ON COCHARACTERS ASSOCIATED TO NILPOTENT ELEMENTS OF REDUCTIVE GROUPS

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Abstract. Let G be a connected reductive linear algebraic group defined over an algebraically closed field of characteristic p. Assume that p is good for G. In this note we consider particular classes of connected reductive subgroups H of G and show that the cocharacters of H that are associated to a given nilpotent element e in the Lie algebra of H are precisely the cocharacters of G associated to e that take values in H. In particular, we show that this is the case provided H is a connected reductive subgroup of G of maximal rank; this answers a question posed by J. C. Jantzen.

§1. Introduction

Let G be a connected reductive linear algebraic group defined over some algebraically closed field k, let $\mathfrak{g}$ be its Lie algebra and $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$. The Jacobson-Morozov Theorem allows one to associate an $\mathfrak{sl}(2)$-triple to any given non-zero nilpotent element in $\mathcal{N}$ in characteristic zero or large positive characteristic. This is an indispensable tool in the Dynkin-Kostant classification of the nilpotent orbits in characteristic zero as well as in the Bala-Carter classification of unipotent conjugacy classes of G in large prime characteristic, see [5, §5.9]. In good characteristic there is a replacement for $\mathfrak{sl}(2)$-triples, so called *associated cocharacters*; see Definition 2.9 below. These cocharacters have become important tools in the classification theory of unipotent and nilpotent classes of reductive algebraic groups in good characteristic, see for instance [7, §5], [10], [12], [15], and [16] for more details.

In [7, §5.6], J. C. Jantzen studies the behaviour of cocharacters associated to nilpotent elements under elementary operations of algebraic groups such as passing to derived subgroups, taking direct products and isogenies of reductive groups. While these cocharacters also behave well with respect to inclusions of reductive subgroups in characteristic zero, this is not the

case in general in positive characteristic, [7, §5.12]; given that they serve as a replacement for $\mathfrak{sl}(2)$-triples, this is somewhat surprising.

More precisely, in [7, Claim 5.12], Jantzen showed that if char $k = 0$ and H is a connected reductive subgroup of G with Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$, then

(†) the cocharacters of H associated to $e \in \mathfrak{h} \cap \mathcal{N}$ are precisely the cocharacters of G associated to e that take values in H.

Jantzen continues to give an example in [7, §5.12] which shows that (†) fails in general in positive characteristic, even when char k is good for G.

G. McNinch pointed out that this failure ultimately stems from the fact that the representations of H are not semisimple in general in positive characteristic, as is the case in Jantzen's counterexample. The following construction due to McNinch shows that (†) fails generically: Let H be a connected reductive group whose Coxeter number satisfies $h < p = \mathrm{char}\, k$. Let $K \leq H$ be a *principal* SL_2-subgroup of H and let V be a faithful representation of H for which there is a K-composition factor of the restriction of V to K having a non-restricted highest weight. See [11] for a definition, existence and uniqueness up to conjugacy of principal SL_2-subgroups, see also [20]. A maximal torus of K determines a cocharacter λ of H associated to some nilpotent element e of $\mathrm{Lie}\, K$. However, viewed as a cocharacter of $\mathrm{GL}(V)$, λ is not associated to e. In this sense, (†) fails for $H \leq \mathrm{GL}(V)$ for most representations V of H.

Nevertheless, a calculation of Jantzen shows that for G of classical type and char k a good prime for G, (†) does hold provided the subgroup H of G is of maximal rank, see [7, §5.12]. In this note we give a general proof showing that this is indeed always the case irrespective of the type of G:

THEOREM 1.1. *Suppose* char k *is good for* G. *Let* H *be a connected reductive subgroup of* G *of maximal rank. Then* (†) *holds.*

This answers a question posed by J. C. Jantzen, see [7, §5.12]. We then extend this result to the case when H is a regular reductive subgroup of G, see Theorem 3.29.

In Theorem 3.18 we show that (†) holds provided there is at least one cocharacter of G that is associated to e and lies in H. In our principal result we show that this is the case for a special class of subgroups:

THEOREM 1.2. *Let* G *be a connected reductive algebraic group. Suppose that* char k *is a good prime for* G. *Let* S *be a linearly reductive group acting on* G *by automorphisms and set* $H = C_G(S)^\circ$. *Then* (†) *holds.*

Theorem 1.2 is a consequence of Theorem 3.18 and Lemma 3.20. Our proof uses ideas due to R. W. Richardson that he used in order to show that any S-stable parabolic subgroup of G admits an S-stable Levi subgroup in [17, Prop. 6.1, §6.2, §6.3].

In turn Theorem 1.1 is a direct consequence of Theorem 1.2.

In Subsection 3.1 we consider arbitrary connected reductive subgroups H of G and give criteria for a given nilpotent element $e \in \mathfrak{h}$ that will ensure (†) to hold. For instance, in Lemma 3.1 we prove that this is the case provided $e \in \mathfrak{h}$ is distinguished in $\mathfrak{g}$. In another result, Theorem 3.14, we show that (†) holds given that the ranks of the centralizers of e in H and G coincide.

We assume throughout that char k is good for G. Then there exists a G-equivariant homeomorphism $\mathcal{N} \to \mathcal{U}$ between the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ and the unipotent variety $\mathcal{U}$ of G. Such a map is called a *Springer isomorphism*, see [25, III, 3.12] and [1, Cor. 9.3.4]. By means of such a map, all the results below admit analogues for associated cocharacters of unipotent elements in G.

For general results on algebraic groups we cite Borel's book [2], and for basic results on cocharacters associated to nilpotent elements, we refer the reader to Jantzen's monograph [7, Ch. 5] and the articles by McNinch-Sommers [12] and Premet [16].

§2. Preliminaries

2.1. Notation

Throughout, G is a connected reductive algebraic group defined over an algebraically closed field k and $p = \operatorname{char} k$ is a good prime for G, although many results hold without this assumption. We denote the Lie algebra of G by $\operatorname{Lie} G$ or by $\mathfrak{g}$; likewise for closed subgroups of G. For $e \in \mathfrak{g}$ and $g \in G$ we denote the adjoint action of g on e by $\operatorname{Ad}(g)e$. The centralizers of e in G and $\mathfrak{g}$ are $C_G(e) = \{g \in G \mid \operatorname{Ad}(g)e = e\}$ and $\mathfrak{c}_{\mathfrak{g}}(e) = \{x \in \mathfrak{g} \mid [x, e] = 0\}$, respectively. We write $Z(G)$ for the centre of G.

Let H be a closed subgroup of G. We write H° for the identity component of H and $C_G(H) = \{g \in G \mid ghg^{-1} = h \text{ for all } h \in H\}$ for the centralizer of H in G. The normalizer of H in G is $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. The derived subgroup of H is denoted by $\mathcal{D}H$ and we write $\operatorname{rank} H$ for the dimension of a maximal torus of H. The unipotent radical of H is denoted by $R_u(H)$. A *Levi subgroup* of H is a complement to $R_u(H)$ in H, [2, Defn. 11.22]; in contrast to *loc. cit.*, we do not require H to be connected;

we also refer to the semi-direct product of a Levi subgroup of H and $R_u(H)$ as a *Levi decomposition* of H.

By a Levi subgroup of G we mean a Levi subgroup of a parabolic subgroup of G. Let S be a torus of G. Then $C_G(S)$ is a Levi subgroup of G, [2, Thm. 20.4]. Note that $C_G(S)$ is connected, [2, Cor. 11.12]. Moreover, by [2, Prop. 8.18] there exists an element $s \in S$ such that $C_G(s) = C_G(S)$. Conversely, every Levi subgroup of G is of this form, e.g., see Lemma 2.6(ii).

