8

Comparison dynamics

The expansion of the self-force suggests that if we are willing to accept an error of order ε^2, the trajectory of the charged particle is governed by an autonomous equation – a substantial simplification of the hitherto coupled problem. An error of order ε^2 in the equation does *not* imply an error of the same order in the solution. This point must be discussed, but let us proceed for a while in good faith and simply ignore the error in Eq. (7.22). Then we obtain the following approximate equation for the motion of the charge,

$$\begin{aligned}
&\dot{\boldsymbol{q}} = \boldsymbol{v}\,, \\
&m(\boldsymbol{v})\dot{\boldsymbol{v}} = e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}) + \boldsymbol{v} \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q})\big) + \varepsilon(e^2/6\pi)\big[\gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} \\
&\qquad\qquad + 3\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v} + 3\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^2\ddot{\boldsymbol{v}}\big]\,.
\end{aligned} \tag{8.1}$$

Here $m(\boldsymbol{v})$ is the effective velocity-dependent mass. It is the sum of the bare mass and the mass (7.23) induced by the field,

$$m(\boldsymbol{v}) = m_{\mathrm{b}}(\gamma\mathbb{1} + \gamma^3\boldsymbol{v}\otimes\boldsymbol{v}) + m_{\mathrm{f}}(\boldsymbol{v})\,. \tag{8.2}$$

As anticipated in section 4.1, via a distinct route, the leading contribution to (8.1) is derived from the effective Lagrangian

$$L_{\mathrm{eff}}(\boldsymbol{q},\dot{\boldsymbol{q}}) = T(\dot{\boldsymbol{q}}) - e\big(\phi_{\mathrm{ex}}(\boldsymbol{q}) - \dot{\boldsymbol{q}}\cdot\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{q})\big)\,, \tag{8.3}$$

or equivalently from the Hamiltonian

$$H_{\mathrm{eff}}(\boldsymbol{q},\boldsymbol{p}) = E_{\mathrm{eff}}\big(\boldsymbol{p} - e\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{q})\big) + e\phi_{\mathrm{ex}}(\boldsymbol{q})\,. \tag{8.4}$$

For later purposes it is more convenient to work with the energy function

$$H(\boldsymbol{q},\boldsymbol{v}) = E_{\mathrm{s}}(\boldsymbol{v}) + e\phi_{\mathrm{ex}}(\boldsymbol{q})\,, \tag{8.5}$$

which is conserved by the solutions to (8.1) with $\varepsilon = 0$; compare with (4.14).

The term of order ε in (8.1) describes the radiation reaction. If included, the energy of the particle fails to be conserved and the energy balance becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{q}, \boldsymbol{v}) - \frac{\mathrm{d}}{\mathrm{d}t} \varepsilon \,(e^2/6\pi)\, \gamma^4 (\boldsymbol{v} \cdot \dot{\boldsymbol{v}}) = -\varepsilon\, (e^2/6\pi) \left[\gamma^4 \dot{\boldsymbol{v}}^2 + \gamma^6 (\boldsymbol{v} \cdot \dot{\boldsymbol{v}})^2\right]. \quad (8.6)$$

The term $-\varepsilon(e^2/6\pi)\gamma^4(\boldsymbol{v} \cdot \dot{\boldsymbol{v}}) = E_{\mathrm{schott}}(\boldsymbol{v}, \dot{\boldsymbol{v}})$ is the *Schott energy*. It has no definite sign. The Schott energy is stored in the near field and can be reversibly exchanged with the mechanical energy of the charge. The right-hand side of (8.6) is the irreversible loss of energy through radiation; compare with section 8.4. Equation (8.6) is analogous to the balance equations in hydrodynamics and a familiar way to rewrite it is

$$e\boldsymbol{v} \cdot E_{\mathrm{ex}}(\boldsymbol{q}) = \frac{\mathrm{d}}{\mathrm{d}t}\big(E_{\mathrm{s}}(\boldsymbol{v}) + E_{\mathrm{schott}}(\boldsymbol{v}, \dot{\boldsymbol{v}})\big) + \varepsilon\, (e^2/6\pi) \left[\gamma^4 \dot{\boldsymbol{v}}^2 + \gamma^6 (\boldsymbol{v} \cdot \dot{\boldsymbol{v}})^2\right]. \quad (8.7)$$

In other words, the work done by the external electric field acting on the charge is divided up into the change in its kinetic energy, the change of the Schott energy, and radiation.

If we set $G_\varepsilon = E_{\mathrm{s}} + E_{\mathrm{schott}}$, then G_ε is decreasing in time, and integrating both sides of (8.6) yields

$$\begin{aligned} &-G_\varepsilon\left(\boldsymbol{q}(t), \boldsymbol{v}(t), \dot{\boldsymbol{v}}(t)\right) + G_\varepsilon\left(\boldsymbol{q}(0), \boldsymbol{v}(0), \dot{\boldsymbol{v}}(0)\right) \\ &\quad = \varepsilon\, (e^2/6\pi) \int_0^t \mathrm{d}s \left[\gamma^4\, \dot{\boldsymbol{v}}(s)^2 + \gamma^6\, (\boldsymbol{v}(s) \cdot \dot{\boldsymbol{v}}(s))^2\right]. \end{aligned} \quad (8.8)$$

The mechanical energy is bounded from below, but the Schott energy does not have a definite sign. *If* (!) the Schott energy remains bounded in the course of time, then

$$\int_0^\infty \mathrm{d}t \left[\gamma^4\, \dot{\boldsymbol{v}}(t)^2 + \gamma^6\, (\boldsymbol{v}(t) \cdot \dot{\boldsymbol{v}}(t))^2\right] < \infty\,, \quad (8.9)$$

which implies

$$\lim_{t\to\infty} \dot{\boldsymbol{v}}(t) = 0\,. \quad (8.10)$$

The same conclusion was already reached for the Abraham model in Theorem 5.1, with no adiabatic limit there. Instead of (8.9) we used the bounded energy dissipation (5.9). Since both the approximate and the true solutions have the same long-time asymptotics, we expect no further time scale, i.e. higher corrections to (8.1) should not change the qualitative behavior of solutions and merely increase

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in precision. One important difference must be stressed, however: Theorem 5.1 holds for every solution, whereas (8.10) holds only for those with bounded Schott energy.

