t^{w(((0,-2,0),2))},$$

Since all the entries of $d_{1,21}$ are positive, $\lambda^{\sigma}\Theta_{L_1}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

(2) Consider L_2.

Let $I=I_0 \setminus \{(1,2,1), (1,3,1)\}$. By Lemma (1.2.6) [8], for any $N \geq 1$,

$$\Theta_{L_2}(\Psi, \underline{\lambda t})t^{-2\rho} \ll \lambda^{-N}t^{-N\gamma_{1,31}}h_I(\lambda, t)t^{-2\rho}.$$

By Lemma (3.7),

$$h_2(\lambda, t) \ll \sup(1, \lambda^{-10})t^{w(((3,2,3),3))},$$
$$h_2(\lambda, t)t^{-2\rho} \ll \sup(1, \lambda^{-10})t^{w(((0,-2,0),2))},$$

So for any $N \geq 1$,

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$$\Theta_{L_2}(\Psi, \underline{\lambda t})t^{-2\rho} \ll \lambda^{-N} \sup(1, \lambda^{-10}) t^{w(((0,-2,0),2)-N((\frac{1}{2},0,\frac{1}{2}),\frac{1}{2}))}.$$

Since all the entries of $((0, -2, 0), 2) - N\left(\left(\frac{1}{2}, 0, \frac{1}{2}\right), \frac{1}{2}\right)$ are negative if $N > 4$, $\lambda^{\sigma}\Theta_{L_1}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

(3) Consider L_3.

Suppose $x \in L_3$. Then since x_1 is non-singular, $x_{1,32}, x_{1,41} \neq 0$. We define $I = I_0 \setminus \{(1,2,1), (1,3,1), (1,3,2), (1,4,1)\}$. Then by Lemma (1.2.6) [8], for any $N \geq 1$,

$$\Theta_{L_3}(\Psi, \underline{\lambda t})t^{-2\rho} \ll \lambda^{-2N} t^{-N(\gamma_{1,32}+\gamma_{1,41})} h_I(\lambda, t).$$

By Lemma (3.7),

$$h_I(\lambda, t) \ll \sup(1, \lambda^{-8}) t^{w((\frac{5}{2},2,\frac{5}{2}),3)},$$
$$h_I(\lambda, t)t^{-2\rho} \ll \sup(1, \lambda^{-8}) t^{w((-\frac{1}{2},-2,-\frac{1}{2}),2)}.$$

So for any $N \geq 1$,

$$\Theta_{L_3}(\Psi, \underline{\lambda t})t^{-2\rho} \ll \lambda^{-2N} \sup(1, \lambda^{-8}) t^{w(((-\frac{1}{2},-2,-\frac{1}{2}),2)-N(d_{1,32}+d_{1,41}))}.$$

Since $d_{1,32} + d_{1,41} = ((0,0,0), 1)$, all the entries of

$$\left(\left(-\frac{1}{2}, -2, -\frac{1}{2}\right), 2\right) - N(d_{1,32} + d_{1,41})$$

are negative if $N > 2$. Therefore, $\lambda^{\sigma}\Theta_{L_3}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

This completes the proof of Theorem (3.1) for the case (3).

Next, we consider the case (4). The following lemma is easy to verify and the proof is left to the reader.

LEMMA (3.11). (1) $d_{1,21} = \left(\left(\frac{2}{3}, \frac{4}{3}, 1, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(2) $d_{1,31} = \left(\left(\frac{2}{3}, \frac{1}{3}, 1, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(3) $d_{1,41} = \left(\left(\frac{2}{3}, \frac{1}{3}, 0, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(4) $d_{1,51} = \left(\left(\frac{2}{3}, \frac{1}{3}, 0, -\frac{1}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(5) $d_{1,61} = \left(\left(\frac{2}{3}, \frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}\right), \frac{1}{2}\right)$.

(6) $d_{1,32} = \left(\left(-\frac{1}{3}, \frac{1}{3}, 1, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(7) $d_{1,42} = \left(\left(-\frac{1}{3}, \frac{1}{3}, 0, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(8) $d_{1,52} = \left(\left(-\frac{1}{3}, \frac{1}{3}, 0, -\frac{1}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(9) $d_{1,62} = \left(\left(-\frac{1}{3}, \frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}\right), \frac{1}{2}\right)$.

(10) $d_{1,43} = \left(\left(-\frac{1}{3}, -\frac{2}{3}, 0, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(11) $d_{1,53} = \left(\left(-\frac{1}{3}, -\frac{2}{3}, 0, -\frac{1}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(12) $d_{1,63} = \left(\left(-\frac{1}{3}, -\frac{2}{3}, 0, -\frac{1}{3}, -\frac{2}{3}\right), \frac{1}{2}\right)$.

(13) $d_{1,54} = \left(\left(-\frac{1}{3}, -\frac{2}{3}, -1, -\frac{1}{3}, \frac{1}{3}\right), \frac{1}{2}\right)$.

(14) $d_{1,64} = \left(\left(-\frac{1}{3}, -\frac{2}{3}, -1, -\frac{1}{3}, -\frac{2}{3}\right), \frac{1}{2}\right)$.

(15) $d_{1,65} = \left(\left(-\frac{1}{3}, -\frac{2}{3}, -1, -\frac{4}{3}, -\frac{2}{3}\right), \frac{1}{2}\right)$.

Note that $d_{2,jk}$ can be obtained by replacing the last $\frac{1}{2}$ in $d_{1,jk}$ by $-\frac{1}{2}$. Also if we put $d_0 = ((-5, -8, -9, -8, -5), -1)$, then $-2\rho = w(d_0)$. By Lemma (3.7), $h_{I_0}(\lambda, t) \ll \sup(1, \lambda^{-30}) t^{-w(c_0)}$, where

$$c_0 = c_{I_0} = -\left(\left(\frac{20}{3}, 8, 6, 8, \frac{20}{3}\right), \frac{15}{2}\right). \tag{3.12}$$

Definition (3.13). (1) $L_1 = \{x \in V_k^{ss} \mid x_{1,21} \text{ or } x_{1,31} \text{ or } x_{1,41} \neq 0\}$.

(2) $L_2 = \{x \in V_k^{ss} \mid x_{1,21} = x_{1,31} = x_{1,41} = 0,\ x_{1,32} \text{ or } x_{1,42} \neq 0\}$.

(3) $L_3 = \{x \in V_k^{ss} \mid x_{1,21} = x_{1,31} = x_{1,41} = x_{1,32} = x_{1,42} = 0\}$.

(4) $L_4 = \{x \in L_3 \mid x_{1,43}, x_{1,51} \neq 0\}$.

(5) $L_5 = \{x \in L_3 \mid x_{1,51} = 0,\ x_{1,43}, x_{1,52} \neq 0\}$.

(6) $L_6 = \{x \in L_3 \mid x_{1,51} = x_{1,52} = 0,\ x_{1,43}, x_{1,61} \neq 0\}$.

(7) $L_7 = \{x \in L_3 \mid x_{1,51} = x_{1,52} = x_{1,61} = 0,\ x_{1,43}, x_{1,62} \neq 0\}$.

(8) $L_8 = \{x \in L_3 \mid x_{1,51} = x_{1,52} = x_{1,61} = x_{1,62} = 0,\ x_{1,43}, x_{2,21} \neq 0\}$.