Let $Y(G) = \mathrm{Hom}(k^*, G)$ denote the set of cocharacters of G. For $\mu \in Y(G)$ we write $C_G(\mu)$ for $C_G(\mu(k^*))$. For $\mu \in Y(G)$ and $g \in G$ we define the *conjugate cocharacter* $g \cdot \mu \in Y(G)$ by $(g \cdot \mu)(t) = g\mu(t)g^{-1}$; this gives a left action of G on $Y(G)$. For H a (connected) reductive subgroup of G, let $Y(H) = \mathrm{Hom}(k^*, H)$ denote the set of cocharacters of H. There is an obvious inclusion $Y(H) \subseteq Y(G)$.

If S is a linear algebraic group acting on G by automorphisms, then we say that G is an *S-group*; we write $s \cdot g$ for the action of $s \in S$ on $g \in G$. The subgroup of G consisting of the S-fixed points is denoted by $C_G(S) = \{g \in G \mid s \cdot g = g \text{ for all } s \in S\}$. If G is an S-group, then S acts naturally by means of Lie algebra automorphisms on $\mathfrak{g}$. By abuse of notation, we simply denote the action of S on $\mathfrak{g}$ by $s \cdot e$ for $s \in S$ and $e \in \mathfrak{g}$. We denote the subalgebra of $\mathfrak{g}$ consisting of the S-fixed points for the induced action on $\mathfrak{g}$ by $\mathfrak{c}_{\mathfrak{g}}(S) = \{e \in \mathfrak{g} \mid s \cdot e = e \text{ for all } s \in S\}$. Also, S acts on $Y(G)$ by acting on the image of a cocharacter in G: $(s \cdot \lambda)(t) = s \cdot \lambda(t)$ for $s \in S$, $\lambda \in Y(G)$, and $t \in k^*$.

More generally, if S and the S-group G both act morphically on an algebraic variety X, then, following [17, (2.1)], the actions of G and S are said to be *compatible*, provided

$$(2.1) \qquad s \cdot (g \cdot x) = (s \cdot g) \cdot (s \cdot x)$$

for all $s \in S$, $g \in G$ and $x \in X$. This is the unique action that defines a morphic action of the semidirect product of G and S on X which extends the actions of both G and S on X, see [17, §2]. All actions by an S-group G together with S considered in this paper are compatible in the sense of (2.1).

Let T be a maximal torus of G. Let $\Psi = \Psi(G, T)$ denote the set of roots of G with respect to T. Fix a Borel subgroup B of G containing T and let $\Pi = \Pi(G, T)$ be the set of simple roots of Ψ defined by B. Then $\Psi^+ = \Psi(B)$ is the set of positive roots of G. For $\beta \in \Psi^+$ write $\beta = \sum_{\alpha \in \Pi} c_{\alpha\beta}\alpha$ with

$c_{\alpha\beta} \in \mathbb{N}_0$. A prime p is said to be *good* for G if it does not divide $c_{\alpha\beta}$ for any α and β, [25, Defn. 4.1].

2.2. Linearly reductive groups

We refer to Richardson's article [17] for information on centralizers of the action of linearly reductive groups on connected reductive groups. Recall that a linear algebraic group S, not necessarily connected, is said to be *linearly reductive* if every rational representation of S is semisimple. It is well known that in characteristic zero, S is linearly reductive if and only if S° is reductive. In characteristic $p > 0$, S is linearly reductive if and only if every element of S is semisimple if and only if S° is a torus and $|S/S^\circ|$ is coprime to p, see [14, §4, Thm. 2]. In particular, a torus is linearly reductive.

In the sequel we require some fundamental results concerning centralizers of linearly reductive groups acting on connected reductive groups; the following facts are [17, Lem. 4.1, Prop. 10.1.5].

PROPOSITION 2.2. *Let G be a connected reductive algebraic group and S a linearly reductive algebraic group acting on G so that G is an S-group. Then we have*

(i) $C_G(S)^\circ$ *is reductive;*
(ii) $\operatorname{Lie} C_G(S) = \mathfrak{c}_{\mathfrak{g}}(S)$.

The following result is due to R. W. Richardson, cf. [17, Prop. 6.1].

PROPOSITION 2.3. *Let G be a connected reductive algebraic group and S a linearly reductive algebraic group acting on G so that G is an S-group. Let K be a (not necessarily connected) closed S-stable subgroup of G. Suppose that K admits a Levi decomposition such that $R_u(K)$ acts simply transitively on the set of all Levi subgroups of K. Then K admits an S-stable Levi subgroup.*

Proof. Although [17, Prop. 6.1] is only stated for parabolic subgroups of G, Richardson's proof applies in this slightly more general setting *mutatis mutandis*. The given conditions on the Levi subgroups of K are precisely the relevant properties of the set of Levi subgroups of a parabolic subgroup of G in Richardson's proof, see [17, §6.3]. In addition, a general result on the vanishing of the (non-commutative) cohomology group $H^1(S, R_u(K))$ is needed; this is proved by Richardson in [17, Lem. 6.2.6]. $\qquad\square$

Remark 2.4. If S is a subgroup of K in Proposition 2.3, then the conclusion is that there exists a Levi subgroup of K containing S, see Jantzen's generalization [7, Lem. 11.24] of Mostow's theorem [13, Thm. 7.1]. In that case the assumption that $R_u(K)$ acts simply transitively on the set of all Levi subgroups of K is not necessary.

The essence of the arguments in the proofs of [7, Lem. 11.24], [13, Thm. 7.1], and [17, Lem. 6.2.6] is the vanishing of the cohomology $H^1(F, U)$, where F is a finite group whose order is coprime to p and U is a (finite) unipotent group.

Remark 2.5. Let σ be a semisimple automorphism of G, [23, §7]. Then there is an embedding $G \leq \mathrm{GL}_n$ of algebraic groups for some n such that σ is given by conjugation by a semisimple element, say s, in GL_n. Thus s belongs to some maximal torus of GL_n, so the algebraic subgroup S of GL_n generated by s consists of semisimple elements. Thus S is linearly reductive, by [14, §4, Thm. 2]. Clearly, S depends on the choice of the ambient group GL_n, but the fact that S is linearly reductive does not.

2.3. Regular reductive subgroups

Let H be a closed (not necessarily connected) subgroup of G normalized by some maximal torus T of G; that is, a *regular* subgroup of G (reductive regular subgroups are often also referred to as *subsystem subgroups* in the literature). In this case the root spaces of $\mathfrak{h}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of H with respect to T, $\Psi(H) = \Psi(H, T) = \{\alpha \in \Psi \mid \mathfrak{g}_\alpha \subseteq \mathfrak{h}\}$, is a subset of Ψ, where $\mathfrak{g}_\alpha$ denotes the root space in $\mathfrak{g}$ corresponding to α. If char k does not divide any of the structure constants of the Chevalley commutator relations of G, then $\Psi(H)$ is closed under addition. In particular, this is the case when char k is a good prime for G. If H is reductive and regular, then $\Psi(H)$ is a semisimple subsystem of Ψ.

Recall that for $s \in G$ semisimple, $H = C_G(s)^\circ$ is called a *pseudo-Levi subgroup* of G, cf. [12, §6]. Since s is contained in a maximal torus T of G, it follows that H is regular of maximal rank. Moreover, $\Psi(H) = \Psi(H, T) = \{\alpha \in \Psi(G, T) \mid \alpha(s) = 0\}$.

2.4. Kempf-Rousseau theory

Let G be a reductive group acting on an affine variety X. For $x \in X$ let $G \cdot x$ denote the G-orbit of x in X and $C_G(x)$ the stabilizer of x in G. Let $\phi : k^* \to X$ be a morphism of algebraic varieties. We say that $\lim_{t \to 0} \phi(t)$

exists if there exists a morphism $\widehat{\phi} : k \rightarrow X$ (necessarily unique) whose restriction to k^* is ϕ; if this limit exists, then we set $\lim_{t \rightarrow 0} \phi(t) = \widehat{\phi}(0)$.