Unfortunately, the energy balance (8.7) by itself does not tell the full story. As noticed apparently first by Dirac (1938), Eq. (8.1) has solutions which run away exponentially fast. This does not contradict (8.8). $G_\varepsilon(t)$ diverges to $-\infty$ and the time integral diverges to $+\infty$ as $t \to \infty$. The occurrence of runaway solutions can be seen most easily in the approximation of small velocities, setting $\boldsymbol{B}_{\mathrm{ex}} = 0$, and linearizing ϕ_{ex} around a stable minimum, say at $\boldsymbol{q} = 0$. Then (8.1) becomes

$$\dot{\boldsymbol{q}} = \boldsymbol{v}\,, \quad m\dot{\boldsymbol{v}} = -m\,\omega_0^2\,\boldsymbol{q} + \varepsilon\,km\,\ddot{\boldsymbol{v}} \tag{8.11}$$

with $km = e^2/6\pi$. The three components of the linear equation (8.11) decouple and for each component there are three modes of the form e^{zt}. The characteristic equation is $z^2 = -\omega_0^2 + \varepsilon k z^3$ and to leading order the eigenvalues are $z_\pm = \pm\,\mathrm{i}\omega_0 - \varepsilon(k\omega_0^2/2)$, $z_3 = (1/\varepsilon k) + \mathcal{O}(1)$. Thus in the nine-dimensional phase space for (8.11) there is a stable six-dimensional hyperplane, $\mathcal{C}_\varepsilon$. On $\mathcal{C}_\varepsilon$ the motion is weakly damped, with friction coefficient $\varepsilon\,(k\omega_0^2/2)$, and relaxes as $t \to \infty$ to rest at $\boldsymbol{q} = 0$. Transversal to $\mathcal{C}_\varepsilon$ the solution runs away as $\mathrm{e}^{(t/\varepsilon k)}$.

Clearly such runaway solutions violate the stability estimate (7.15). Thus the full Maxwell–Newton equations do not have runaways. They somehow appear as an artifact of the Taylor expansion of $\boldsymbol{F}^{\varepsilon}_{\mathrm{self}}(t)$ from (7.6). Dirac simply postulated that physical solutions must satisfy the *asymptotic condition*

$$\lim_{t\to\infty}\ \dot{\boldsymbol{v}}(t) = 0\,. \tag{8.12}$$

In the linearized version (8.11) this means that the initial conditions have to lie in $\mathcal{C}_\varepsilon$. In Theorem 5.1 we proved the asymptotic condition to hold for the Abraham model. Thus only those solutions to (8.1) satisfying the asymptotic condition can serve as a comparison dynamics to the true solution. We then have to understand how the asymptotic condition arises, even more so the global structure of the solution flow to (8.1).

We note that in (8.1) the highest derivative is multiplied by a small prefactor. Such equations have been studied in great detail under the header of (geometric) singular perturbation theory. The main conclusion is that the structure found for the linear equation (8.11) persists for the nonlinear equation (8.1). Of course the hyperplane $\mathcal{C}_\varepsilon$ is now deformed into some manifold, called the critical (or center) manifold. We explain a standard example in the following section and then apply the theory to (8.1).

8.1 An example for singular perturbation theory

As a purely mathematical example we consider the coupled system

$$\dot{x} = f(x, y) , \quad \varepsilon \dot{y} = y - h(x) . \tag{8.13}$$

h and f are bounded, smooth functions. The phase space is $\mathbb{R}^2$. The question we address is to understand how the solutions to (8.13) behave for small ε. If we set $\varepsilon = 0$, then $y = h(x)$ and we obtain the autonomous equation

$$\dot{x} = f(x, h(x)) . \tag{8.14}$$

Geometrically this means that the two-dimensional phase space has been squeezed to the line $y = h(x)$ and the base point, $x(t)$, is governed by (8.14). $\{(x, h(x)) | \, x \in \mathbb{R}\}$ is the critical manifold to zeroth order in ε.

To see some motion appear in the phase space ambient to $\mathcal{C}_0$ we change from t to the fast time scale $\tau = \varepsilon^{-1} t$. Denoting differentiation with respect to τ by $'$, (8.13) goes over to

$$x' = \varepsilon f(x, y) , \quad y' = y - h(x) . \tag{8.15}$$

In the limit $\varepsilon \to 0$ we now have $x' = 0$, i.e. $x(\tau) = x_0$ and $y' = y - h(x_0)$ with solution $y(\tau) = (y_0 - h(x_0))\mathrm{e}^{\tau} + h(x_0)$. Thus on this time scale, $\mathcal{C}_0$ consists exclusively of repelling fixed points. This is why $\mathcal{C}_0$ is called critical. The linearization at $\mathcal{C}_0$ has the eigenvalue 1 transverse and the eigenvalue 0 tangential to $\mathcal{C}_0$. In the theory of dynamical systems zero eigenvalues in the linearization turn out to be linked to center manifolds, and thus $\mathcal{C}_0$ is also called the center manifold (at $\varepsilon = 0$). The basic result of singular perturbation theory is that for small ε the critical manifold deforms smoothly into $\mathcal{C}_\varepsilon$; compare with figure 8.1. Thus $\mathcal{C}_\varepsilon$ is invariant under the solution flow to (8.13). Its linearization at $(x, y) \in \mathcal{C}_\varepsilon$ has an eigenvalue of $\mathcal{O}(1)$ with eigenvector tangential to $\mathcal{C}_\varepsilon$ and an eigenvalue $1/\varepsilon$ with eigenvector transverse to $\mathcal{C}_\varepsilon$. Thus for an initial condition slightly away from $\mathcal{C}_\varepsilon$ the solution very rapidly diverges to infinity. Since $\mathcal{C}_0$ is deformed by order ε, also $\mathcal{C}_\varepsilon$ is of the form $\{(x, h_\varepsilon(x)) | \, x \in \mathbb{R}\}$. According to (8.13) the base point evolves as

$$\dot{x} = f(x, h_\varepsilon(x)) . \tag{8.16}$$

Since h_ε is smooth in ε, it can be Taylor-expanded as

$$h_\varepsilon(x) = \sum_{j=0}^{m} \varepsilon^j \, h_j(x) + \mathcal{O}(\varepsilon^{m+1}) . \tag{8.17}$$

By (8.13) and (8.16) we have the identity

$$\varepsilon \, \partial_x h_\varepsilon(x) \, f(x, h_\varepsilon(x)) = h_\varepsilon(x) - h(x) . \tag{8.18}$$

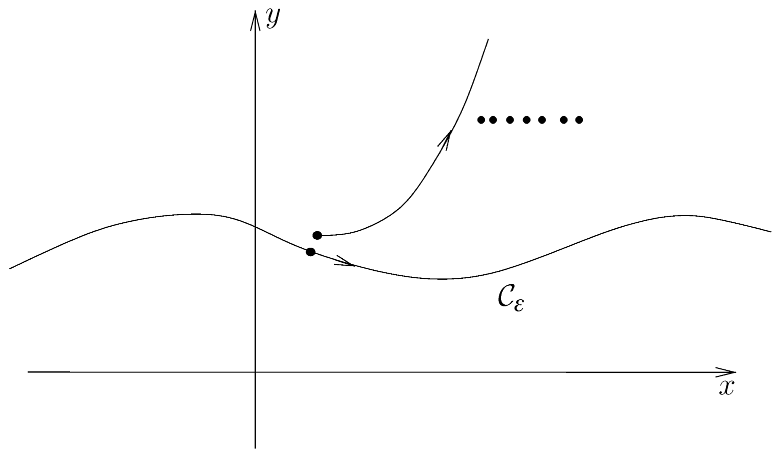

Figure 8.1: Repulsive center manifold $\mathcal{C}_\varepsilon$. The motion on $\mathcal{C}_\varepsilon$ is slow and the motion away from $\mathcal{C}_\varepsilon$ is fast.