(9) $L_9 = \{x \in L_3 \mid x_{1,43} = 0,\ x_{1,51} \neq 0\}$.
(10) $L_{10} = \{x \in L_3 \mid x_{1,43} = x_{1,51} = 0,\ x_{1,52},\ x_{1,61} \neq 0\}$.
(11) $L_{11} = \{x \in L_3 \mid x_{1,43} = x_{1,51} = x_{1,61} = 0,\ x_{1,52} \neq 0\}$.
(12) $L_{12} = \{x \in L_3 \mid x_{1,43} = x_{1,51} = x_{1,52} = 0,\ x_{1,61},\ x_{1,53} \neq 0\}$.
(13) $L_{13} = \{x \in L_3 \mid x_{1,43} = x_{1,51} = x_{1,52} = x_{1,53} = 0,\ x_{1,61},\ x_{1,54} \neq 0\}$.
(14) $L_{14} = \{x \in L_3 \mid x_{1,43} = x_{1,51} = x_{1,52} = x_{1,61} = 0\}$.
(15) $L_{15} = \{x \in L_{14} \mid x_{1,62},\ x_{1,53} \neq 0\}$.
(16) $L_{16} = \{x \in L_{14} \mid x_{1,53} = 0,\ x_{1,62},\ x_{1,54} \neq 0\}$.
(17) $L_{17} = \{x \in L_{14} \mid x_{1,62} = 0,\ x_{1,53} \neq 0\}$.
(18) $L_{18} = \{x \in L_{14} \mid x_{1,62} = x_{1,53} = 0,\ x_{1,63},\ x_{1,54} \neq 0\}$.

PROPOSITION (3.14). (1) $V_k^{ss} = \coprod_{\substack{1 \le i \le 18 \\ i \neq 3,14}} L_i$.
(2) *If* $x \in V_k^{ss}$, *there exist* $1 \le i \le 2,\ 2 \le j \le 6$ *such that* $x_{i,j1} \neq 0$.
(3) *If* $x \in L_6$, *there exist* $2 \le j \le 5,\ 1 \le k \le 2 (j > k)$ *such that* $x_{2,jk} \neq 0$.
(4) *If* $x \in L_7$, *there exists* $2 \le j \le 5$ *such that* $x_{2,j1} \neq 0$.
(5) *If* $x \in L_9$ *or* L_{10}, *there exist* $1 \le k \le j \le 4$ *such that* $x_{2,jk} \neq 0$.
(6) *If* $x \in L_{11}$, $x_{2,21}$ *or* $x_{2,31}$ *or* $x_{2,41} \neq 0$.
(7) *If* $x \in L_{12}$, *there exist* $2 \le j \le 4,\ 1 \le k \le 2 (j > k)$ *such that* $x_{2,jk} \neq 0$.
(8) *If* $x \in L_{13}$, $x_{2,21}$ *or* $x_{2,31}$ *or* $x_{2,32} \neq 0$.
(9) *If* $x \in L_{15}$, $x_{2,21}$ *or* $x_{2,31}$ *or* $x_{2,41} \neq 0$.
(10) *If* $x \in L_{16}$, $x_{2,21}$ *or* $x_{2,31} \neq 0$.
(11) *If* $x \in L_{17}$ *or* L_{18}, $x_{2,21} \neq 0$.

Proof. Note that if $1 \le i, j \le 18$, $i, j \neq 3,14$, and $i \neq j$, then $L_i \cap L_j = \emptyset$.
It is easy to see that if $x \in V_k^{ss}$ and $x \notin L_1, L_2$, then $x \in L_3$. Suppose that $x \in L_3$ and $x_{1,43} \neq 0$. Then if $x \notin L_4, \cdots, L_7$, $x_{1,ij} = 0$ for $i = 1,2$, $j = 2, \cdots, 6$. Suppose $x_{2,21} = 0$. Then by considering the cofactor expansion with respect to the first two columns, $\det M_x(v)$ is a product of v_2^2 and a sum of determinants of matrices of the form

$$v_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{bmatrix} + v_2 \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

The determinant of the above matrix clearly is divisible by v_2^2. So $F_x(v)$ is divisible by v_2^2, which contradicts to the assumption $x \in V_k^{ss}$. This implies that $x_{2,21} \neq 0$ and $x \in L_8$.

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Suppose that $x \in L_3$ and $x_{1,43} = 0$. If $x_{1,51}$ or $x_{1,52} \neq 0$, then $x \in \coprod_{i=9}^{11} L_i$. So we assume that $x \in L_3$ and $x_{1,43} = x_{1,51} = x_{1,52} = 0$. Then $x_{1,61} \neq 0$ or $x \in L_{14}$. If $x_{1,61}, x_{1,53} \neq 0$, $x \in L_{12}$. Suppose $x_{1,61} \neq 0$, $x_{1,53} = 0$. Then if $x_{1,54} = 0$, $x_{1,jk} = 0$ for $j, k = 1, \cdots, 5$. So by the cofactor expansion with respect to the last row and the last column, $\det M_x(v)$ is divisible by v_2^4, which is a contradiction. Therefore, $x_{1,54} \neq 0$, which implies that $x \in L_{13}$.

Suppose $x \in L_{14}$. If $x_{1,62}, x_{1,53} \neq 0$, then $x \in L_{15}$. Also if $x_{1,62} = 0$, $x_{1,53} \neq 0$, then $x \in L_{17}$. Suppose $x_{1,62} \neq 0$, $x_{1,53} = 0$. If $x_{1,54} = 0$, then $x_{1,jk} = 0$ for $j, k = 1, \cdots, 5$, which cannot happen. So $x_{1,54} \neq 0$, which implies that $x \in L_{16}$. Suppose $x_{1,62} = x_{1,53} = 0$. Then $x_{1,54} \neq 0$ for the same reason. If $x_{1,63} = 0$, the first three columns of x_1 are zero. So $\det M_x(v)$ is divisible by v_2^3. But since $M_x(v)$ is an alternating matrix, $\det M_x(v)$ is divisible by v_2^4, which is a contradiction. This proves (1).

The statements (2), (3), (5) are clear.

Consider the statement (4). Let $x \in L_7$. Then the first column of x_1 is zero. Suppose $x_{2,j1} = 0$ for $j = 2, \ldots, 5$. Then by the cofactor expansion with respect to the (6,1), (1,6)-entries, $\det M_x(v)$ is a product of $x_{1,61}^2 v_2^2$ and the determinant of an alternating matrix of the form

$$v_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & * & 0 & * \\ 0 & * & * & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & * & * & * \\ * & 0 & * & * \\ * & * & 0 & * \\ * & * & * & 0 \end{bmatrix}.$$

The determinant of the above matrix is divisible by v_2^2. So $F_x(v)$ is divisible by v_2^2, which is a contradiction. Therefore, if $x \in L_7$, there exists $2 \leq j \leq 5$ such that $x_{2,j1} \neq 0$.