Recall the characterization of parabolic subgroups of G in terms of cocharacters of G, e.g. see [22, Prop. 8.4.5].

LEMMA 2.6. *Given a parabolic subgroup P of G and any Levi subgroup L of P, there exists $\lambda \in Y(G)$ such that the following hold*:

(i) $P = P_\lambda := \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t) g \lambda(t)^{-1} \ exists\}$.
(ii) $L = L_\lambda := C_G(\lambda)$.
(iii) $R_u(P) = \{g \in G \mid \lim_{t \rightarrow 0} \lambda(t) g \lambda(t)^{-1} = 1\}$.

Conversely, given any $\lambda \in Y(G)$ the subset P_λ defined as in part (i) *is a parabolic subgroup of G and L_λ is a Levi subgroup of P_λ.*

Let G act morphically on the affine algebraic variety X. Let $x \in X$ and let C be the unique closed orbit in the closure of $G \cdot x$. Then there exists a subset $\Omega(x)$ of $Y(G)$ consisting of so called *optimal* cocharacters λ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ belongs to C, [8], [18]; see [10, §3] or [16, §2.2] for the relevant parts of the theory; see also Slodowy's survey article [21]. We record the crucial points of this theory.

THEOREM 2.7. *Let G act morphically on the affine algebraic variety X. Let $x \in X$ and let $\Omega(x) \subseteq Y(G)$ be the optimal class of cocharacters defined by x. Then the following hold*:

(i) $\Omega(x) \neq \varnothing$ *and there exists an* optimal *parabolic subgroup $P = P(x)$ of G so that $P = P_\lambda$ for every $\lambda \in \Omega(x)$.*
(ii) $\Omega(x)$ *is a single P-orbit.*
(iii) *For every $g \in G$, we have $\Omega(g \cdot x) = g \cdot \Omega(x)$ and $P(g \cdot x) = gP(x)g^{-1}$. In particular, $C_G(x) \leq P$.*

2.5. Cocharacters associated to nilpotent elements

Recall that any cocharacter $\lambda \in Y(G)$ of G affords a $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j, \lambda)$$

of $\mathfrak{g}$, where

$$\mathfrak{g}(j, \lambda) := \{e \in \mathfrak{g} \mid \mathrm{Ad}(\lambda(t))e = t^j e \text{ for every } t \in k^*\},$$

cf. [5, §5.5] or [7, §5.1]. We recall the relevant concepts of distinguished nilpotent elements and of cocharacters associated to a nilpotent element following [7, §4.1, §5.3].

DEFINITION 2.8. Let $H \leq G$ be a closed connected reductive subgroup of G and $e \in \mathcal{N} \cap \mathfrak{h}$. We call e *distinguished* in $\mathfrak{h}$ if each torus contained in $C_H(e)$ is contained in the centre of H.

DEFINITION 2.9. Let $e \in \mathcal{N}$. A cocharacter $\lambda : k^* \to G$ of G is called *associated* to e provided $e \in \mathfrak{g}(2, \lambda)$ and there exists a Levi subgroup L of G such that e is distinguished nilpotent in $\mathrm{Lie}\, L$ and $\lambda(k^*) \leq \mathcal{D}L$.

Remarks 2.10. Let $e \in \mathcal{N}$ and let $\lambda \in Y(G)$ that is associated to e.

(i) For $g \in C_G(e)$ the conjugate cocharacter $g \cdot \lambda$ is also associated to e, cf. [7, §5.3]. Proposition 2.11(ii) gives a converse to this property.

(ii) Let S be a maximal torus of $C_G(e)$. Then e is distinguished in $\mathfrak{c}_\mathfrak{g}(S) = \mathrm{Lie}\, C_G(S)$, for S is the unique maximal torus of $C_{C_G(S)}(e)$. Proposition 2.11(iii) gives a converse to this.

We require some basic facts about cocharacters associated to nilpotent elements; the following results are [7, Rem. 4.7; Lem. 5.3], see also [16, Prop. 2.5].

PROPOSITION 2.11. *Let $e \in \mathcal{N}$.*

(i) *Suppose* $\mathrm{char}\, k$ *is good for G. Then cocharacters of G associated to e exist.*

(ii) *Any two cocharacters of G associated to e are conjugate by an element of $C_G(e)^\circ$.*

(iii) *If L is a Levi subgroup of G with e distinguished in $\mathrm{Lie}\, L$, then the connected centre of L is a maximal torus of $C_G(e)^\circ$.*

Remark 2.12. The Dynkin-Kostant classification theory giving a bijection between nilpotent G-classes and G-conjugacy classes of $\mathfrak{sl}(2)$-triples is also valid in large positive characteristic (more precisely, when $\mathrm{char}\, k > 3(h-1)$, where h denotes the Coxeter number of G, cf. [5, §§5.3–5.6]). For $e \in \mathcal{N}$, the cocharacters of G constructed from the semisimple elements of $\mathfrak{sl}(2)$-triples containing e (cf. [5, §5.5]) are all associated to e in the sense of Definition 2.9, see [7, Rem. 5.5].

Let $e \in \mathcal{N}$. In [16, §2.4], A. Premet explicitly defines a cocharacter of G which is associated to e, thanks to [16, Prop. 2.5]. Moreover, in [16, Thm. 2.3], Premet shows that each of these associated cocharacters belongs to the optimal class determined by e. Premet proves this under the so called *standard hypotheses* on G, see [7, §2.9]. These restrictions were subsequently removed by G. McNinch in [10, Prop. 16] so that this fact holds for any reductive G in good characteristic. It thus follows from [10, Prop. 16], Theorem 2.7, and Proposition 2.11(ii) that all the cocharacters of G associated to $e \in \mathcal{N}$ belong to the optimal class $\Omega(e)$ defined by e; see also [10, Prop. 18, Thm. 21]. This motivates and justifies the following notation which we frequently use in the sequel.

DEFINITION 2.13.   Let $e \in \mathcal{N}$. Then we define

$$\Omega^a(e) := \{\lambda \in Y(G) \mid \lambda \text{ is associated to } e\} \subseteq \Omega(e).$$

Further, we sometimes write $\Omega^a_G(e)$ for $\Omega^a(e)$, to indicate that this is a set of cocharacters of G, and if H is a reductive subgroup of G with $e \in \mathfrak{h}$ nilpotent we also write $\Omega^a_H(e)$ to denote the cocharacters of H that are associated to e.

For H a connected reductive subgroup of G and $e \in \mathfrak{h} \cap \mathcal{N}$, in the notation of Definition 2.13, property (†) from page 106 becomes the equality

$$\Omega^a_H(e) = \Omega^a_G(e) \cap Y(H).$$

Let $e \in \mathcal{N}$ and let $\lambda \in \Omega^a(e)$. Let $P = P(e)$ be the canonical parabolic subgroup defined by e. Then $P = C_G(\lambda)R_u(P)$ is a Levi decomposition of P. Thanks to Theorem 2.7(iii), $C_G(e) \leq P$. Following [7, §5.10] and [16, §2.4], we define the subgroups

(2.14) $C_G(e, \lambda) := C_G(e) \cap C_G(\lambda) \quad \text{and} \quad R_e := C_G(e) \cap R_u(P)$

of $C_G(e)$.

In view of [16, Prop. 2.5], our next result is [16, Thm. 2.3(iii)], see also [7, Prop. 5.10, Prop. 5.11].

PROPOSITION 2.15.   *Suppose that* $\operatorname{char} k$ *is good for* G. *Let* $e \in \mathcal{N}$ *and let* $\lambda \in \Omega^a(e)$. *Then* $C_G(e)$ *is the semidirect product of* $C_G(e, \lambda)$ *and* R_e, *where* $C_G(e, \lambda)^\circ$ *is reductive and* $R_e = R_u(C_G(e))$.