Substituting into (8.17) and comparing powers of ε one can thus determine $h_j(x)$ recursively. To lowest order we obtain

$$h_0(x) = h(x)\,, \quad h_1(x) = h'(x)\, f(x, h(x)) \tag{8.19}$$

and to order ε the base point is governed by

$$\dot{x} = f(x, h(x)) + \varepsilon\, \partial_y f(x, h(x))\, h'(x)\, f(x, h(x))\,. \tag{8.20}$$

Given the geometric picture of the center manifold, the stable (i.e. not runaway) solutions to (8.13) can be determined to any required precision.

8.2 The critical manifold

Our task is to cast (8.1) into the canonical form used in singular perturbation theory. We set $(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{v}) \in \mathbb{R}^3 \times \mathbb{V}$, $\boldsymbol{y} = \dot{\boldsymbol{v}} \in \mathbb{R}^3$,

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}_2, \boldsymbol{y}) \in \mathbb{V} \times \mathbb{R}^3 \tag{8.21}$$

and

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \varepsilon) = \gamma^{-2} \kappa(\boldsymbol{x}_2)^{-1} \Big(&(6\pi/e^2) \big[m(\boldsymbol{x}_2)\boldsymbol{y} - \boldsymbol{F}_{\mathrm{ex}}(\boldsymbol{x}) \big] \\ &- \varepsilon \big[3\gamma^6 (\boldsymbol{x}_2 \cdot \boldsymbol{y})^2 \boldsymbol{x}_2 + 3\gamma^4 (\boldsymbol{x}_2 \cdot \boldsymbol{y}) \boldsymbol{y} \big] \Big)\,, \end{aligned} \tag{8.22}$$

where $\gamma = (1 - \boldsymbol{x}_2^2)^{-1/2}$ as before, $\boldsymbol{F}_{\mathrm{ex}}(\boldsymbol{x}) = e(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{x}_1) + \boldsymbol{x}_2 \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{x}_1))$, and $\kappa(\boldsymbol{v})$ is the 3×3 matrix $\kappa(\boldsymbol{v}) = \mathbb{1} + \gamma^2\, \boldsymbol{v} \otimes \boldsymbol{v}$ with inverse matrix $\kappa(\boldsymbol{v})^{-1} = \mathbb{1} - \boldsymbol{v} \otimes \boldsymbol{v}$. With this notation Eq. (8.1) reads

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})\,, \quad \varepsilon\, \dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \varepsilon)\,. \tag{8.23}$$

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We set $\boldsymbol{h}(\boldsymbol{x}) = m(\boldsymbol{x}_2)^{-1} \boldsymbol{F}_{\mathrm{ex}}(\boldsymbol{x})$. Then for $\varepsilon = 0$ the critical manifold, $\mathcal{C}_0$, is given by

$$\mathcal{C}_0 = \{(\boldsymbol{x}, \boldsymbol{h}(\boldsymbol{x}))|\, \boldsymbol{x} \in \mathbb{R}^3 \times \mathbb{V}\} = \{(\boldsymbol{q}, \boldsymbol{v}, \dot{\boldsymbol{v}})|\, m(\boldsymbol{v})\dot{\boldsymbol{v}} = \boldsymbol{F}_{\mathrm{ex}}(\boldsymbol{q}, \boldsymbol{v})\}\,, \quad (8.24)$$

which means that, for $\varepsilon = 0$, it is spanned by the solutions of the leading Hamiltonian part of Eq. (8.1). Linearizing at $\mathcal{C}_0$ the repelling eigenvalue is dominated by $\gamma^{-2}\kappa(\boldsymbol{x}_2)^{-1} m(\boldsymbol{x}_2)$ which tends to zero as $|\boldsymbol{x}_2| \to 1$. Therefore $\mathcal{C}_0$ is not uniformly hyperbolic, which is one of the standard assumptions of singular perturbation theory.

To overcome this difficulty we modify $\boldsymbol{g}$ to $\boldsymbol{g}_\delta$, δ small, which agrees with $\boldsymbol{g}$ on $\mathbb{R}^3 \times \{\boldsymbol{v}|\, |\boldsymbol{v}| \le 1-\delta\} \times \mathbb{R}^3$ and which is constantly extended to values $|\boldsymbol{v}| \ge 1-\delta$. Thus for $|\boldsymbol{x}_2(t)| \le 1-\delta$ the solution to $\dot{\boldsymbol{x}} = \boldsymbol{f},\ \varepsilon\dot{\boldsymbol{y}} = \boldsymbol{g}_\delta$ agrees with the solution to $\dot{\boldsymbol{x}} = \boldsymbol{f},\ \varepsilon\dot{\boldsymbol{y}} = \boldsymbol{g}$. For sufficiently small ε the modified equation then has a critical manifold $\mathcal{C}_\varepsilon$ with the properties discussed in the example of section 8.1. We only have to make sure that the modification is never seen by the solution. Thus, for the initial condition $|\boldsymbol{v}(0)| \le \bar{v}$, we have to find a $\delta = \delta(\bar{v})$ such that $|\boldsymbol{v}(t)| \le 1-\delta$ for all times. To do so, one needs the energy balance (8.7).

We consider the modified evolution with vector field $(\boldsymbol{f}, \boldsymbol{g}_\delta)$ and choose the initial velocity such that $|\boldsymbol{v}(0)| \le \bar{v} < 1$. For ε small enough this dynamics has a critical manifold of the form $\dot{\boldsymbol{v}} = \boldsymbol{h}_\varepsilon(\boldsymbol{q}, \boldsymbol{v})$ and $|\boldsymbol{h}_\varepsilon(\boldsymbol{q}, \boldsymbol{v})| \le c_1 = c_1(\delta)$. We start the dynamics on $\mathcal{C}_\varepsilon$. According to (8.8), for all $t \ge 0$,

$$\begin{aligned} G_\varepsilon\,(\boldsymbol{q}(t), \boldsymbol{v}(t), \boldsymbol{h}_\varepsilon(t)) &\le G_\varepsilon(0) = H(\boldsymbol{q}(0), \boldsymbol{v}(0)) - \varepsilon(e^2/6\pi)(\boldsymbol{v}(0)\cdot \boldsymbol{h}_\varepsilon(0)) \\ &\le E_{\mathrm{s}}(\bar{v}) + e\phi_{\mathrm{ex}}\,(\boldsymbol{q}(0)) + \varepsilon c_1\,. \end{aligned} \quad (8.25)$$