Consider the statement (6). Suppose $x \in L_{11}$. If the statement of (6) in false, there exists $x \in L_3$ such that

$$x_{1,43} = x_{1,51} = x_{1,61} = x_{2,21} = x_{2,31} = x_{2,41} = 0,\ x_{1,52} \neq 0. \tag{3.15}$$

We show that (3.15) cannot happen. Suppose (3.15) is satisfied. Then $M_x(v)$ is of the following form

$$v_1 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & * & * & * & 0 & * \\ 0 & * & * & * & * & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & * & * & * & * \\ 0 & * & 0 & * & * & * \\ 0 & * & * & 0 & * & * \\ * & * & * & * & 0 & * \\ * & * & * & * & * & 0 \end{bmatrix}.$$

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If $x_{2,51} = x_{2,61} = 0$, $\det M_x(v)$ is identically zero, which is a contradiction. So we assume that $x_{2,51}$ or $x_{2,61} \neq 0$. Let $g_1 \in GL(6)_k$ be an element of the form

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & A \end{bmatrix},$$

where $A \in GL(2)_k$. By applying an element of the form $g = (g_1, I_2) \in GL(6)_k \times GL(2)_k$, we may assume that $x_{2,21} = x_{2,31} = x_{2,41} = x_{2,51} = 0$, $x_{2,61} \neq 0$. Note that by the action of g, $\det M_x(v)$ changes by a non-zero constant and the form of x_1 does not change.

Therefore, $\det M_x(v)$ is a product of $x_{2,61}^2 v_2^2$ and the determinant of an alternating matrix of the form

$$v_1 \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & * & * & * \\ * & 0 & * & * \\ * & * & 0 & * \\ * & * & * & 0 \end{bmatrix}.$$

The determinant of the above matrix is divisible by v_2^2. This implies that $F_x(v)$ is divisible by v_2^2, which is a contradiction.

Consider the statement (7). Let $x \in L_{12}$. If $x_{2,jk} = 0$ for $j = 2,3,4$, $k = 1,2$, $M_x(v)$ is of the form

$$v_1 \begin{bmatrix} 0 & A_2 \\ A_1 & * \end{bmatrix} + v_1 \begin{bmatrix} 0 & B_2 \\ B_1 & * \end{bmatrix},$$

where A_1, B_1 are 2×2 and A_2, B_2 are 4×4. Also the first now of A_1, the first and the second columns of A_2 are zero. Since $\det M_x(v) = \det(v_1A_1 + v_2B_1)\det(v_1A_2 + v_2B_2)$, $\det M_x(v)$ is divisible by v_2^3. Since $M_x(v)$ is an alternating matrix, $\det M_x(v)$ is divisible by v_2^4, which is a contradiction. Therefore, there exists $1 \leq k < j \leq 4$ such that $x_{2,jk} \neq 0$.

Consider the statement (8). Let $x \in L_{13}$. If $x_{2,21} = x_{2,31} = x_{2,32} = 0$, $M_x(v)$ is of the form

$$v_1 \begin{bmatrix} 0 & -{}^tA \\ A & * \end{bmatrix} + v_1 \begin{bmatrix} 0 & -{}^tB \\ B & * \end{bmatrix},$$

where A, B are 3×3 and the first and the second rows of A are zero. Since $\det M_x(v) = \det(v_1A + v_2B)^2$, $\det M_x(v)$ is divisible by v_2^4, which is a contradiction.

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Therefore, $x_{2,21}$ or $x_{2,31}$ or $x_{2,32} \neq 0$.

Consider the statement (9). Let $x \in L_{15}$. Suppose $x_{2,21} = x_{2,31} = x_{2,41} = 0$. Then by the cofactor expansion with respect to the first row and first column, $\det M_x(v)$ is a product of v_2^2 and the determinant of a matrix of the form

$$v_1 \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ * & * & * & * \end{bmatrix} + v_2 \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Therefore, $\det M_x(v)$ is divisible by v_2^4, which is a contradiction. So $x_{2,21}$ or $x_{2,31}$ or $x_{2,41} \neq 0$.

Consider the statement (10). Let $x \in L_{16}$. Suppose $x_{2,21} = x_{2,31} = 0$. Then by the cofactor expansion with respect to the first row and the first column, $\det M_x(v)$ is a product of v_2^2 and the determinant of a matrix of the form

$$v_1 \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & * & * \end{bmatrix} + v_2 \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

Therefore, $\det M_x(v)$ is divisible by v_2^3. Since $M_x(v)$ is an alternating matrix, $\det M_x(v)$ is divisible by v_2^4, which is a contradiction. So $x_{2,21}$ or $x_{2,31} \neq 0$.

Consider the statement (11). Let $x \in L_{17}$ or L_{18}. Then $x_{1,62} = 0$. Suppose $x_{2,21} = 0$. Then by considering the cofactor expansion with respect to the first row and the first column, $\det M_x(v)$ is a product of v_2^2 and the determinant of a matrix of the form

$$v_1 \begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} + v_2 \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

So, $\det M_x(v)$ is divisible by v_2^3, which is a contradiction. Therefore, $x_{2,21} \neq 0$.

This completes the proof of Proposition (3.14). Q.E.D.

The following proposition is an immediate consequence of Proposition (3.14).

PROPOSITION (3.16).

$$\Theta_{V_k^{ss}}(\Psi, \underline{\lambda t}) = \sum_{\substack{1 \le i \le 18 \\ i \neq 3,14}} \Theta_{L_i}(\Psi, \underline{\lambda t}).$$

We now consider individual cases.

(1) Consider L_1.

Note that $t^{-\gamma_{i,21}}, t^{-\gamma_{i,31}} \ll t^{-\gamma_{i,41}}$. So by Lemma (1.2.6) [8] and Lemma (3.7), for any $N \geq 1$,

$$\Theta_{L_1}(\Psi, \underline{\lambda t}) t^{-2\rho} \ll \lambda^{-N} t^{-N\gamma_{1,41}} h_{I_0}(\lambda, t) t^{w(d_0)}$$
$$\ll \lambda^{-N} \sup(1, \lambda^{-30}) t^{w(d_0 - c_0 - Nd_{1,41})}$$

Since

$$d_0 - c_0 - Nd_{1,41} = \left(\left(\frac{5}{3}, 0, -3, 0, \frac{5}{3}\right), \frac{15}{2}\right) - N\left(\left(\frac{2}{3}, \frac{1}{3}, 0, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right),$$

all the entries of $-Nd_{1,41} - c_0 + d_0$ are negative if N is large. Also if N is large, the exponent of λ tends to $-\infty$.

Therefore, $\lambda^{\sigma} \Theta_{L_1}(\Psi, \underline{\lambda t}) t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T^0_{\varepsilon}$ for $\sigma \gg 0$ and on $[1, \infty) \times T^0_{\varepsilon}$ for all σ.

(2) Consider L_2.

Let $I = I_0 \setminus \{(1,2,1), (1,3,1), (1,4,1)\}$. For $2 \leq \beta \leq 6$, we define

$$L_{2,1\beta} = \{x \in L_2 \mid x_{1,32}, x_{1,\beta 1} \neq 0\},$$
$$L_{2,2\beta} = \{x \in L_2 \mid x_{1,42}, x_{1,\beta 1} \neq 0\},$$
$$L_{2,3\beta} = \{x \in L_2 \mid x_{1,32}, x_{2,\beta 1} \neq 0\},$$
$$L_{2,4\beta} = \{x \in L_2 \mid x_{1,42}, x_{2,\beta 1} \neq 0\}.$$

(We only consider $\beta = 5,6$ for $L_{2,1\beta}, L_{2,2\beta}$.)

By Proposition (3.14)(2), $L_2 = \cup_{\alpha,\beta} L_{2,\alpha\beta}$. So

$$\Theta_{L_2}(\Psi, \underline{\lambda t}) \leq \sum_{\alpha,\beta} \Theta_{L_{2,\alpha\beta}}(\Psi, \underline{\lambda t}).$$

We consider $L_{2,1\beta}$ first.