Proposition 2.15 says that $C_G(e) = C_G(e, \lambda)R_e$ is a Levi decomposition of $C_G(e)$, so that different choices of associated cocharacters in $\Omega^a(e)$ give conjugate Levi subgroups $C_G(e, \lambda)$ of $C_G(e)$, by Proposition 2.11(ii). Our next result now readily follows from these two propositions.

COROLLARY 2.16. *Suppose that* char k *is good for* G. *Let* $e \in \mathcal{N}$. *Then* R_e *acts simply transitively on* $\Omega^a(e)$.

Remark 2.17. It follows from Propositions 2.11, 2.15 and Corollary 2.16 that the map $\lambda \mapsto C_G(e, \lambda)$ is a bijection between $\Omega^a(e)$ and the set of Levi subgroups of $C_G(e)$.

Let $e \in \mathfrak{g}$ be nilpotent and let $\lambda \in \Omega^a(e)$. It follows readily from Definition 2.9 that $C_G(e)$ is normalized by $\lambda(k^*)$, thus we may define the subgroup

$$N_e := \lambda(k^*)C_G(e)$$

of G, cf. [7, §5.3(2)]. According to Proposition 2.11(ii), N_e does not depend on the choice of $\lambda \in \Omega^a(e)$; this is also apparent, since $N_e = \{g \in G \mid g \cdot ke = ke\}$, see [7, §2.10(2)]. Clearly, $\lambda(k^*)$ also normalizes $R_e = R_u(C_G(e))$. Thus we may define the subgroup

(2.18) $$Q_e := \lambda(k^*)R_e$$

of N_e. By Corollary 2.16, equally Q_e does not depend on the choice of λ in $\Omega^a(e)$.

Let H be a connected reductive subgroup of G. Since the nilpotent cone of Lie $\mathcal{D}H$ is a closed subvariety of the nilpotent variety of $\mathfrak{h}$, and both are irreducible of the same dimension, we have $\mathfrak{h} \cap \mathcal{N} = \text{Lie}(\mathcal{D}H) \cap \mathcal{N}$.

LEMMA 2.19. *Let* H *be a connected reductive subgroup of* G. *Let* $e \in \mathfrak{h} \cap \mathcal{N} = \text{Lie}(\mathcal{D}H) \cap \mathcal{N}$ *be nilpotent. Then the cocharacters of* $\mathcal{D}H$ *associated to* e *are precisely the cocharacters of* H *associated to* e.

Proof. Assume that λ is a cocharacter of $\mathcal{D}H$ associated to e. Note that $H = Z(H)^\circ \mathcal{D}H$. Let $L' = C_{\mathcal{D}H}(S')$ be a Levi subgroup of $\mathcal{D}H$ satisfying the conditions of Definition 2.9, where S' is a maximal torus of $C_{\mathcal{D}H}(e)$. Then $S = Z(H)^\circ S'$ is a maximal torus of $C_H(e)$ containing S'. Set $L = C_H(S) = Z(H)^\circ L'$. It follows easily that e is distinguished in Lie L. Further, $\lambda(k^*) \leq \mathcal{D}(L') = \mathcal{D}L$, and so λ is associated to e, viewed as a cocharacter of H.

The reverse implication of the lemma is shown in [7, §5.6].   □

§3. Cocharacters associated to nilpotent elements of reductive subgroups

We maintain the notation and assumption of the previous sections. In particular, G is a connected reductive algebraic group defined over an algebraically closed field k, $\operatorname{char} k$ is a good prime for G, and H is a closed connected reductive subgroup of G.

3.1. Local conditions

In this subsection we study conditions on a given nilpotent element e in $\mathfrak{h}$ (or nilpotent H-class in $\mathfrak{h}$) that ensure (†) from page 106 holds for e without further assumptions on H itself. Firstly we consider nilpotent elements $e \in \mathfrak{h}$ that are distinguished in $\mathfrak{g}$, Lemma 3.1, and secondly we study the case when the centralizers in H and G of e have the same rank, Theorem 3.14.

LEMMA 3.1. *Suppose that $e \in \mathfrak{h}$ is nilpotent and distinguished in $\mathfrak{g}$. Then e is distinguished in $\mathfrak{h}$ and $\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$.*

Proof. Let S be a torus of $C_H(e)$. Since e is distinguished in $\mathfrak{g}$ and $C_H(e) \leq C_G(e)$, we have $S \leq Z(G)$. So $S \leq Z(G) \cap H \leq Z(H)$ and thus e is distinguished in $\mathfrak{h}$.

First let $\lambda \in \Omega_H^a(e)$. Then $e \in \mathfrak{h}(2,\lambda) \subseteq \mathfrak{g}(2,\lambda)$. By Lemma 2.19, we have $\lambda(k^*) \leq \mathcal{D}H \leq \mathcal{D}G$, and so $\lambda \in \Omega_G^a(e)$, as e is distinguished in $\mathfrak{g}$.

Conversely, let $\lambda \in \Omega_G^a(e)$ with $\lambda(k^*) \leq H$. Since e is distinguished in $\mathfrak{h}$, and since $e \in \mathfrak{h} \cap \mathfrak{g}(2,\lambda) = \mathfrak{h}(2,\lambda)$, it suffices to show that $\lambda(k^*) \leq \mathcal{D}H$. Since e is distinguished in $\mathfrak{g}$ and $Z(H)^\circ \leq C_H(e) \leq C_G(e)$, we have $Z(H)^\circ \leq Z(G)^\circ \cap H$. Note that $\mathcal{D}H \leq \mathcal{D}G \cap H \leq H$. Since H is reductive, we have $H = Z(H)^\circ \mathcal{D}H$, so that $\mathcal{D}G \cap H = A\mathcal{D}H$, where A is a subgroup of $Z(H)^\circ$. By assumption, $\lambda(k^*) \leq \mathcal{D}G \cap H$. Since $\lambda(k^*)$ is connected, we have $\lambda(k^*) \leq (\mathcal{D}G \cap H)^\circ = A^\circ \mathcal{D}H$. Because $A^\circ \leq Z(H)^\circ \leq Z(G)^\circ \cap H$ and $A^\circ \leq \mathcal{D}G \cap H$, we have $A^\circ \leq \mathcal{D}G \cap Z(G)^\circ$, and so A° is trivial. Consequently, $\lambda(k^*) \leq \mathcal{D}H$, as desired. □

We give an example for Lemma 3.1.

EXAMPLE 3.2. Let G be simple of type E_6 and let H be the fixed point subgroup of the non-trivial graph automorphism of G; so that H is of type F_4. Let C' be the nilpotent H-class with Bala-Carter label $F_4(a_2)$ and let C be the nilpotent G-class with Bala-Carter label $E_6(a_3)$. According to [9,

Table A], we have $C' \subset C$. Note that each of these classes is distinguished, see [5, §5.9]. Thus Lemma 3.1 applies. Another example is given by the regular nilpotent class in H which belongs to the regular G-class in $\mathcal{N}$; they are obviously both distinguished. Although the results in [5, §5.9] and [9] concern unipotent classes in G, since p is good for G, they equally apply to nilpotent classes in $\mathfrak{g}$.

Corollary 3.24 below implies that in this case the conclusion of Lemma 3.1 holds for any nilpotent class of H.