We now choose δ such that $\bar{v} \le 1 - 2\delta$. Since the initial conditions are on $\mathcal{C}_\varepsilon$, the solution will stay for a while on $\mathcal{C}_\varepsilon$ until the first time, τ, when $|\boldsymbol{v}(\tau)| = 1-\delta$ occurs. After that time the modification becomes visible. At time τ we have, using the lower bound on the energy and (8.25),

$$\begin{aligned} E_{\mathrm{s}}(\boldsymbol{v}(\tau)) + e\bar{\phi} &\le H(\boldsymbol{q}(\tau), \boldsymbol{v}(\tau)) = G_\varepsilon(\tau) + \varepsilon(e^2/6\pi)\,\gamma^4\,(\boldsymbol{v}(\tau)\cdot \boldsymbol{h}_\varepsilon(\tau)) \\ &\le E_{\mathrm{s}}(\bar{v}) + e\phi_{\mathrm{ex}}\,(\boldsymbol{q}(0)) + 2\varepsilon c_1 \end{aligned} \quad (8.26)$$

and therefore

$$E_{\mathrm{s}}(1-\delta) \le E_{\mathrm{s}}(1-2\delta) + e\,(\phi_{\mathrm{ex}}\,(\boldsymbol{q}(0)) - \bar{\phi}) + 2\varepsilon c_1\,. \quad (8.27)$$

$E_{\mathrm{s}}(1-\delta) \cong 1/\sqrt{\delta}$ for small δ, which implies

$$\frac{1}{\sqrt{\delta}} \le c_2 + 4\,\varepsilon c_1 \quad (8.28)$$

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with $c_2 = 2e\,(\phi_{\mathrm{ex}}(\boldsymbol{q}(0)) - \bar{\phi})$. We now choose δ so small that $1/\sqrt{\delta} > c_2 + 1$ and then ε so small that $4\varepsilon c_1 < 1$. Then (8.28) is a contradiction to the assumption that $|\boldsymbol{v}(\tau)| = 1 - \delta$. We thus conclude that $\tau = \infty$ and the solution trajectory stays on $\mathcal{C}_\varepsilon$ for all times.

Equipped with this information we have for small ε the critical manifold

$$\dot{\boldsymbol{v}} = \boldsymbol{h}_\varepsilon\,(\boldsymbol{q}, \boldsymbol{v})\,. \tag{8.29}$$

On the critical manifold the Schott energy is bounded and from the argument leading to (8.10) we conclude that Dirac's asymptotic condition holds on $\mathcal{C}_\varepsilon$. On the other hand, slightly off $\mathcal{C}_\varepsilon$ the solution diverges with a rate of order $1/\varepsilon$. Therefore the asymptotic condition singles out, for given $\boldsymbol{q}(0)$, $\boldsymbol{v}(0)$, the *unique* $\dot{\boldsymbol{v}}(0)$ on $\mathcal{C}_\varepsilon$.

The motion on the critical manifold is governed by an effective equation which can be determined approximately following the scheme of section 8.1. We define

$$\boldsymbol{h}(\boldsymbol{q}, \boldsymbol{v}) = m(\boldsymbol{v})^{-1}\, e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}) + \boldsymbol{v} \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q})\big)\,. \tag{8.30}$$

Then, up to errors of order ε^2,

$$\begin{aligned} m(\boldsymbol{v})\dot{\boldsymbol{v}} &= e\,\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}) + \boldsymbol{v} \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q})\big) \\ &\quad + \varepsilon\,(e^2/6\pi)\,\big[\gamma^2\kappa(\boldsymbol{v})\big((\boldsymbol{v}\cdot\nabla_{\boldsymbol{q}})\boldsymbol{h} + (\boldsymbol{h}\cdot\nabla_{\boldsymbol{v}})\boldsymbol{h} + 3\gamma^2(\boldsymbol{v}\cdot\boldsymbol{h})\boldsymbol{h}\big)\big]\,. \end{aligned} \tag{8.31}$$

The physical solutions of (8.1), in the sense of satisfying the asymptotic condition, are governed by Eq. (8.31). Thus it, and *not* Eq. (8.1), must be regarded as the correct comparison dynamics to the true microscopic evolution equations (6.11). Note that the error accumulated in going from (8.1) to (8.31) is of the same order as the error made in the derivation of Eq. (8.1).

Because of the special structure of (8.1), on a formal level the final result (8.31) can be deduced without the help of geometric perturbation theory. We regard $m(\boldsymbol{v})\dot{\boldsymbol{v}} = e\,(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}) + \boldsymbol{v} \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}))$ as the "unperturbed" equation and substitute for the terms inside the square bracket, which means replacing $\dot{\boldsymbol{v}}$ by $\boldsymbol{h}$ and $\ddot{\boldsymbol{v}}$ by $\dot{\boldsymbol{h}} = (\boldsymbol{v}\cdot\nabla_{\boldsymbol{q}})\boldsymbol{h} + (\boldsymbol{h}\cdot\nabla_{\boldsymbol{v}})\boldsymbol{h}$. While yielding the correct answer, one misses the geometrical picture of the critical manifold and the associated motion in phase space.

8.3 Tracking of the true solution

From (6.11) we have the true solution $\boldsymbol{q}^\varepsilon(t)$, $\boldsymbol{v}^\varepsilon(t)$ with initial conditions $\boldsymbol{q}^0$, $\boldsymbol{v}^0$ and correspondingly adapted field data. We face the problem of how well this solution is tracked by the comparison dynamics (8.1) on its critical manifold. Let us first disregard the radiation reaction. From our a priori estimates we know that

$$\dot{\boldsymbol{q}}^\varepsilon = \boldsymbol{v}^\varepsilon, \quad m(\boldsymbol{v}^\varepsilon)\dot{\boldsymbol{v}}^\varepsilon = e\,\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}^\varepsilon) + \boldsymbol{v}^\varepsilon \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}^\varepsilon)\big) + \mathcal{O}(\varepsilon) \tag{8.32}$$

which should be compared to

$$\dot{\boldsymbol{r}} = \boldsymbol{u}\,, \quad m(\boldsymbol{u})\dot{\boldsymbol{u}} = e\left(\boldsymbol{E}_{\rm ex}(\boldsymbol{r}) + \boldsymbol{u} \times \boldsymbol{B}_{\rm ex}(\boldsymbol{u})\right). \tag{8.33}$$

We switched to the variables $\boldsymbol{r}, \boldsymbol{u}$ instead of $\boldsymbol{q}, \boldsymbol{v}$ so as to distinguish more clearly between the true and comparison dynamics.

Theorem 8.1 (Adiabatic limit, conservative tracking dynamics). *For the Abraham model satisfying the conditions* (C), (P), *and* (I) *let* $|e| \leq \bar{e}$ *and* $\varepsilon \leq \varepsilon_0$ *be sufficiently small. Let* $\boldsymbol{r}(t), \boldsymbol{u}(t)$ *be the solution to the comparison dynamics (8.33) with initial conditions* $\boldsymbol{r}(0) = \boldsymbol{q}^0$, $\boldsymbol{u}(0) = \boldsymbol{v}^0$. *Then for every* $\tau > 0$ *there exist constants* $c(\tau)$ *such that*

$$|\boldsymbol{q}^\varepsilon(t) - \boldsymbol{r}(t)| \leq c(\tau)\varepsilon\,, \quad |\boldsymbol{v}^\varepsilon(t) - \boldsymbol{u}(t)| \leq c(\tau)\varepsilon \tag{8.34}$$

for $0 \leq t \leq \tau$.