Let $I' = \{(1,3,2), (1, \beta, 1)\}$, $I'' = I \setminus \{(1,3,2), (1, \beta, 1)\}$. We define

$$V' = \{x \in V \mid x_{i,jk} = 0 \text{ for } (i, j, k) \notin I'\},$$
$$V'' = \{x \in V \mid x_{i,jk} = 0 \text{ for } (i, j, k) \notin I''\}.$$

The subsets V', V'' are subspaces of V, and $L_{2,1\beta}$ can be considered as a subset of $V'_k \oplus V''_k$. For $x \in V' \oplus V''$, let $p'(x)$, $p''(x)$ be the projections to the first factor and the second factor respectively.

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By Lemma (1.2.5) [8], there exist $0 \leq \Psi' \in \mathcal{S}(V'_{\mathbf{A}})$, $0 \leq \Psi'' \in \mathcal{S}(V''_{\mathbf{A}})$ such that

$$\Theta_{L_{2,1\beta}}(\Psi, \underline{\lambda} t) \ll \sum_{x \in L_{2,1}\beta} \Psi'(p'(\underline{\lambda} tx))\Psi''(p''(\underline{\lambda} tx)).$$

We define

$$\Theta'(\Psi, \underline{\lambda} t) = \sum_{\substack{x \in V'_k \\ x_{1,32}, x_{1,\beta 1} \in k^{\times}}} \Psi'(\underline{\lambda} tx),$$

$$\Theta''(\Psi'', \underline{\lambda} t) = \sum_{x \in V''_k} \Psi''(\underline{\lambda} tx).$$

Then

$$\Theta_{L_{2,1\beta}}(\Psi, \underline{\lambda} t) \ll \Theta'(\Psi', \underline{\lambda} t)\Theta''(\Psi'', \underline{\lambda} t).$$

By Lemma (1.2.6) [8],

$$\Theta''(\Psi'', \underline{\lambda} t) \ll h_{I''}(\lambda, t) \leq h_I(\lambda, t).$$

We estimate $h_I(\lambda, t)$. We define

$$I_1 = \{(2, j, 1) \text{ for } j = 2,3,4\},$$

$$I_2 = \left\{ \begin{matrix} (i, 5, 1), (i, 6, 1), (i, 3, 2), (i, 4, 2), (i, 5, 2), \\ (i, 6, 2), (i, 4, 3), (i, 5, 3), (i, 5, 4) \text{ for } i = 1,2 \end{matrix} \right\},$$

$$I_3 = I \setminus (I_1 \cup I_2) = \{(i, 6, k) \text{ for } i = 1,2,\ k = 3,4,5\}.$$

Then $I = I_1 \amalg I_2 \amalg I_3$ If $(i, j, k) \in I_1$, all the entries except for the last of $d_{i,jk}$ are positive. If $(i, j, k) \in I_2$, $d_{i,jk}$ is of the form $((c_1, *, *, *, -c_1), *)$ or $((*, c_2, *, -c_2, *), *)$.

By Lemma (3.7),

$$h_{I_1}(\lambda, t) \ll \sup(1, \lambda^{-3}) t^{w(((0,0,0,0,0),\frac{3}{2}))},$$

$$h_{I_3}(\lambda, t) \ll \sup(1, \lambda^{-6}) t^{w(((2,4,4,4,4),\frac{3}{2}))}.$$

We have to be a little more careful about $h_{I_2}(\lambda, t)$. By Lemma (3.6),

$$h_{I_2}(\lambda, t) \ll \sup(1, \lambda^{-18}) \sup_{I'_2 \subset I_2} \prod_{(i,j,k) \in I'_2} t^{-\gamma_{i,jk}}.$$

By the proof of Lemma (3.7), for each $I'_2 \subset I_2$, there exist $a, b \in \mathbf{R}$ such that

$$\prod_{(i,j,k) \in I'_2} t^{-\gamma_{i,jk}} \ll t^{w(((\frac{2}{3},\frac{8}{3},2,\frac{4}{3},\frac{4}{3}),\frac{9}{2}) + ((a,b,0,-b,-a),0))}.$$

So there exist a finite number of real numbers $a_1, \cdots, a_l, b_1, \cdots, b_l$ such that

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$$h_{I_2}(\lambda, t) \ll \sup(1, \lambda^{-18}) \sum_{h=1}^{l} t^{w((\frac{2}{3},\frac{8}{3},2,\frac{4}{3},\frac{4}{3}),\frac{9}{2})+((a_h,b_h,0,-b_h,-a_h))}.$$

Moreover, $-\dfrac{4}{3} \leq a_h \leq 4$, $-\dfrac{8}{3} \leq b_h \leq \dfrac{4}{3}$ for all h.

Let

$$(3.17) \qquad p_h = \left(\left(a_h - \frac{7}{3}, 0, -3, 0, -a_h + \frac{1}{3}\right), \frac{13}{2}\right),$$

$$q_h = \left(\left(a_h - \frac{7}{3}, b_h - \frac{4}{3}, -3, -b_h - \frac{8}{3}, -a_h + \frac{1}{3}\right), \frac{13}{2}\right).$$

Then we get the following lemma by the above considerations.

LEMMA (3.18).

$$h_I(\lambda, t)t^{-2\rho} \ll \sup(1, \lambda^{-27}) \sum_{h=1}^{l} t^{w(q_h)} \ll \sup(1, \lambda^{-27}) \sum_{h=1}^{l} t^{w(p_h)}.$$

Therefore,

$$\Theta_{L_{2,1\beta}}(\Psi, \underline{\lambda}t)t^{-2\rho} \ll \sup(1, \lambda^{-27}) \sum_{h=1}^{l} \Theta'(\Psi', \underline{\lambda}t)t^{w(p_h)}.$$

This implies that we only have to estimate functions of the form $\Theta'(\Psi', \underline{\lambda}t)t^{w(p_h)}$.

For $L_{2,2\beta}$, $L_{2,3\beta}$, $L_{2,4\beta}$, exactly the same argument works replacing I' by $I' = \{(1, 4, 2), (1, \beta, 1)\}$, $\{(1, 3, 2), (2, \beta, 1)\}$, $\{(1, 4, 2), (2, \beta, 1)\}$ respectively.

By Lemma (1.2.6) [8], for any N_2, $N_2 \geq 1$,

$$\Theta'(\Psi', \underline{\lambda}t)t^{w(p_h)} \ll \lambda^{-N_1-N_2} \sup(1, \lambda^{-27})t^{w(p_h - N_1 d_{1,32} - N_2 d_{1,\beta 1})}.$$

For $L_{2,3\beta}$ etc., we get the same estimate replacing $d_{1,32}$ or $d_{1,\beta 1}$ by $d_{1,42}$ or $d_{2,\beta 1}$. However, since $t^{-\gamma_{1,32}} \ll t^{-\gamma_{1,42}}$ and $t^{-\gamma_{1,\beta 1}}$, $t^{-\gamma_{2,\beta 1}} \ll t^{-\gamma_{2,61}}$ for all β, we only have to consider functions of the form

$$\lambda^{-N_1-N_2} \sup(1, \lambda^{-27})t^{w(p_h - N_1 d_{1,42} - N_2 d_{2,61})}.$$

The point here is that we can choose N_1, N_2 for each h separately. If we had used Lemma (1.2.6) [8] directly to $\Theta_{L_{2,1\beta}}(\Psi, \underline{\lambda}t)$, we get an estimate by the function

$$\lambda^{-N_1-N_2} \sup(1, \lambda^{-27})t^{-N_1\gamma_{1,32} - N_2\gamma_{1,\beta 1}} \sum_{h=1}^{l} t^{w(p_h)},$$

and the choice of N_1, N_2 must be the same for all h. This is the reason why we

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had to separate the two non-zero coordinates to start with.