LEMMA 3.3. *Let $e \in \mathfrak{h}$ be nilpotent. Suppose that $\operatorname{rank} C_G(e) = \operatorname{rank} C_H(e)$. Then $\Omega_H^a(e) \subseteq \Omega_G^a(e) \cap Y(H)$.*

Proof. Let $\lambda \in \Omega_H^a(e)$. Since $e \in \mathfrak{h}(2, \lambda) \subseteq \mathfrak{g}(2, \lambda)$, it suffices to find a Levi subgroup L of G such that e is distinguished in $\operatorname{Lie} L$ and $\lambda(k^*) \leq \mathcal{D}L$. Let M be a Levi subgroup of H with the properties as in Definition 2.9. Thanks to Proposition 2.11(iii), we have $M = C_H(S)$, where S is a maximal torus of $C_H(e)$. Since $C_H(e) \leq C_G(e)$ and by our hypothesis, S is also a maximal torus of $C_G(e)$. Thus e is distinguished in $\operatorname{Lie} C_G(S)$, cf. Remark 2.10(ii). Finally, since $\lambda(k^*) \leq \mathcal{D}C_H(S) \leq \mathcal{D}C_G(S)$, we see that $\lambda \in \Omega_G^a(e)$, as desired. $\square$

Let $e \in \mathfrak{g}$ be nilpotent. Let λ be a cocharacter of G associated to e. Define

$$(3.4) \quad \Upsilon_\lambda(e) = \{S \leq C_G(e) \mid S \text{ is a maximal torus of } C_G(e)$$
$$\text{and } \lambda(k^*) \leq \mathcal{D}C_G(S)\}.$$

Note that by Proposition 2.11(iii), $\Upsilon_\lambda(e)$ is non-empty.

For our next result recall the definition of $C_G(e, \lambda) = C_G(e) \cap C_G(\lambda)$ from (2.14).

LEMMA 3.5. *Let $e \in \mathfrak{g}$ be nilpotent. Let λ be a cocharacter of G associated to e. Then $\Upsilon_\lambda(e)$ consists precisely of the maximal tori of $C_G(e, \lambda)$.*

Proof. Let $S \in \Upsilon_\lambda(e)$. Then S is a maximal torus of $C_G(e)$ and $\lambda(k^*) \leq \mathcal{D}C_G(S) \leq C_G(S)$. In particular, $S \leq C_G(\lambda)$ and so $S \leq C_G(e, \lambda)$. Thus S is a maximal torus of $C_G(e, \lambda)$.

Conversely, let S be a maximal torus of $C_G(e, \lambda)$. Then, by what we have just shown, S is conjugate in $C_G(e, \lambda)$ to some $S' \in \Upsilon_\lambda(e)$, and so

in particular, S is a maximal torus of $C_G(e)$. Let $g \in C_G(e, \lambda)$ so that $S = gS'g^{-1}$. Since $\lambda(k^*) \leq \mathcal{D}C_G(S')$, we have $g\lambda(k^*)g^{-1} \leq g\mathcal{D}C_G(S')g^{-1} = \mathcal{D}C_G(S)$. Finally, since $g \in C_G(\lambda)$, we obtain $g\lambda(k^*)g^{-1} = \lambda(k^*)$, and therefore, $S \in \Upsilon_\lambda(e)$, as desired.   □

LEMMA 3.6. *Let $e \in \mathfrak{h}$ be nilpotent. Let $\lambda \in \Omega_G^a(e)$ with $\lambda(k^*) \leq H$. Suppose there exists a maximal torus of $C_H(e)$ which is also a maximal torus of $C_G(e, \lambda)$. Then $\lambda \in \Omega_H^a(e)$.*

Proof. Since $e \in \mathfrak{h} \cap \mathfrak{g}(2, \lambda) = \mathfrak{h}(2, \lambda)$, it suffices to find a Levi subgroup M of H such that e is distinguished in Lie M and $\lambda(k^*) \leq \mathcal{D}M$. Let L be a Levi subgroup of G such that e is distinguished in Lie L and $\lambda(k^*) \leq \mathcal{D}L$. By Proposition 2.11(iii), we have $L = C_G(S)$, where S is a maximal torus of $C_G(e)$. By Remark 2.10(ii), λ is a cocharacter of $C_G(S)$ that is associated to e and e is distinguished in $\mathfrak{c}_\mathfrak{g}(S) = \mathrm{Lie}\, C_G(S)$. By our hypothesis and Lemma 3.5, we may assume without loss of generality that S is a maximal torus of $C_H(e)$. Then $S \leq H$ and so $C_H(S)$ is a Levi subgroup of H. Consequently, by Lemma 3.1 applied to $C_H(S) \leq C_G(S)$, we get that λ is a cocharacter of $C_H(S)$ associated to e. Since S is a maximal torus of $C_H(e)$ as well as of $C_{C_H(S)}(e)$, we can apply Lemma 3.3 to $C_H(S) \leq H$ and so $\lambda \in \Omega_H^a(e)$.   □

Let $e \in \mathfrak{h}$ be nilpotent and let λ be a cocharacter of G associated to e with $\lambda(k^*) \leq H$. Let P be the parabolic subgroup of G defined by λ, i.e., $P = P_\lambda$, cf. Lemma 2.6(i). Since λ is optimal for e, we have $P = P(e)$ is the optimal parabolic subgroup determined by e, cf. Theorem 2.7(i). Define $P_H = P \cap H$. Since $\lambda(k^*) \leq H$, we see P_H is a parabolic subgroup of H. Also P_H has Lie algebra $\bigoplus_{i \geq 0} \mathfrak{h}(i, \lambda)$ and the unipotent radical $R_u(P_H)$ of P_H is $R_u(P) \cap H$ and $C_H(\lambda) = C_G(\lambda) \cap H$ is a Levi subgroup of P_H. The unipotent radical of P_H and the Levi subgroup $C_H(\lambda)$ have Lie algebras $\bigoplus_{i > 0} \mathfrak{h}(i, \lambda)$ and $\mathfrak{h}(0, \lambda)$, respectively, see [7, §5.1]. Analogous to (2.14) define the subgroups

$$(3.7) \qquad C_H(e, \lambda) := C_H(e) \cap C_H(\lambda) \quad \text{and} \quad U_e := C_H(e) \cap R_u(P_H)$$

of $C_H(e)$. Note that $C_H(e, \lambda) \cap U_e = \{1\}$, so $C_H(e, \lambda)U_e$ is a semidirect product of algebraic groups. Next we require the following facts, see [7,

§5.10]. For all $i \in \mathbb{Z}$ we have

$$(3.8) \qquad \operatorname{Ad}(x)(\mathfrak{h}(i, \lambda)) = \mathfrak{h}(i, \lambda) \text{ for all } x \in C_H(\lambda);$$

$$(3.9) \qquad (\operatorname{Ad}(y) - 1)(\mathfrak{h}(i, \lambda)) \subseteq \bigoplus_{j > i} \mathfrak{h}(j, \lambda) \text{ for all } y \in R_u(P_H).$$

LEMMA 3.10. *With the notation introduced in* (3.7), *we have the following.*

(i) $C_H(e) \leq P_H$.

(ii) U_e *is a normal connected unipotent subgroup of* $C_H(e)$.

(iii) $C_H(e)$ *is the semidirect product of* $C_H(e, \lambda)$ *and* U_e.

Proof. (i) Let $x \in C_H(e) = C_G(e) \cap H$. Since $\lambda \in \Omega^a(e)$, we have $C_G(e) \subseteq P$, by Theorem 2.7(iii). Therefore, $x \in P \cap H = P_H$.

(ii) By definition, U_e is unipotent. Let $x \in U_e$ and $y \in C_H(e)$. Now $x \in R_u(P_H)$ and $y \in P_H$. So $yxy^{-1} \in R_u(P_H)$, as $R_u(P_H)$ is normal in P_H. Also $x, y \in C_H(e)$ so $yxy^{-1} \in R_u(P_H) \cap C_H(e) = U_e$. Since U_e is normalized by $\lambda(k^*)$, it is connected: For any $x \in U_e$ we have a morphism $\phi_x : k \to U_e$ given by $\phi_x(t) = \lambda(t)x\lambda(t)^{-1}$ for $t \in k^*$ and $\phi_x(0) = \lim_{t \to 0} \lambda(t)x\lambda(t)^{-1} = 1$, cf. Lemma 2.6(iii). Thus for each $x \in U_e$ the image of ϕ_x is a connected subvariety of U_e containing 1 and $x = \phi_x(1)$.