Proof: Let $\delta(t) = |\boldsymbol{q}^\varepsilon(t) - \boldsymbol{r}(t)| + |\boldsymbol{v}^\varepsilon(t) - \boldsymbol{u}(t)|$. Converting the differential equations (8.32), (8.33) into their integral form, one obtains

$$\begin{aligned}\delta(t) &\leq \delta(0) + C\int_0^t \mathrm{d}s\,\delta(s) + \varepsilon\int_0^t \mathrm{d}s\,C\big(1 + \varepsilon(\varepsilon + s)^{-2}\big)\\ &\leq \delta(0) + \varepsilon C(t+1) + C\int_0^t \mathrm{d}s\,\delta(s)\,.\end{aligned} \tag{8.35}$$

Since $\delta(0) = 0$ by assumption, Gronwall's lemma yields the bound $\delta(t) \leq \varepsilon C \mathrm{e}^{Ct}$. □

Theorem 8.1 states that, up to an error of order ε, the true solution is well approximated by the Hamiltonian dynamics (8.33).

In the next order the comparison dynamics reads

$$\begin{aligned}&\dot{\boldsymbol{r}} = \boldsymbol{u}\,,\\ &m(\boldsymbol{u})\dot{\boldsymbol{u}} = e\left(\boldsymbol{E}_{\rm ex}(\boldsymbol{r}) + \boldsymbol{u} \times \boldsymbol{B}_{\rm ex}(\boldsymbol{r})\right)\\ &\qquad + \varepsilon(e^2/6\pi)\big[\gamma^4\,(\boldsymbol{u}\cdot\ddot{\boldsymbol{u}})\boldsymbol{u} + 3\gamma^6(\boldsymbol{u}\cdot\dot{\boldsymbol{u}})^2\,\boldsymbol{u} + 3\gamma^4(\boldsymbol{u}\cdot\dot{\boldsymbol{u}})\dot{\boldsymbol{u}} + \gamma^2\ddot{\boldsymbol{u}}\big]\end{aligned} \tag{8.36}$$

restricted to its critical manifold $\mathcal{C}_\varepsilon$. Since the radiation reaction is proportional to ε, the solution $\boldsymbol{r}(t), \boldsymbol{u}(t)$ depends now on ε, a dependence which is suppressed in our notation. Naively one would expect that improving the equation by a term of order ε increases the precision to order ε^2, i.e.

$$|\boldsymbol{q}^\varepsilon(t) - \boldsymbol{r}(t)| + |\boldsymbol{v}^\varepsilon(t) - \boldsymbol{u}(t)| = \mathcal{O}(\varepsilon^2)\,. \tag{8.37}$$

An alternative option to keeping track of the ε-correction is to consider longer times, of the order $\varepsilon^{-1}\tau$ on the macroscopic time scale. Then the radiative effects add up to deviations of order one from the Hamiltonian trajectory. Thus

$$|\boldsymbol{q}^{\varepsilon}(t) - \boldsymbol{r}(t)| = \mathcal{O}(\varepsilon) \quad \text{for} \quad 0 \le t \le \varepsilon^{-1}\tau \,. \tag{8.38}$$

One should be somewhat careful here. In a scattering situation the charged particle reaches the force-free region after a finite macroscopic time. According to (8.37) the error in the velocity is then $\mathcal{O}(\varepsilon^2)$, which builds up an error in the position of order ε over a time span $\varepsilon^{-1}\tau$. Thus we cannot hope to do better than (8.38). On the other hand, when the motion remains bounded, as e.g. in a uniform external magnetic field, the charge comes to rest at some point $\boldsymbol{q}^*$ in the long-time limit and the rest point $\boldsymbol{q}^*$ is the same for the true and the comparison dynamics. At least, for an external electrostatic potential with a discrete set of critical points we have already established such behavior and presumably it holds in general. Thus for small ε we have $\boldsymbol{q}^{\varepsilon}(\varepsilon^{-1}\tau) \cong \boldsymbol{q}^*$ and also $\boldsymbol{r}^{\varepsilon}(\varepsilon^{-1}\tau) \cong \boldsymbol{q}^*$. Therefore, in the case of bounded motion, we conjecture that (8.38) holds for *all* times.

Conjecture 8.2 (Adiabatic limit including friction). *For the Abraham model satisfying* (C), (P), *and* (I) *let* $\boldsymbol{q}(t)$ *be bounded, i.e.* $|\boldsymbol{q}(t)| \le C$ *for all* $t \ge 0$, *and* $\varepsilon \le \varepsilon_0$. *Then there exists* $(\boldsymbol{r}(0)\,, \boldsymbol{u}(0), \dot{\boldsymbol{u}}(0)) \in \mathcal{C}_\varepsilon$ *such that*

$$\sup_{t \ge 0} |\boldsymbol{q}^{\varepsilon}(t) - \boldsymbol{r}(t)| = \mathcal{O}(\varepsilon)\,, \tag{8.39}$$

where $\boldsymbol{r}(t)$ *is the solution to (8.36) with the initial conditions given before.*

In a more descriptive mode, the true solution $\boldsymbol{q}^{\varepsilon}(t)$ is ε-shadowed for all times by one solution (and thus by many solutions) of the comparison dynamics.

At present we are far from such strong results. The problem is that an error of order ε^2 in (8.36) is generically amplified as $\varepsilon^2 e^{t/\varepsilon}$. Although such an increase violates the a priori bounds, it renders a proof of (8.39) difficult. We seem to be back to (8.34) which carries no information on the radiation reaction. Luckily the radiation correction in (8.36) can be seen in the energy balance.

Theorem 8.3 (Adiabatic limit including friction). *Under the assumptions of Theorem 8.1 one has*

$$\Big|[E_{\mathrm{s}}(\boldsymbol{v}^{\varepsilon}(t)) + e\,\phi_{\mathrm{ex}}(\boldsymbol{q}^{\varepsilon}(t))] - [E_{\mathrm{s}}(\boldsymbol{u}(t)) + e\phi_{\mathrm{ex}}(\boldsymbol{r}(t))]\Big| \le Cc(\tau)\varepsilon^2 \tag{8.40}$$

for $t_\varepsilon \le t \le \tau$. *Here* $(\boldsymbol{r}(t), \boldsymbol{u}(t))$ *is the solution to (8.36) with initial data* $\boldsymbol{r}(t_\varepsilon) = \boldsymbol{q}^{\varepsilon}(t_\varepsilon)$, $\boldsymbol{u}(t_\varepsilon) = \boldsymbol{v}^{\varepsilon}(t_\varepsilon)$, $\dot{\boldsymbol{u}}^{\varepsilon}(t_\varepsilon) = h_\varepsilon(\boldsymbol{q}^{\varepsilon}(t_\varepsilon), \boldsymbol{v}^{\varepsilon}(t_\varepsilon))$ *and* $t_\varepsilon = \varepsilon^{1/3}$.

To achieve a precision of order ε^2, the initial slip in (7.15) does not allow one to match the true and comparison dynamics at $t = 0$. One needs $|\dddot{\boldsymbol{q}}^{\varepsilon}(t)|$ uniformly bounded, which is ensured only for $t \geq C\varepsilon^{1/3}$, i.e. $t \geq t_\varepsilon$ with the arbitrary choice $C = 1$.