It is easy to see that

$$2d_{1,42} + d_{2,61} = \left((0,1,0,1,0),\ \frac{1}{2}\right).$$

We choose N_1, N_2 of the form $N_1 = 2N_3 + N_4$, $N_2 = N_3 + N_5$, where $N_3 \geq 1$, N_4, $N_5 \geq 0$. Then

$$\begin{aligned} &p_h - N_1 d_{1,42} - N_2 d_{2,61} \\ &= \left(\left(a_h - \frac{7}{3},\ -N_3,\ -3,\ -N_3,\ -a_h + \frac{1}{3}\right),\ \frac{13 - N_3}{2}\right) - N_4 d_{1,42} - N_5 d_{2,61}. \end{aligned}$$

If $a_h \geq 0$, we choose $N_4 = 4 + 3a_h$, $N_5 = 0$. Then

$$\begin{aligned} &p_h - N_1 d_{1,42} - N_2 d_{2,61} \\ &= \left(\left(-1,\ -\frac{4+3a_h}{3},\ -N_3,\ -3,\ -\frac{8+6a_h}{3} - N_3,\ -1\right),\ -\frac{4+3a_h}{2} + \frac{13-N_3}{2}\right). \end{aligned}$$

Since $a_h \geq 0$,

$$-\frac{4+3a_h}{3},\ -\frac{8+6a_h}{3},\ -\frac{4+3a_h}{2} \leq 0.$$

Therefore,

$$t^{w(p_h - N_1 d_{1,42} - N_2 d_{2,61})} \ll t^{w(((-1,-N_3,-3,-N_3,-1),\frac{13-N_3}{2}))},$$
$$\lambda^{-N_4 - N_5} \ll \sup(\lambda^{-4},\ \lambda^{-8}) = \lambda^{-4} \sup(1,\ \lambda^{-4}).$$

If $a_h \leq 0$, we choose $N_4 = 4$, $N_5 = -\dfrac{3a_h}{2}$. Then

$$\begin{aligned} &p_h - N_1 d_{1,42} - N_2 d_{2,61} \\ &= \left(\left(-1,\ -\frac{4}{3} + \frac{a_h}{2},\ -N_3,\ -3,\ -\frac{8}{3} - \frac{a_h}{2} - N_3,\ -1\right),\ -2 + \frac{3a_h}{4} + \frac{13-N_3}{2}\right). \end{aligned}$$

Since $-\dfrac{4}{3} \leq a_h$ for all h,

$$-\frac{4}{3} + \frac{a_h}{2},\ -\frac{8}{3} - \frac{a_h}{2},\ -2 + \frac{3a_h}{4} \leq 0.$$

Therefore,

$$t^{w(p_h - N_1 d_{1,42} - N_2 d_{2,61})} \ll t^{w(((-1,-N_3,-3,-N_3,-1),\frac{13-N_3}{2}))},$$

$$\lambda^{-N_4-N_5} \ll \sup(\lambda^{-4}, \lambda^{-10}) = \lambda^{-4} \sup(1, \lambda^{-6}).$$

By the above considerations,

$$\Theta'(\Psi, \underline{\lambda t}) t^{w(p_h)} \ll \sup(1, \lambda^{-33}) \lambda^{-3N_3-4} t^{w((-1,-N_3,-3,-N_3,-1),\frac{13-N_3}{2})}.$$

This bound does not depend on α, β, h. So for any $N_3 \geq 1$,

$$\Theta_{L_2}(\Psi, \underline{\lambda t}) t^{-2\rho} \ll \sup(1, \lambda^{-33}) \lambda^{-3N_3-4} t^{w((-1,-N_3,-3,-N_3,-1),\frac{13-N_3}{2})}.$$

Therefore, $\lambda^{\sigma}\Theta_{L_2}(\Psi, \underline{\lambda t}) t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

(3) Consider L_4.

Let $I = I_0 \setminus \{(1,2,1), \cdots, (1,5,1), (1,3,2), (1,4,2), (1,4,3)\}$. Then by Lemma (1.2.6) [8], for any $N \geq 1$,

$$\Theta_{L_4}(\Psi, \underline{\lambda t}) t^{-2\rho} \ll \lambda^{-3N} t^{-N(\gamma_{1,43}+2\gamma_{1,51})} h_I(\lambda, t) t^{-2\rho}.$$

By Lemma (3.7),

$$h_I(\lambda, t) \ll \sup(1, \lambda^{-23}) t^{w((\frac{17}{3},8-\frac{2}{3},3,8,-\frac{1}{3},\frac{20}{3}),\frac{15}{2})},$$

$$h_I(\lambda, t) t^{-2\rho} \ll \sup(1, \lambda^{-23}) t^{w((\frac{2}{3},-\frac{2}{3},-3,-\frac{1}{3},\frac{5}{3}),\frac{13}{2})}.$$

It is easy to see that

$$d_{1,43} + 2d_{1,51} = \left((1,0,0,0,1), \frac{3}{2}\right).$$

Since the first, the fifth, and the last entries are positive, all the entries of

$$\left(\left(\frac{2}{3}, -\frac{2}{3}, -3, -\frac{1}{3}, \frac{5}{3}\right), \frac{13}{2}\right) - N(d_{1,43} + 2d_{1,51})$$

are negative.

Therefore, $\lambda^{\sigma}\Theta_{L_4}(\Psi, \underline{\lambda t}) t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

(4) Consider L_5.

Let $L_{5,\alpha\beta} = \{x \in L_5 \mid x_{\alpha,\beta 1} \neq 0$ for $\alpha = 1,2, \beta = 2, \cdots, 6\}$. Then $L_5 = \cup_{\alpha,\beta} L_{5,\alpha\beta}$.

We define

$$I = I_0 \setminus \{(1,2,1), (1,3,1), (1,4,1), (1,5,1), (1,3,2), (1,4,2), (1,5,2)\}.$$

Then $h_I(\lambda, t)$ has the same bound as in Lemma (3.18) with $a_h, b_h, q_h \in \mathbf{R}$ for

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$h=1,\cdots,l$.

We fix α, β. Let $I'=\{(1,4,3),(1,5,2),(\alpha,\beta,1)\}$. For $\Psi'\in\mathcal{S}(V'_{\mathbf{A}})$, we define

$$\Theta'(\Psi',\underline{\lambda t})=\sum_{\substack{x\in V'_k\\ x_{1,43},x_{1,52},x_{\alpha,\beta,1}}}\Psi'(\underline{\lambda t}x).$$

By a similar consideration as before, there exists $0\le\Psi'\in\mathcal{S}(V'_{\mathbf{A}})$ such that

$$\Theta_{L_{5,\alpha\beta}}(\Psi,\underline{\lambda t})t^{-2\rho}\ll\sup(1,\lambda^{-27})\sum_{l=1}^{h}\Theta'(\Psi',\underline{\lambda t})t^{w(q_h)}.$$

We consider each term. By Lemma (1.2.6) [8], for any N_1, N_2, $N_3\ge1$,

$$\Theta'(\Psi',\underline{\lambda t})t^{w(q_h)}\ll\lambda^{-N_1-N_2-N_3}t^{w(q_h-N_1d_{1,43}-N_2d_{1,52}-N_3d_{\alpha,\beta1})}.$$

Since $t^{-\gamma_{\alpha,\beta1}}\ll t^{-\gamma_{2,61}}$ for all α, β as above, we only consider the case $(\alpha,\beta)=(2,6)$.