(iii) Let $z \in C_H(e)$. Thanks to part (i), we can write z as $z = xy$ with $x \in C_H(\lambda)$ and $y \in R_u(P_H)$. Since $e \in \mathfrak{h}(2, \lambda)$, it follows from (3.9) that $\operatorname{Ad}(y)(e) = e + e'$ with $e' \in \bigoplus_{i \geq 3} \mathfrak{h}(i, \lambda)$. Since $z \in C_H(e)$, we have $\operatorname{Ad}(x)(e + e') = e$. It follows that $\operatorname{Ad}(x)(e') = 0$, thanks to (3.8). As $\operatorname{Ad}(x) : \mathfrak{h} \to \mathfrak{h}$ is a Lie algebra automorphism of $\mathfrak{h}$, we infer that $e' = 0$. Consequently, $y \in C_H(e) \cap R_u(P_H)$ and thus $x \in C_H(e) \cap C_H(\lambda)$. Therefore, $C_H(e) = C_H(e, \lambda)U_e$. $\qquad \square$

Remark 3.11. Despite the analogy between Proposition 2.15 and Lemma 3.10, the semidirect product $C_H(e) = C_H(e, \lambda)U_e$ need not be a Levi decomposition of $C_H(e)$; we cannot invoke Proposition 2.15, as we do not know whether λ lies in $\Omega^a_H(e)$. It follows from Theorem 3.21 that this is the case for the particular class of subgroups H of G considered there.

LEMMA 3.12. *Let* $e \in \mathfrak{h}$ *be nilpotent. Suppose that* $\operatorname{rank} C_G(e) = \operatorname{rank} C_H(e)$. *Let* $\lambda \in \Omega^a_G(e) \cap Y(H)$. *Then there exists a maximal torus of* $C_H(e)$ *which is also a maximal torus of* $C_G(e, \lambda)$.

Proof. It follows from Lemma 3.10(iii) that $\operatorname{rank} C_H(e) = \operatorname{rank} C_H(e, \lambda)$. Obviously, we have $C_H(e, \lambda) \leq C_G(e, \lambda) \leq C_G(e)$. It follows from the assumption on the rank of the centralizers of e in H and in G that a maximal torus of $C_H(e, \lambda)$, therefore of $C_H(e)$, is also a maximal torus of $C_G(e, \lambda)$. □

Our next result is a consequence of Lemmas 3.6 and 3.12.

PROPOSITION 3.13. *Let $e \in \mathfrak{h}$ be nilpotent. Suppose that $\operatorname{rank} C_G(e) = \operatorname{rank} C_H(e)$. Then $\Omega_G^a(e) \cap Y(H) \subseteq \Omega_H^a(e)$.*

Our next result follows immediately from Lemma 3.3 and Proposition 3.13.

THEOREM 3.14. *Let $e \in \mathfrak{h}$ be nilpotent. Suppose that $\operatorname{rank} C_G(e) = \operatorname{rank} C_H(e)$. Then $\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$.*

We give an example for Theorem 3.14.

EXAMPLE 3.15. We return to the case of Example 3.2. Let G be simple of type E_6 and let H be the standard subgroup of G of type F_4. Let D' be the nilpotent H-class with Bala-Carter label $\tilde{A}_2$ and let D be the nilpotent G-class with Bala-Carter label $2A_2$. According to [9, Table A], we have $D' \subset D$. It is known that the reductive parts of the centralizers of these classes in H and G are of type G_2 in each case, see [5, Ch. 13]. In particular, the rank condition in Theorem 3.14 is satisfied for an element belonging to D'. This also applies to the classes with Bala-Carter labels C_3 in H and A_5 in G, here the reductive parts of the centralizers are of type A_1 in each case. There is one further pair of classes with the same property.

It follows from Corollary 3.24 below that in this example the conclusion of Theorem 3.14 holds for any nilpotent element of $\mathfrak{h}$ irrespective of the condition on the ranks of the respective centralizers.

See also Example 3.33 below for another application of Theorem 3.14.

3.2. Global conditions

We maintain the notation from the previous sections. In this subsection we study conditions on the subgroup H of G (rather than on a given nilpotent H-class in $\mathfrak{h}$) that ensure that (†) holds for all $e \in \mathcal{N} \cap \mathfrak{h}$.

Our first result shows that the reverse inclusion of (†) always holds in good characteristic without any restrictions on H.

PROPOSITION 3.16. *Let $e \in \mathfrak{h}$ be nilpotent. Then $\Omega_G^a(e) \cap Y(H) \subseteq \Omega_H^a(e)$.*

Proof. Let $\lambda \in \Omega_G^a(e)$ with $\lambda(k^*) \leq H$. First assume that e is distinguished in $\mathfrak{h}$. Since $e \in \mathfrak{h} \cap \mathfrak{g}(2,\lambda) = \mathfrak{h}(2,\lambda)$ and e is distinguished in $\mathfrak{h}$, it suffices to show that $\lambda(k^*) \leq \mathcal{D}H$. Set $Z = Z(H)^\circ$. Then $Z \leq C_H(e,\lambda) \leq C_G(e,\lambda)$. Choose a maximal torus S of $C_G(e,\lambda)$ so that $Z \leq S$. Thanks to Lemma 3.5, S is a maximal torus of $C_G(e)$. Since $S \leq C_G(e,\lambda)$, we have $\lambda(k^*) \leq \mathcal{D}C_G(S)$, by Lemma 3.5. Also, as $Z \leq S$, we have $C_G(S) \leq C_G(Z)$. Thus $\lambda(k^*) \leq \mathcal{D}C_G(Z)$ and so $\lambda(k^*) \leq H \cap \mathcal{D}C_G(Z)$. As $H \leq C_G(Z)$, we get $\mathcal{D}H \leq H \cap \mathcal{D}C_G(Z) \leq H$. Since H is reductive, we have $H = Z\mathcal{D}H$, so $H \cap \mathcal{D}C_G(Z) = A\mathcal{D}H$, where $A \leq Z$. As $\lambda(k^*)$ is connected, $\lambda(k^*) \leq A^\circ \mathcal{D}H$. Clearly, we have $A^\circ \leq A\mathcal{D}H = H \cap \mathcal{D}C_G(Z)$ and $A^\circ \leq Z$ so $A^\circ \leq Z \cap \mathcal{D}C_G(Z)$. Since $C_G(Z)$ is a connected reductive subgroup of G and Z is contained in the connected centre of $C_G(Z)$, it follows that $Z \cap \mathcal{D}C_G(Z)$ is finite. Thus A° is trivial and hence $\lambda(k^*) \leq \mathcal{D}H$, as desired.

Now we consider the general case where e is not necessarily distinguished in $\mathfrak{h}$. Let S be a maximal torus of $C_H(e,\lambda)$. By Lemma 3.10(iii), S is then also a maximal torus of $C_H(e)$. Since $S \leq C_G(\lambda)$, we have $\lambda(k^*) \leq C_H(S)$ and that e is distinguished in $\text{Lie } C_H(S)$, cf. Remark 2.10(ii). Thus, by the distinguished case just proved, λ is a cocharacter of $C_H(S)$ associated to $e \in \mathfrak{c}_\mathfrak{h}(S) = \text{Lie } C_H(S)$. Observe that $\dim S = \text{rank } C_{C_H(S)}(e) = \text{rank } C_H(e)$. It thus follows from Lemma 3.3 applied to $C_H(S) \leq H$ that λ is a cocharacter of H associated to e, as claimed. □

Our next result shows that the forward inclusion of (†) holds provided there is at least one cocharacter of G that is associated to $e \in \mathcal{N} \cap \mathfrak{h}$ and takes values in H.