Proof: Let us use the estimate (7.22) on the self-force and denote the error term by $\boldsymbol{f}^\varepsilon(t)$. Then $|\boldsymbol{f}^\varepsilon(t)| \leq C\varepsilon^2$ for $t_\varepsilon \leq t$. As in (8.7),

$$\frac{\mathrm{d}}{\mathrm{d}t} G_\varepsilon(\boldsymbol{q}^\varepsilon, \boldsymbol{v}^\varepsilon, \dot{\boldsymbol{v}}^\varepsilon) = \boldsymbol{f}^\varepsilon(t) \cdot \boldsymbol{v}^\varepsilon - \varepsilon\,(e^2/6\pi)\big[\gamma^4(\dot{\boldsymbol{v}}^\varepsilon)^2 + \gamma^6(\boldsymbol{v}^\varepsilon \cdot \dot{\boldsymbol{v}}^\varepsilon)^2\big] \tag{8.41}$$

and therefore

$$\begin{aligned} |H(\boldsymbol{q}^\varepsilon, \boldsymbol{v}^\varepsilon) - H(\boldsymbol{r}, \boldsymbol{u})| \leq \varepsilon\,(e^2/6\pi)|\gamma(\boldsymbol{v}^\varepsilon)^4(\boldsymbol{v}^\varepsilon \cdot \dot{\boldsymbol{v}}^\varepsilon) - \gamma(\boldsymbol{u})^4(\boldsymbol{u} \cdot \dot{\boldsymbol{u}})| \\ + \int_{t_\varepsilon}^{t} \mathrm{d}s\big(|\boldsymbol{f}^\varepsilon \cdot \boldsymbol{v}^\varepsilon| + \varepsilon(e^2/6\pi)|\gamma(\boldsymbol{v}^\varepsilon)^4(\dot{\boldsymbol{v}}^\varepsilon)^2 \\ + \gamma(\boldsymbol{v}^\varepsilon)^6(\boldsymbol{v}^\varepsilon \cdot \dot{\boldsymbol{v}}^\varepsilon)^2 - \gamma(\boldsymbol{u})^4(\dot{\boldsymbol{u}})^2 - \gamma(\boldsymbol{u})^6(\boldsymbol{u} \cdot \dot{\boldsymbol{u}})^2|\big)\,. \end{aligned} \tag{8.42}$$

Since $|\boldsymbol{v}^\varepsilon|$, $|\boldsymbol{u}|$ remain bounded away from 1, the γ-factors are uniformly bounded, and it suffices to estimate the difference on the Hamiltonian level of precision. From Theorem 8.1 one has the bound $|\boldsymbol{v}^\varepsilon(t) - \boldsymbol{u}(t)| \leq c(\tau)\varepsilon$. Inserting (8.34) into (8.32) and (8.33), we obtain the same bound for the first derivative, $|\dot{\boldsymbol{v}}^\varepsilon(t) - \dot{\boldsymbol{u}}(t)| \leq c(\tau)\varepsilon$. Moreover $\int_{t_\varepsilon}^t \mathrm{d}s\,|\boldsymbol{f}^\varepsilon(s)| \leq Ct\varepsilon^2$. Working out the differences in (8.42), one concludes

$$|H(\boldsymbol{q}^\varepsilon(t), \boldsymbol{v}^\varepsilon(t)) - H(\boldsymbol{r}(t), \boldsymbol{u}(t))| \leq C(t + c(t))\varepsilon^2\,, \tag{8.43}$$

as claimed. □

8.4 Electromagnetic fields in the adiabatic limit

So far we have concentrated on the Lorentz force with retarded fields and have obtained approximate evolution equations for the charged particle. Such an approximate solution can be reinserted into the inhomogeneous Maxwell–Lorentz equations in order to obtain the electromagnetic fields in the adiabatic limit.

As before, let $(\boldsymbol{q}^\varepsilon(t), \boldsymbol{v}^\varepsilon(t))$, $t \geq 0$, be the true solution. We extend it to $\boldsymbol{q}^\varepsilon(t) = \boldsymbol{q}^0 + \boldsymbol{v}^0 t$, $\boldsymbol{v}^\varepsilon(t) = \boldsymbol{v}^0$ for $t \leq 0$. According to (4.31), (4.32) and using the scaled fields as in (6.8), one has

$$\frac{1}{\sqrt{\varepsilon}}\,\boldsymbol{E}(t) = -\int_{-\infty}^{t} \mathrm{d}s\big(\nabla G_{t-s} * \rho_\varepsilon(s) + \partial_t G_{t-s} * \boldsymbol{j}_\varepsilon(s)\big) \tag{8.44}$$

with $\rho_\varepsilon(\boldsymbol{x},t) = e\varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{q}^\varepsilon(t))$, $\boldsymbol{j}_\varepsilon(\boldsymbol{x},t) = e\varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{q}^\varepsilon(t))\boldsymbol{v}^\varepsilon(t)$. Inserting from (2.15) and by partial integration,

$$\begin{aligned}\frac{1}{\sqrt{\varepsilon}}\boldsymbol{E}(\boldsymbol{x},t) = &-\int_{-\infty}^{t} \mathrm{d}s \int \mathrm{d}^3y \frac{1}{4\pi(t-s)}\delta(|\boldsymbol{x}-\boldsymbol{y}| - (t-s))\nabla\rho_\varepsilon(\boldsymbol{y},s)\\ &-\int_{-\infty}^{t} \mathrm{d}s \int \mathrm{d}^3y \frac{1}{4\pi(t-s)^2}\delta(|\boldsymbol{x}-\boldsymbol{y}| - (t-s))\\ &\quad\times[(\boldsymbol{y}-\boldsymbol{x})\cdot\nabla\boldsymbol{j}_\varepsilon(\boldsymbol{y},s) + \boldsymbol{j}_\varepsilon(\boldsymbol{y},s)]\\ = &-e\int \mathrm{d}^3y\Big(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\nabla\varphi_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|))\boldsymbol{v}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|)\\ &+\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^2}\boldsymbol{v}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|)(1+(\boldsymbol{y}-\boldsymbol{x})\cdot\nabla)\\ &\quad\varphi_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|))\Big). \end{aligned} \tag{8.45}$$

In the same fashion

$$\frac{1}{\sqrt{\varepsilon}}\boldsymbol{B}(\boldsymbol{x},t) = -e\int \mathrm{d}^3y \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{v}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|)\times\nabla\varphi_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|)). \tag{8.46}$$