It is easy to see that

$$d_{1,43}+d_{1,52}+d_{2,61}=\left((0,0,0,0,0),\frac{1}{2}\right).$$

So we put $N_1=N_4+N_5$, $N_2=N_4+N_6$, $N_3=N_4+N_7$, where $N_4\ge1$, N_5, N_6, $N_7\ge0$.

Let $W=\{(a,b,0,-b,-a)\mid a,b\in\mathbf{R}\}\subset\mathbf{R}^5$. The following lemma and its corollary are easy to verify and the proofs are left to the reader.

LEMMA (3.19). *The convex hull of*

$$\left\{\left(-\frac{1}{3},-\frac{2}{3},0,\frac{2}{3},\frac{1}{3}\right),\left(-\frac{1}{3},\frac{1}{3},0,-\frac{1}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{1}{3},0,-\frac{1}{3},-\frac{2}{3}\right)\right\}$$

contains a neighborhood of the origin of W.

COROLLARY (3.20). *For any* a, $b\in\mathbf{R}$, *there exist* c_1, c_2, $c_3\ge0$ *such that*

$$c_1\left(-\frac{1}{3},-\frac{2}{3},0,\frac{2}{3},\frac{1}{3}\right)+c_2\left(-\frac{1}{3},\frac{1}{3},0,-\frac{1}{3},\frac{1}{3}\right)+c_3\left(\frac{2}{3},\frac{1}{3},0,-\frac{1}{3},-\frac{2}{3}\right)$$
$$=(a,b,0,-b,-a).$$

By the above Corollary, we choose N_5, N_6, $N_7\ge0$ so that

$$N_5\left(-\frac{1}{3},-\frac{2}{3},0,\frac{2}{3},\frac{1}{3}\right)+N_6\left(-\frac{1}{3},\frac{1}{3},0,-\frac{1}{3},\frac{1}{3}\right)+N_7\left(\frac{2}{3},\frac{1}{3},0,-\frac{1}{3},-\frac{2}{3}\right)$$

$$= \left(a_h - \frac{4}{3},\ b_h,\ 0,\ -b_h,\ -a_h + \frac{4}{3}\right).$$

Then

$$\Theta'(\Psi', \underline{\lambda} t)\, t^{w(q_h)} \ll \lambda^{-3N_4-N_5-N_6-N_7} t^{w(((-1,-\frac{4}{3},-3,-\frac{8}{3},-1),\frac{13-N_4}{2}))}.$$

Since there are finitely many possibilities for h, there exist c_1, $c_2 \geq 0$ such that

$$\Theta_{L_{5,\alpha\beta}}(\Psi, \underline{\lambda} t)\, t^{-2\rho} \ll \lambda^{-3N_4-c_1} \sup(1, \lambda^{-27-c_2})\, t^{w(((-1,\frac{4}{3},-3,-\frac{8}{3},-1),\frac{13-N_4}{2}))}.$$

The right hand side does not depend on α, β, h.

Therefore, $\lambda^{\sigma}\Theta_{L_5}(\Psi, \underline{\lambda} t)\, t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_{\varepsilon}^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_{\varepsilon}^0$ for all σ.

(5) Consider L_6.

For $\alpha = 2, \cdots, 5$, $\beta = 1,2$, $\alpha > \beta$, we define $L_{6,\alpha\beta} = \{x \in L_6 \mid x_{2,\alpha\beta} \neq 0\}$. Then by Proposition (3.14)(3), $L_6 = \cup_{\alpha,\beta} L_{6,\alpha\beta}$. Since $t^{-\gamma_{2,\alpha\beta}} \ll t^{-\gamma_{2,52}}$ for all α, β as above, we only consider the case $(\alpha, \beta) = (5,2)$. If $x \in L_{6,52}$, $x_{1,43}$, $x_{1,61}$, $x_{2,52} \neq 0$. So by the same argument as in (4), $\lambda^{\sigma}\Theta_{L_6}(\Psi, \underline{\lambda} t)\, t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_{\varepsilon}^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_{\varepsilon}^0$ for all σ.

(6) Consider L_7.

For $\alpha = 2, \cdots, 5$, we define $L_{7,\alpha} = \{x \in L_7 \mid x_{2,\alpha 1} \neq 0\}$. Then by Proposition (3.14)(4), $L_7 = \cup_{\alpha} L_{7,\alpha}$. Since for all $t^{-\gamma_{2,\alpha 1}} \ll t^{-\gamma_{2,51}}$ for all α as above, we only consider the case $\alpha = 5$.

Let

$$I = I_0 \setminus \{(1, j, k) \text{ for } j = 2, \cdots, 6,\ k = 1,2,\ j > k,\ (1,4,3)\}.$$

Then by Lemma (3.7),

$$h_I(\lambda, t) \ll \sup(1, \lambda^{-20})\, t^{w(((5,8-\frac{2}{3},6,8-\frac{4}{3},\frac{16}{3}),\frac{15}{2}))}$$
$$h_I(\lambda, t)\, t^{-2\rho} \ll \sup(1, \lambda^{-20})\, t^{w(((0,-\frac{2}{3},-3,-\frac{4}{3},\frac{1}{3}),\frac{13}{2}))}.$$

By Lemma (1.2.6) [8], for any N_1, N_2, $N_3 \geq 1$,

$$\Theta_{L_{7,5}}(\Psi, \underline{\lambda} t) \ll \sup(1, \lambda^{-20})\lambda^{-N_1-N_2-N_3} \times t^{w(((0,-\frac{2}{3},-3,-\frac{4}{3},\frac{1}{3}),\frac{13}{2})-N_1 d_{1,43}-N_2 d_{1,62}-N_3 d_{2,51})}.$$

It is easy to see that

$$3d_{1,43} + 2d_{1,62} + 4d_{2,51} = \left((1,0,0,0,1),\ \frac{1}{2}\right).$$

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So if we choose $N_1 = 3N_4$, $N_2 = 2N_4$, $N_3 = 4N_4$ and $N_4 \gg 0$, all the entries of

$$\begin{aligned}&\left(\left(0, -\frac{2}{3}, -3, \frac{4}{3}, \frac{1}{3}\right), \frac{13}{2}\right) - N_1 d_{1,43} - N_2 d_{1,62} - N_3 d_{2,51}\\ &= \left(\left(-N_4, -\frac{2}{3}, -3, \frac{4}{3}, \frac{1}{3} - N_4\right), \frac{13 - N_4}{2}\right)\end{aligned}$$

are negative.

Therefore, $\lambda^{\sigma}\Theta_{L_7}(\Psi, \underline{\lambda} t)t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

(7) Consider L_8.