LEMMA 3.17. *Let $e \in \mathfrak{h} \cap \mathcal{N}$. Then $\Omega_H^a(e) \subseteq \Omega_G^a(e) \cap Y(H)$, provided $\Omega_G^a(e) \cap Y(H)$ is non-empty.*

Proof. Let $\lambda \in \Omega_G^a(e) \cap Y(H)$ and let $\mu \in \Omega_H^a(e)$. Then, by Proposition 3.16, λ is a cocharacter of H associated to e. So, by Proposition 2.11(ii), there exists an $x \in C_H(e)^\circ$ such that $x\lambda x^{-1} = \mu$. Clearly, $C_H(e)^\circ \leq C_G(e)^\circ$. By Remark 2.10(i) it follows that μ is thus a cocharacter of G associated to e. □

Combining Proposition 3.16 and Lemma 3.17 gives our next result.

THEOREM 3.18. *Let $e \in \mathfrak{h} \cap \mathcal{N}$. If $\Omega_G^a(e) \cap Y(H)$ is non-empty, then $\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$.*

Let S be a linearly reductive group acting on G by automorphisms, so that G is an S-group. Set $H = C_G(S)^\circ$ which is reductive, by Proposition 2.2(i). Let $e \in \mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(S)$ be nilpotent. In the proof of [17, Thm. C], R. W. Richardson showed that there always exists a cocharacter of G which belongs to the optimal class $\Omega(e)$ defined by e so that its image lies in H, see [17, §8]. For our purpose we need a variant of this result: we require the existence of a cocharacter of G which is associated to e (rather than merely being optimal) so that its image lies in H. That is, we need to construct an S-fixed cocharacter of G which is associated to e. This is done in Lemma 3.20 with the aid of Richardson's result Proposition 2.3. For our next result, recall the subgroup Q_e of N_e from (2.18).

LEMMA 3.19. *Let $e \in \mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(S)$ be nilpotent. Then*

(i) $\Omega^a(e)$ *is S-stable*;
(ii) $C_G(e)$ *is S-stable*;
(iii) Q_e *is S-stable*.

Proof. (i) Let $\lambda \in \Omega^a(e)$. Since the induced actions of S and G on $\mathfrak{g}$ are compatible in the sense of (2.1) and linear, we have

$$\mathrm{Ad}((s \cdot \lambda)(t))e = s \cdot \mathrm{Ad}(\lambda(t))e = s \cdot (t^2 e) = t^2 e,$$

for every $s \in S$ and $t \in k^*$; so $e \in \mathfrak{g}(2, s \cdot \lambda)$ for every $s \in S$. Clearly, if e is distinguished in $\mathrm{Lie}\, L$, then $e = s \cdot e$ is distinguished in $\mathrm{Lie}(s \cdot L)$ and $s \cdot L$ is another Levi subgroup of G for $s \in S$. Finally, $\lambda(k^*) \leq \mathcal{D}L$ implies that $(s \cdot \lambda)(k^*) \leq s \cdot \mathcal{D}L = \mathcal{D}(s \cdot L)$ for $s \in S$. It follows that $s \cdot \lambda \in \Omega^a(e)$ for any $s \in S$.

(ii) Again, by the compatibility of the actions of S and G on $\mathfrak{g}$, we obtain

$$\mathrm{Ad}(s \cdot g)e = s \cdot \mathrm{Ad}(g)e = s \cdot e = e,$$

for any $g \in C_G(e)$, $s \in S$.

(iii) Since $C_G(e)$ is S-stable, so is $R_e = R_u(C_G(e))$. Since $\Omega^a(e)$ is S-stable, by part (i), it follows from Corollary 2.16 that Q_e is also S-stable. $\quad\square$

LEMMA 3.20.   *Let $e \in \mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(S)$ be nilpotent. Then the following hold.*

(i)   *There exists an S-stable Levi subgroup $\lambda(k^*)$ of Q_e.*
(ii) *There exists an S-fixed cocharacter in $\Omega^a(e)$.*
(iii) *There exists an S-stable Levi subgroup $C_G(e, \lambda)$ of $C_G(e)$.*

Proof.   (i) Let $\lambda' \in \Omega^a(e)$ so that $Q_e = \lambda'(k^*)R_e$. Then $R_u(Q_e) = R_e$. According to Corollary 2.16, R_e acts simply transitively on $\Omega^a(e)$. Thus R_e acts simply transitively on the set of Levi subgroups of Q_e, Remark 2.17. Thanks to Lemma 3.19(iii), Q_e is S-stable. The desired result now follows from Proposition 2.3.

(ii) Let $\lambda(k^*)$ be as in (i), i.e., $(s \cdot \lambda)(k^*) = s \cdot (\lambda(k^*)) = \lambda(k^*)$ for every $s \in S$. Since $e \in \mathfrak{g}(2, \lambda) \cap \mathfrak{g}(2, s \cdot \lambda)$, it follows from [7, Lem. 4.11] that $s \cdot \lambda = \lambda$ for every $s \in S$.

(iii) Let λ in $\Omega^a(e)$ be S-fixed as in (ii). Since S and G act compatibly on $Y(G)$ in the sense of (2.1), $C_G(\lambda)$ is S-stable. For,

$$(s \cdot g) \cdot \lambda(t) = s \cdot (g \cdot \lambda)(t) = s \cdot \lambda(t) = \lambda(t),$$

for all $s \in S$, $g \in C_G(\lambda)$, and $t \in k^*$. Consequently, since $C_G(e)$ is S-stable, by Lemma 3.19(ii), so is $C(e, \lambda) = C_G(e) \cap C_G(\lambda)$. The result now follows from Proposition 2.15. □

Finally, Theorem 1.2 follows from Theorem 3.18 and Lemma 3.20(ii):

THEOREM 3.21.   *Let S be a linearly reductive group acting on G by automorphisms and set $H = C_G(S)^\circ$. Let $e \in \mathfrak{h} \cap \mathcal{N}$. Then $\Omega^a_H(e) = \Omega^a_G(e) \cap Y(H)$.*

We record various special cases of Theorem 3.21 as separate corollaries.

Since a Levi subgroup of G is of the form $C_G(S)$ for some torus S of G, our next result is immediate from Theorem 3.21.

COROLLARY 3.22.   *Let H be a Levi subgroup of G. Let $e \in \mathfrak{h} \cap \mathcal{N}$. Then $\Omega^a_H(e) = \Omega^a_G(e) \cap Y(H)$.*

Likewise, our next result is immediate from Theorem 3.21 and Remark 2.5.

COROLLARY 3.23.   *Let σ be a semisimple automorphism of G. Let $H = C_G(\sigma)^\circ$. Let $e \in \mathfrak{h} \cap \mathcal{N}$. Then $\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$.*

Here is a special case of Corollary 3.23. Assume that G is simple and let γ be a non-trivial graph automorphism of G. If char k is coprime to the order of γ, then γ is a semisimple automorphism of G.

COROLLARY 3.24.   *Let G be simple and let γ be a non-trivial graph automorphism of G. Suppose that char k is coprime to the order of γ. Let $H = C_G(\gamma)^\circ$. Let $e \in \mathfrak{h} \cap \mathcal{N}$. Then $\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$.*

We give two examples for Corollary 3.24.

EXAMPLE 3.25.   Suppose that $p \neq 2$. Let V be a finite-dimensional k-vector space. Let H be either $\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$. Observe that H is the fixed point subgroup of an involution of $\mathrm{SL}(V)$ (cf. [24, §11 p. 169]) and thus Corollary 3.24 applies. That is, the cocharacters of the classical groups $H = \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ associated to a given nilpotent element e in the Lie algebra of H are precisely the cocharacters of the ambient linear group $\mathrm{SL}(V)$ associated to e whose image lies in H.

EXAMPLE 3.26.   Suppose that char $k > 3$. Let G be of type D_4 and let γ be the triality graph automorphism of G. Then $H = C_G(\gamma)^\circ$ is of type G_2 and so Corollary 3.24 applies. Thus, for a given nilpotent element $e \in \mathfrak{h}$ we can realize the cocharacters of H associated to e as the cocharacters of G associated to e that are γ-fixed.