In the limit $\varepsilon \to 0$ one has $\varphi_\varepsilon(\boldsymbol{x}) \to \delta(\boldsymbol{x})$ and, by Theorem 8.1, $\boldsymbol{q}^\varepsilon(t) \to \boldsymbol{r}(t)$, $\boldsymbol{v}^\varepsilon(t) \to \boldsymbol{u}(t)$, where $\boldsymbol{r}(t) = \boldsymbol{q}^0 + \boldsymbol{v}^0 t$, $\boldsymbol{u}(t) = \boldsymbol{v}^0$ for $t \le 0$. We substitute $\boldsymbol{y}' = \boldsymbol{y} - \boldsymbol{q}^\varepsilon(t - |\boldsymbol{x}-\boldsymbol{y}|)$ with volume element $\det(\mathrm{d}\boldsymbol{y}/\mathrm{d}\boldsymbol{y}') = [1 - \boldsymbol{v}^\varepsilon(t-|\boldsymbol{x}-\boldsymbol{y}|)\cdot(\boldsymbol{x}-\boldsymbol{y})/|\boldsymbol{x}-\boldsymbol{y}|]^{-1}$. Then $\delta(\boldsymbol{y}')$ leads to the constraint $0 = \boldsymbol{y} - \boldsymbol{r}(t-|\boldsymbol{x}-\boldsymbol{y}|)$ which has the unique solution $\boldsymbol{y} = \boldsymbol{r}(t_{\mathrm{ret}})$; compare with (2.22). In particular the volume element $\det(\mathrm{d}\boldsymbol{y}/\mathrm{d}\boldsymbol{y}')$ becomes $[1 - \widehat{\boldsymbol{n}}\cdot\boldsymbol{u}(t_{\mathrm{ret}})]^{-1}$ in the limit, with $\widehat{\boldsymbol{n}} = \widehat{\boldsymbol{n}}(\boldsymbol{x},t) = (\boldsymbol{x} - \boldsymbol{r}(t_{\mathrm{ret}}))/|\boldsymbol{x} - \boldsymbol{r}(t_{\mathrm{ret}})|$.

We conclude that

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\boldsymbol{E}(\boldsymbol{x},t) = \bar{\boldsymbol{E}}(\boldsymbol{x},t), \quad \lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon}}\boldsymbol{B}(\boldsymbol{x},t) = \bar{\boldsymbol{B}}(\boldsymbol{x},t), \tag{8.47}$$

where $\bar{\boldsymbol{E}}, \bar{\boldsymbol{B}}$ are the Liénard–Wiechert fields (2.24), (2.25) generated by a point charge moving along the trajectory $t \mapsto \boldsymbol{r}(t)$. The convergence in (8.47) is pointwise, except for the Coulomb singularity at $\boldsymbol{x} = \boldsymbol{r}(t)$.

8.5 Larmor's formula

We want to determine the energy per unit time radiated to infinity and consider, for this purpose, a ball of radius R centered at $\boldsymbol{q}^\varepsilon(t)$. At time $t + R$ the energy in this

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ball is

$$\mathcal{E}_{R,\boldsymbol{q}^\varepsilon(t)}\,(t+R) = \mathcal{E}(0) - \frac{1}{2}\int\limits_{\{|\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t)|\geq R\}} \mathrm{d}^3x\,\big(\boldsymbol{E}(\boldsymbol{x},t+R)^2 + \boldsymbol{B}(\boldsymbol{x},t+R)^2\big) \tag{8.48}$$

using conservation of total energy. The radiation emitted from the charge at time t reaches the surface of the ball at time $t+R$, and the energy loss per unit time is given by

$$\begin{aligned}
I_{R,\varepsilon}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{E}_{R,\boldsymbol{q}^\varepsilon(t)}\\
&= \int \mathrm{d}^3x\,\delta(|\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t)|-R)\,\Big(\frac{1}{2}(\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{v}^\varepsilon(t))\,\big(\boldsymbol{E}(\boldsymbol{x},t+R)^2\\
&\quad + \boldsymbol{B}(\boldsymbol{x},t+R)^2\big) + \boldsymbol{E}(\boldsymbol{x},t+R)\cdot\big(\boldsymbol{n}(\boldsymbol{x})\times\boldsymbol{B}(\boldsymbol{x},t+R)\big)\Big)\\
&= \frac{1}{2}\,R^2\int \mathrm{d}^2\omega\,\Big((\widehat{\boldsymbol{\omega}}\cdot\boldsymbol{v}^\varepsilon(t))\big(\boldsymbol{E}(\boldsymbol{q}^\varepsilon(t)+R\widehat{\boldsymbol{\omega}},t+R)^2\\
&\quad + \boldsymbol{B}(\boldsymbol{q}^\varepsilon(t)+R\widehat{\boldsymbol{\omega}},t+R)^2\big) + 2\boldsymbol{E}(\boldsymbol{q}^\varepsilon(t)+R\widehat{\boldsymbol{\omega}},t+R)\\
&\quad \cdot\big(\widehat{\boldsymbol{\omega}}\times\boldsymbol{B}(\boldsymbol{q}^\varepsilon(t)+R\widehat{\boldsymbol{\omega}},t+R)\big)\Big)\,,
\end{aligned} \tag{8.49}$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the outer normal of the ball and $|\widehat{\boldsymbol{\omega}}|=1$, with $\mathrm{d}^2\omega$ the integration over the unit sphere. Equation (8.49) holds for sufficiently large R, since we used $\{\boldsymbol{x}|\ |\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t)|\geq R\}\cap\{\boldsymbol{x}|\ |\boldsymbol{x}-\boldsymbol{q}^\varepsilon(t+R)|\leq\varepsilon R_\varphi\}=\emptyset$, which is the case for $(1-\bar{v})R\geq\varepsilon R_\varphi$.

Equation (8.49) still contains the reversible energy transport between the considered ball and its complement. To isolate that part of the energy which is irreversibly lost one has to take the limit $R\to\infty$. For this purpose we first partially integrate in (8.45), (8.46) by using the identity

$$\nabla\varphi = \nabla_{\boldsymbol{y}}\varphi - \frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}\left(1+\frac{(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{v}^\varepsilon}{|\boldsymbol{y}-\boldsymbol{x}|}\right)^{-1}(\boldsymbol{v}^\varepsilon\cdot\nabla_{\boldsymbol{y}})\varphi \tag{8.50}$$

at the argument $\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t-|\boldsymbol{y}-\boldsymbol{x}|)$. For large R the fields in (8.49) then become

$$\begin{aligned}
R\boldsymbol{E}(\boldsymbol{q}^\varepsilon(t)+R\widehat{\boldsymbol{\omega}},t+R) &\cong \sqrt{\varepsilon}\int \mathrm{d}^3y\,\frac{e}{4\pi}\varphi_\varepsilon(\boldsymbol{y}-\boldsymbol{q}^\varepsilon)\big[-(1-\widehat{\boldsymbol{\omega}}\cdot\boldsymbol{v}^\varepsilon)^{-1}\dot{\boldsymbol{v}}^\varepsilon\\
&\quad -(1-\widehat{\boldsymbol{\omega}}\cdot\boldsymbol{v}^\varepsilon)^{-2}(\widehat{\boldsymbol{\omega}}\cdot\dot{\boldsymbol{v}}^\varepsilon)(\boldsymbol{v}^\varepsilon-\widehat{\boldsymbol{\omega}})\big]\big|_{t+\widehat{\boldsymbol{\omega}}\cdot(\boldsymbol{y}-\boldsymbol{q}^\varepsilon(t))}\,,
\end{aligned} \tag{8.51}$$