By Lemma (1.2.6) [8], for any $N \geq 1$,

$$\Theta_{L_8}(\Psi, \underline{\lambda} t)t^{-2\rho} \ll \lambda^{-5N} t^{w(d_0 - N(3d_{1,43} + 2d_{2,21}))} h_{I_0}(\lambda, t).$$

It is easy to see that

$$3d_{1,43} + 2d_{2,21} = \left(\left(\frac{1}{3}, \frac{2}{3}, 2, \frac{10}{3}, \frac{5}{3}\right), \frac{1}{2}\right).$$

Since all the entries of the above element are positive, $\lambda^{\sigma}\Theta_{L_8}(\Psi, \underline{\lambda} t)t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

(8) Consider L_9.

Let

$$I = I_0 \setminus \{(1, j, k) \text{ for } 1 \leq k < j \leq 4, (1,5,1)\},$$

Then by Lemma (3.7),

$$\begin{aligned}h_I(\lambda, t) &\ll \sup(1, \lambda^{-23}) t^{w(((\frac{17}{3}, 8 - \frac{2}{3}, 6, 8, -\frac{1}{3}, \frac{20}{3}), \frac{15}{2}))},\\ h_I(\lambda, t)t^{-2\rho} &\ll \sup(1, \lambda^{-23}) t^{w(((\frac{4}{3}, -\frac{2}{3}, -3, -\frac{1}{3}, \frac{5}{3}), \frac{13}{2}))}.\end{aligned}$$

For $1 \leq \beta < \alpha \leq 4$, we define $L_{9,\alpha\beta} = \{x \in L_9 \mid x_{2,\alpha\beta} \neq 0\}$. Then by Proposition (3.14)(5), $L_9 = \cup_{\alpha,\beta} L_{9,\alpha\beta}$. Since $t^{-\gamma_{2,\alpha\beta}} \ll t^{-\gamma_{2,43}}$ for all α, β as above, by Lemma (1.2.6) [8], for any $N \geq 1$,

$$\Theta_{L_{9,\alpha\beta}}(\Psi, \underline{\lambda} t)t^{-2\rho} \ll \sup(1, \lambda^{-21})\lambda^{-3N} t^{w(((0, -\frac{2}{3}, -3, -1, 1), \frac{13}{2}) - N(2d_{1,51} - d_{2,43}))}.$$

It is easy to see that

$$2d_{1,51} + d_{2,43} = \left((1,0,0,0,1), \frac{1}{2}\right).$$

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So all the entries of

$$\left(\left(\frac{4}{3}, -\frac{2}{3}, -3, -\frac{1}{3}, \frac{5}{3}\right), \frac{13}{2}\right) - N(2d_{1,51} + d_{2,43})$$
$$= \left(\left(\frac{4}{3} - N, -\frac{2}{3}, -3, -\frac{1}{3}, \frac{4}{3} - N\right), \frac{13-N}{2}\right)$$

are negative if $N \gg 0$.

Therefore, $\lambda^{\sigma}\Theta_{L_9}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $\mathbf{R}_+ \neq T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

(9) Consider L_{10}.

For $1 \leq \beta < \alpha \leq 4$, we define $L_{10,\alpha\beta} = \{x \in L_{10} \mid x_{2,\alpha\beta} \neq 0\}$. Then by Proposition (3.14)(5), $L_{10} = \cup_{\alpha,\beta} L_{10,\alpha\beta}$. Since $t^{-\gamma_{2,\alpha\beta}} \ll t^{-\gamma_{2,43}}$ for all α, β as above, we only consider the case $(\alpha, \beta) = (4,3)$.

Let $I' = \{(1,5,2), (1,6,2), (2,4,3)\}$, and

$$V' = \{x \in V \mid x_{i,jk} = 0 \text{ for } (i, j, k) \notin I'\}.$$

For $\Psi' \in \mathcal{S}(V'_{\mathbf{A}})$, we define

$$\Theta'(\Psi', \underline{\lambda t}) = \sum_{\substack{x \in V'_k \\ x_{1,52}, x_{1,61}, x_{2,\alpha\beta} \neq 0}} \Psi'(\underline{\lambda t} x).$$

Then as before, there exists $0 \leq \Psi' \in \mathcal{S}(V'_{\mathbf{A}})$ such that

$$\Theta_{L_{10,43}}(\Psi, \underline{\lambda t})t^{-2\rho} \ll \sup(1, \lambda^{-27})\Theta'(\Psi', \underline{\lambda t}) \sum_{h=1}^{l} t^{w(q_h)}.$$

By Lemma (1.2.6) [8], for any $N_1, N_2, N_3 \geq 1$,

$$\Theta'(\Psi', \underline{\lambda t})t^{w(q_h)} \ll \sup(1, \lambda^{-27})\lambda^{-N_1-N_2-N_3} t^{w(q_h - N_1 d_{1,61} - N_2 d_{1,52} - N_3 d_{2,43})}.$$

It is easy to see that

$$d_{1,61} + d_{1,52} + d_{2,43} = \left((0,0,0,0,0), \frac{1}{2}\right).$$

Therefore, by the same argument as in (4), $\lambda^{\sigma}\Theta_{L_{10}}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^\alpha$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^\alpha$ for all σ.

(10) Consider L_{11}.

For $\alpha = 2,3,4$, we define $L_{11,\alpha} = \{x \in L_{11} \mid x_{2,\alpha 1} \neq 0\}$. Then by Proposition (3.14)(6), $L_{11} = \cup_{\alpha} L_{11,\alpha}$. Since $t^{-\gamma_{2,\alpha 1}} \ll t^{-\gamma_{2,43}}$ for $\alpha = 2,3,4$, we only consider the case $\alpha = 4$.

It is easy to see that

$$3d_{1,52} + 2d_{1,41} = \left(\left(\frac{1}{3}, \frac{5}{3}, 0, \frac{1}{3}, \frac{5}{3}\right), \frac{1}{2}\right).$$

So by Lemma (1.2.6) [8], for any $N \geq 1$,

$$\begin{aligned}\Theta_{L_{11,4}}(\Psi, \underline{\lambda t})t^{-2\rho} &\ll \lambda^{-5N} t^{w(d_0 - N(3d_{1,52}+2d_{1,41}))} h_{I0}(\underline{\lambda t}) \\ &\ll \lambda^{-5N} \sup(1, \lambda^{-30}) t^{w(((\frac{5-N}{3}, -\frac{5N}{3}, -3, -\frac{N}{3}, \frac{5-5N}{3}), \frac{13-N}{2}))}.\end{aligned}$$

Therefore, $\lambda^{\sigma}\Theta_{L_{11}}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

(11) Consider L_{12}.

For $\alpha = 2,3,4$, $\beta = 1,2$, $\alpha > \beta$, we define $L_{12,\alpha\beta} = \{x \in L_{12} \mid x_{2,\alpha\beta} \neq 0\}$. Then by Proposition (3.14)(7), $L_{12} = \cup_{\alpha,\beta} L_{12,\alpha\beta}$. Since $t^{-\gamma_{2,\alpha\beta}} \ll t^{-\gamma_{2,42}}$ for all α, β as above, we only consider the case $(\alpha, \beta) = (4.2)$.