Recall from subsection 2.3 that for $s \in G$ semisimple, $C_G(s)^\circ$ is a pseudo-Levi subgroup of G. The next result is again a special case of Corollary 3.23.

COROLLARY 3.27.   *Let H be a pseudo-Levi subgroup of G. Let $e \in \mathfrak{h} \cap \mathcal{N}$. Then $\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$.*

Clearly, each pseudo-Levi subgroup of G is of maximal rank. The subsystems corresponding to maximal rank, semisimple subgroups of a simple group G are determined by means of the algorithm of Borel and de Siebenthal [3], see also [4, Ex. Ch. VI §4.4]. Using Corollary 3.27, the algorithm of Borel and de Siebenthal, as well as Deriziotis' characterization of maximal rank reductive subgroups (cf. [6, §2.15]), we can generalize Corollary 3.27 to arbitrary maximal rank reductive subgroups.

Our next result is Theorem 1.1.

THEOREM 3.28. *Let H be a connected reductive subgroup of G of maximal rank. Let $e \in \mathfrak{h} \cap \mathcal{N}$. Then $\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$.*

Proof. Thanks to Lemma 2.19 and by passing to simple factors, we may assume that G is simple. Let H be a maximal rank reductive subgroup of G. Let T be a maximal torus of G contained in H. Let Π be a set of simple roots of $\Psi = \Psi(G, T)$, let ϱ be the highest root of Ψ^+, and let W be the Weyl group of G with respect to T. Let $\Phi = \Phi(H, T)$ be the root system of H; in particular Φ is a semisimple subsystem of Ψ. Since char k is good for G, it follows from Deriziotis' Criterion (cf. [6, §2.15]) that H is of the form $H = C_G(s)^\circ$ if and only if Φ admits a base which is W-conjugate to a proper subset of $\Pi \cup \{-\varrho\}$. This construction coincides with the inductive step in the Borel-de Siebenthal procedure, [3]. Thanks to [12, Prop. 16], since p is good for G, it is also good for H. Because every maximal rank subsystem of Ψ is obtained by an iteration of the Borel-de Siebenthal procedure, the proposition follows by a repeated application of this algorithm, Deriziotis' Criterion, and Corollary 3.27.   □

THEOREM 3.29. *Let H be a connected regular reductive subgroup of G. Let $e \in \mathfrak{h} \cap \mathcal{N}$. Then $\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$.*

Proof. Let T be a maximal torus of G normalizing H. Then TH is a connected reductive subgroup of G of maximal rank. Since $\mathcal{D}(TH) = \mathcal{D}H$ is the semisimple part of H and so $TH = Z(TH)\mathcal{D}H$, the result follows from Lemma 2.19 and Theorem 3.28.   □

Our next two examples show that by iterating our results we can ensure that (†) holds even in cases where we cannot apply Theorem 3.21 directly.

EXAMPLE 3.30. Let G be of type F_4 and let K be a connected simple subgroup of G of type D_4 and H a connected subgroup of K of type G_2. Suppose that p is good for G. Then a successive application of Theorem 3.28 and the conclusion from Example 3.26 show that (†) also holds for the embedding $H \leq G$. Note that H is not a regular subgroup of G so that we cannot invoke Theorem 3.29 directly to the embedding $H \leq G$. This is the *standard* embedding of G_2 in F_4. We discuss a different embedding of G_2 in F_4 in characteristic 7 in Example 3.33 below.

EXAMPLE 3.31.   Let G be of type E_8 and suppose that p is good for G. Let K be a maximal rank subgroup of G of type $D_4 \times D_4$, and let H be a connected subgroup of K of type D_4 embedded diagonally into K. Then H is the fixed point subgroup of the involution interchanging the D_4-factors in K. Note that H is not a regular subgroup of G, so we cannot invoke Theorem 3.29. Nevertheless, it follows from Corollary 3.23 and Theorem 3.28 that (†) holds for the embedding $H \leq G$.

Remarks 3.32.   (i) Theorem 3.28 answers a question posed by J. C. Jantzen, cf. [7, §5.12].

(ii) The forward inclusion of Corollary 3.27 was already proved in [12, Prop. 23, Rem. 25] by different methods.

EXAMPLE 3.33.   In [26, Thm. 1(c)], D. Testerman showed that there is a maximal subgroup of type G_2 in F_4 in characteristic $p = 7$; see also [19, Thm. 1]. In this case let G be the ambient group of type F_4 and let H be the maximal subgroup of type G_2. The fusion of the unipotent classes of this embedding has been determined by R. Lawther (unpublished). Since $p = 7$ is good for G, this also determines the fusion of the nilpotent classes of $\mathfrak{h}$ in $\mathfrak{g}$; this is given in terms of the corresponding Bala-Carter labellings as follows:

G_2	1	A_1	$\tilde{A}_1$	$G_2(a_1)$	G_2
F_4	1	$A_1\tilde{A}_1$	$\tilde{A}_2A_1$	$F_4(a_3)$	$F_4(a_2)$

Table 1: The fusion of nilpotent classes for $G_2 \leq_{\max} F_4$ ($p = 7$).

In contrast to the standard (non-maximal) embedding of G_2 in F_4 which exists in any characteristic (Example 3.30), we cannot deduce (†) for the embedding $H \leq G$ directly or by iterating our results, as H is maximal in G. Nevertheless, our methods allow us to easily deduce that (†) holds for all but one of the nilpotent classes of $\mathfrak{h}$.

Note that the trivial cocharacter is associated to $e = 0$ for both H and G. If e belongs to the regular or subregular class in $\mathfrak{h}$, then according to Table 1 and [5, §5.9], the corresponding G-classes $F_4(a_3)$ and $F_4(a_2)$ in $\mathcal{N}$ are distinguished. Therefore, the result follows by Lemma 3.1. Let e belong to the G_2-class with label $\tilde{A}_1$. According to Table 1 and [5, Ch. 13], the

reductive parts of the centralizers $C_H(e)$ and $C_G(e)$ of this class in G_2 and the corresponding class $\tilde{A}_2A_1$ in F_4 are both of type A_1. In particular, $C_H(e)$ and $C_G(e)$ have the same rank and Theorem 3.14 gives the desired result in this case.

For the remaining pair of nilpotent orbits one can use Theorem 3.18 and show directly that (†) also holds in this case. Fix maximal tori T_H and T of H and G respectively, so that $T_H \leq T$. Let α be the long simple root of H with respect to T_H and let $\{\alpha_1, \ldots, \alpha_4\}$ be the set of simple roots of G with respect to T so that α_1 and α_2 are long. Let $e = e_\alpha$, a non-trivial root vector in the root space of α in $\mathfrak{g}$. Then e belongs to the H-class in $\mathfrak{h}$ with label A_1 and the coroot $\alpha^\vee \in Y(H)$ is an associated cocharacter of e, see [7, §5.13]. Let $L_{\{\alpha_1,\alpha_3\}}$ be the standard Levi subgroup of G of type $A_1\tilde{A}_1$ with root system $\{\pm\alpha_1, \pm\alpha_3\}$. Let $M = {}^{s_{\alpha_2}}(L_{\{\alpha_1,\alpha_3\}})$, where s_{α_2} is the simple reflection in the Weyl group of G corresponding to α_2. Then M is another Levi subgroup of G. Using Lawther's explicit fusion calculations, one can show that $\alpha^\vee(k^*) \leq \mathcal{D}M$ and that e is distinguished in $\mathfrak{m}$. Finally, since $e \in \mathfrak{h}(2, \alpha^\vee) \subset \mathfrak{g}(2, \alpha^\vee)$, it follows that $\alpha^\vee \in \Omega_G^a(e)$. Therefore, $\alpha^\vee \in \Omega_G^a(e) \cap Y(H)$, and consequently, by Theorem 3.18, we have $\Omega_H^a(e) = \Omega_G^a(e) \cap Y(H)$.

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