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$$
\begin{aligned}
RB(q^\varepsilon(t) + R\widehat{\omega}, t+R) &\cong \sqrt{\varepsilon}\int d^3y \frac{e}{4\pi}\varphi_\varepsilon(y - q^\varepsilon)\big[-(1-\widehat{\omega}\cdot v^\varepsilon)^{-1}(\widehat{\omega}\times \dot{v}^\varepsilon) \\
&\quad -(1-\widehat{\omega}\cdot v^\varepsilon)^{-2}(\widehat{\omega}\cdot\dot{v}^\varepsilon)(\widehat{\omega}\times v^\varepsilon)\big]\big|_{t+\widehat{\omega}\cdot(y-q^\varepsilon(t))} \\
&= \widehat{\omega}\times RE(q^\varepsilon(t)+R\widehat{\omega}, t+R)\,, \qquad (8.52)
\end{aligned}
$$

where we used the property that $t + R - |q^\varepsilon(t) + R\widehat{\omega} - y| = t + \widehat{\omega}\cdot(y - q^\varepsilon(t)) + \mathcal{O}(1/R)$ for large R. Inserting in (8.49) yields

$$
\begin{aligned}
\lim_{R\to\infty} I_{R,\varepsilon}(t) &= I_\varepsilon(t) \\
&= -\lim_{R\to\infty}\int d^2\omega(1-\widehat{\omega}\cdot v^\varepsilon(t))\big(RE(q^\varepsilon(t)+R\widehat{\omega}, t+R)\big)^2 \qquad (8.53) \\
&= -\varepsilon\int d^2\omega(1-\widehat{\omega}\cdot v^\varepsilon(t)) \\
&\quad \times\Big(\Big[\frac{e}{4\pi}\int d^3y\varphi_\varepsilon(y-q^\varepsilon)(1-\widehat{\omega}\cdot v^\varepsilon)^{-2}(\widehat{\omega}\cdot\dot{v}^\varepsilon)\Big]^2 \\
&\quad - \Big[\frac{e}{4\pi}\int d^3y\varphi_\varepsilon(y-q^\varepsilon)(1-\widehat{\omega}\cdot v^\varepsilon)^{-1}\dot{v}^\varepsilon \\
&\quad + (1-\widehat{\omega}\cdot v^\varepsilon)^{-2}(\widehat{\omega}\cdot\dot{v}^\varepsilon)v^\varepsilon\Big]^2\Big)\Big|_{t+\widehat{\omega}\cdot(y-q^\varepsilon(t))}. \qquad (8.54)
\end{aligned}
$$

$I_\varepsilon(t)$ is the energy radiated per unit time at ε fixed. As argued before, it is indeed of order ε. From the expression (8.53) it can be seen that $I_\varepsilon(t) \le 0$.

Equation (8.54) is not yet Larmor's formula. To obtain it we have to go to the adiabatic limit $\varepsilon \to 0$. Then $q^\varepsilon(t) \to r(t)$. Since $\varphi_\varepsilon(x) \to \delta(x)$, we have $y \cong q^\varepsilon(t) \cong r(t)$ in (8.54). From the d^3y volume element we get an additional factor of $(1 - \widehat{\omega}\cdot v^\varepsilon)^{-1}$. Thus

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\varepsilon^{-1}I_\varepsilon(t) = I(t) &= -e^2\int d^2\omega(1-\widehat{\omega}\cdot u(t))\big(4\pi(1-\widehat{\omega}\cdot u(t))^{-3}\big)^2 \\
&\quad\times\big((\widehat{\omega}\cdot\dot{u}(t))^2 - [(1-\widehat{\omega}\cdot u(t))\dot{u}(t) + (\widehat{\omega}\cdot\dot{u}(t))u(t)]^2\big) \\
&= -(e^2/6\pi)\big[\gamma^4\dot{u}(t)^2 + \gamma^6(u(t)\cdot\dot{u}(t))^2\big] \\
&= -(e^2/6\pi)\gamma^6\big[\dot{u}(t)^2 - (u(t)\times\dot{u}(t))^2\big]\,, \qquad (8.55)
\end{aligned}
$$

which is the standard textbook formula of Larmor. Note that the same energy loss per unit time was obtained already in (8.6) using only the energy balance for the comparison dynamics.

Starting from (8.49) one could alternatively first take the limit $\varepsilon^{-1} I_{R,\varepsilon}(t) \to I_{R,0}(t)$, which is the change of energy in a ball of radius R centered at the particle's position $\boldsymbol{r}(t)$ in the adiabatic limit. As before the irreversible energy loss is isolated through

$$\lim_{R\to\infty} I_{R,0}(t) = I(t)\,. \tag{8.56}$$

The energy loss does not depend on the order of limits, as it should be.

We recall that in Larmor's treatment the trajectory of the charge, taken as a point charge, is prescribed. In our case the charged particle is guided by external fields and interacts with its own Maxwell field, which is physically somewhat more realistic. Since the charge distribution is extended, by necessity, Larmor's formula holds only in the adiabatic approximation.

Notes and references

Section 8

The radiation damped harmonic oscillator is discussed in Jackson (1999) with a variety of physical applications. The asymptotic condition was first stated in Dirac (1938). It has been reemphasized by Haag (1955) in analogy to a similar condition in quantum field theory.

Section 8.1

Singular, or geometric, perturbation theory is a standard tool in the theory of dynamical systems. Sakamoto (1990) presents the theory at the level of generality needed here. We refer to Jones (1995) for a review with many applications. In the context of synergetics (Haken 1983) one talks of slow and fast variables and the slaving principle, which means that fast variables are enslaved by the slow ones. Within our context this would correspond to an attractive critical manifold. The renormalization group flows in critical phenomena have a structure similar to that discovered here: the critical surface corresponds to critical couplings which then flow to some fixed point governing the universal critical behavior. The critical surface is repelling, and slightly away from that surface the trajectory moves towards either the high- or low-temperature fixed points.

Section 8.2

Particular cases have been studied before, most extensively the one-dimensional potential step of finite width and with linear interpolation (Haag 1955; Baylis and

Huschilt 1976; Carati and Galgani 1993; Carati *et al.* 1995; Blanco 1995; Ruf and Srikanth 2000), head-on collision in the two-body problem (Huschilt and Baylis 1976), the motion in a uniform magnetic field (Endres 1993), and motion in an attractive Coulomb potential (Marino 2003). These authors emphasize that there can be several solutions to the asymptotic condition. From the point of view of singular perturbation theory such behavior is generic. If ε is increased, then the critical manifold is strongly deformed and is no longer given as a graph of a function. For specified $\boldsymbol{q}(0)$, $\boldsymbol{v}(0)$ there are then several $\dot{\boldsymbol{v}}(0)$ on $\mathcal{C}_\varepsilon$, which means that the solution to the asymptotic condition is not unique. However, these authors fail to emphasize that the nonuniqueness in the examples occurs only at such high field strengths that a classical theory has long lost its empirical validity. At moderate field strengths the worked-out examples confirm our findings. The general applicability of singular perturbation theory was first recognized in Spohn (1998).

Sections 8.3, 8.4, and 8.5

The discussion is adapted from Kunze and Spohn (2000a, 2000b).