It is easy to see that

$$3d_{1,61} + 2d_{1,53} + 4d_{2,42} = \left((0,1,0,1,0), \frac{1}{2}\right),$$

$$4d_{1,61} + 2d_{1,53} + 4d_{2,42} = \left(\left(\frac{2}{3}, \frac{4}{3}, 0, \frac{2}{3}, -\frac{2}{3}\right), 1\right).$$

$$d_{1,61} + d_{1,53} + d_{2,42} = \left(\left(-\frac{1}{3}, \frac{1}{3}, 0, \frac{2}{3}, \frac{1}{3}\right), 0\right).$$

Also

$$t^{-w(((\frac{2}{3},\frac{4}{3},0,\frac{2}{3},-\frac{2}{3}),1))} \ll t^{-w(((-\frac{1}{3},0,0,0,\frac{1}{3}),0))},$$

$$t^{-w(((-\frac{1}{3},\frac{1}{3},0,\frac{2}{3},\frac{1}{3}),0))} \ll t^{-w(((-\frac{1}{3},0,0,0,\frac{1}{3}),0))}.$$

We define

$$m_1 = \left((0,1,0,1,0), \frac{1}{2}\right),$$

$$m_2 = \left(\left(\frac{2}{3}, 0,0,0, -\frac{2}{3}\right), 0\right),$$

$$m_3 = \left(\left(-\frac{1}{3}, 0,0,0, \frac{1}{3}\right), 0\right).$$

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Then by the same argument as in (2), we only have to consider functions of the form

$$\lambda^{-10N_1-4N_2-9N_3}\sup(1, \lambda^{-27})t^{w(p_h-N_1m_1-N_2m_2-N_3m_3)},$$

where $N_1 \geq 1$, N_2, $N_3 \geq 0$.

It is easy to see that we can choose N_1, N_2, N_3 so that all the entries of $p_h - N_1m_1 - N_2m_2 - N_3m_3$ are negative.

Therefore, $\lambda^\sigma \Theta_{L_{12}}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[0, \infty) \times T_\varepsilon^0$ for all σ.

(12) Consider L_{13}.

For $(\alpha, \beta) = (2,1), (3,1), (3,2)$, we define $L_{13,\alpha\beta} = \{x \in L_{13} \mid x_{2,\alpha\beta} \neq 0\}$. Then by Proposition (3.14)(8) $L_{13} = \cup_{\alpha,\beta} L_{13,\alpha\beta}$. Since $t^{-\gamma_{2\alpha\beta}} \ll t^{-\gamma_{2,32}}$ for all α, β as above, we only consider the case $(\alpha, \beta) = (3,2)$.

It is easy to see that

$$3d_{1,61} + 2d_{1,54} + 4d_{2,32} = \left((0,1,2,1,0), \frac{1}{2}\right),$$

$$d_{1,61} + d_{2,32} = \left(\left(\frac{1}{3}, \frac{2}{3}, 1, \frac{1}{3}, -\frac{1}{3}\right), 0\right),$$

$$d_{1,61} + d_{1,54} + 2d_{2,32} = \left(\left(-\frac{1}{3}, \frac{1}{3}, 1, \frac{2}{3}, \frac{1}{3}\right), 0\right).$$

Also

$$t^{-w(((\frac{1}{3},\frac{2}{3},1,\frac{1}{3},-\frac{1}{3}),0))} \ll t^{-w(((\frac{1}{3},0,0,0,-\frac{1}{3}),0))},$$

$$t^{-w(((-\frac{1}{3},\frac{1}{3},1,\frac{2}{3},\frac{1}{3}),0))} \ll t^{-w(((-\frac{1}{3},0,0,0,\frac{1}{3}),0))}.$$

Therefore, by the argument of (2) and (11), $\lambda^\sigma \Theta_{L13}(\Psi, \underline{\lambda t})t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T_\varepsilon^0$ for $\sigma \gg 0$ and on $[1, \infty) \times T_\varepsilon^0$ for all σ.

(13) Consider L_{15}.

For $\alpha = 2,3,4$, we define $L_{15,\alpha} = \{x \in L_{15} \mid x_{2,\alpha 1} \neq 0\}$. Then by Proposition (3.14)(9), $L_{15} = \cup_\alpha L_{15,\alpha}$. Since $t^{-\gamma_{2,\alpha 1}} \ll t^{-\gamma_{2,41}}$ for $\alpha = 2,3,4$, we only consider the case $\alpha = 4$.

It is easy to see that

$$2d_{1,62} + 2d_{1,53} + 3d_{2,41} = \left(\left(\frac{2}{3}, \frac{1}{3}, 0, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right).$$

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Then by Lemma (1.2.6) [8], for any $N \geq 1$,

$$\Theta_{L_{15,4}}(\Psi, \underline{\lambda t}) t^{-2\rho} \ll \lambda^{-7N} t^{w(d_0 - N((\frac{2}{3},\frac{1}{3},0,\frac{2}{3},\frac{1}{3}),\frac{1}{2}))} h_{I_0}(\lambda, t)$$

$$\ll \lambda^{-7N} \sup(1, \lambda^{-30}) t^{w(((\frac{5}{3},0,-3,0,\frac{5}{3}),\frac{13}{2}) - N((\frac{2}{3},\frac{1}{3},0,\frac{2}{3},\frac{1}{3}),\frac{1}{2}))}.$$

Therefore, by the argument of (11), $\lambda^\sigma \Theta_{L_{15}}(\Psi, \underline{\lambda t}) t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T^0_\varepsilon$ for $\sigma \gg 0$ and on $[1, \infty) \times T^0_\varepsilon$ for all σ.

(14) Consider L_{16}.

For $\alpha = 2,3$, we define $L_{16,\alpha} = \{x \in L_{16} \mid x_{2,\alpha 1} \neq 0\}$. Then by Proposition (3.14)(10), $L_{16} = \cup_\alpha L_{16,\alpha}$. Since $t^{-\gamma_{2,\alpha 1}} \ll t^{-\gamma_{2,31}}$ for $\alpha = 2,3$, we only consider the case $\alpha = 3$.

It is easy to see that

$$2d_{1,62} + 2d_{1,54} + 3d_{2,31} = \left(\left(\frac{2}{3}, \frac{1}{3}, 1, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right).$$

Since all the entries of the above element are positive, $\lambda^\sigma \Theta_{L_{16}}(\Psi, \underline{\lambda t}) t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T^0_\varepsilon$ for $\sigma \gg 0$ and on $[1, \infty) \times T^0_\varepsilon$ for all σ.

(15) Consider L_{17}.

It is easy to see that

$$3d_{1,53} + 2d_{2,21} = \left(\left(\frac{1}{3}, \frac{2}{3}, 2, \frac{1}{3}, \frac{5}{3}\right), \frac{1}{2}\right).$$

Since all the entries of the above element are positive, $\lambda^\sigma \Theta_{L_{17}}(\Psi, \underline{\lambda t}) t^{-2\rho}$ is integrable on $\mathbf{R}_+ \times T^0_\varepsilon$ for $\sigma \gg 0$ and on $[1, \infty) \times T^0_\varepsilon$ for all σ.

(16) Consider L_{18}.

It is easy to see that

$$2d_{1,63} + 2d_{1,54} + 3d_{2,21} = \left(\left(\frac{2}{3}, \frac{4}{3}, 1, \frac{2}{3}, \frac{1}{3}\right), \frac{1}{2}\right).$$

Since all the entries of the above element are positive, $\lambda^\sigma \Theta_{L_{18}}(\Psi, \underline{\lambda t}) t^{-2\rho}$ is in tegrable on $\mathbf{R}_+ \times T^0_\varepsilon$ for $\sigma \gg 0$ and on $[1, \infty) \times T^0_\varepsilon$ for all σ.

This completes the proof of Theorem (3.1) for the case (4). Q.E.D.

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