

Addendum to: hodge classes on self-products of a variety with an automorphism

CHAD SCHOEN

*Department of Mathematics, Duke University, Box 90320, Durham, NC 27708-0320 U.S.A.
e-mail: schoen@math.duke.edu*

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Abstract. There are infinitely many fundamentally distinct families of polarized Abelian fourfolds of Weil type with multiplication from the cyclotomic field of cube roots of unity. The Hodge conjecture is shown to hold at a sufficiently general fiber in any of these families.

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Let K be an imaginary quadratic field and let A be a complex Abelian variety. We say that A is of Weil type, if there is a unitary ring homomorphism, $\theta: K \rightarrow \text{End}(A) \otimes \mathbb{Q}$. In this situation A. Weil [We] identified a rational Hodge substructure of the middle dimensional cohomology of A which will be referred to here as the Weil cohomology. A generalized Prym variety for an étale $\mathbb{Z}/3$ -cover of a genus 3 curve is an example of an Abelian variety of Weil type with $K = \mathbb{Q}(\mu_3)$. The paper mentioned in the title establishes that the Hodge conjecture holds for the Weil cohomology in this case. The purpose of this addendum is to extend this result to cover the Weil cohomology of infinitely many modular families of Abelian varieties of Weil type. Precisely, we prove

THEOREM. *Suppose that A is a four dimensional Abelian variety of Weil type for the field $K = \mathbb{Q}(\mu_3)$. Assume that the K^* -module $H^{1,0}(A)$ decomposes as a direct sum, $\psi_1^{\oplus 2} \oplus \psi_2^{\oplus 2}$, where $\psi_i: K \rightarrow \mathbb{C}$ are distinct field homomorphisms. Then the Weil cohomology is generated by the fundamental classes of two dimensional algebraic cycles.*

Abelian varieties which satisfy the hypotheses of the theorem admit polarizations which are compatible with the action of K (see (1.1) and 9 below). A general Abelian variety of this type has only one polarization up to positive scalar multiples. It is possible to distinguish infinitely many families of polarized Abelian varieties which satisfy the hypotheses of the theorem by means of an invariant of the polarization which may take any value in the infinite group $(\mathbb{Q}_+)^* / N_{\mathbb{Q}}^K \mathbb{Q}^*$

(\mathbb{Q}_+ := positive rational numbers). We note that this invariant was overlooked in [Sch]. A hypothesis regarding the polarization should have appeared in the statement of [Sch, Theorem 3.2] since the proof implicitly assumes that the polarization invariant of A coincides with that of the generalized Prym.

The proof of the theorem proceeds in three steps. First, the theory of moduli of Abelian varieties of Weil type is recalled with special emphasis on the polarization invariant. Next the Hodge conjecture for the Weil cohomology of infinitely many families of Weil type Abelian 4-folds is deduced from the assumption that the Hodge conjecture holds for a single family of Weil type Abelian 6-folds. Finally the Hodge conjecture for a family of Weil type Abelian 6-folds is proved using the methods of [Sch, Sect. 3].

I am indebted to the referee for drawing attention to the very relevant article of van Geemen [vG]. One may use [vG, 4.11, 6.12] to deduce the assertion in the abstract from the theorem. In [vG, 7.3] one finds an approach to Sections 11–13 of the present paper based on a calculation of Faber [F, Theorem 3.1] which is closely related to Section 13.

Notation

Weil cohomology = Weil Hodge structure. This is defined in [Sch, p. 24], where it is denoted U' .

$\text{cl}(z)$ denotes the cohomology class of an algebraic cycle, z .

K denotes an arbitrary imaginary quadratic field in Sections 1–10.

1. We begin the discussion of moduli of Weil type Abelian varieties with the notion of a Weil pair of rank g . This consists of a torsion free, rank g module, $V_{\mathbb{Z}}$, over some order, $R \subset K$, and an alternating form

$$\xi: V_{\mathbb{Z}} \times V_{\mathbb{Z}} \rightarrow \mathbb{Z},$$

whose extension to $V_{\mathbb{Q}}$ is non-degenerate and satisfies

$$\xi(kv_1, v_2) = \xi(v_1, \bar{k}v_2), \quad \forall k \in K. \quad (1.1)$$

2. Given $(V_{\mathbb{Z}}, \xi)$ and a homomorphism of fields, $\psi: K \rightarrow \mathbb{C}$, there is an associated invariant $(b, f) \in \{0, 1, 2, \dots, g\} \times (\mathbb{Q}^*/\mathbb{N}_{\mathbb{Q}}^K K^*)$ which satisfies $\text{sign}(f) = (-1)^b$. (Here the sign of f is the sign of an arbitrary lift of f to \mathbb{Q}^* .) To define (b, f) we first fix $\eta \in K$ such that $\psi(\eta)$ is a positive multiple of $i \in \mathbb{C}$ and note that $\phi(\cdot, \cdot) := \xi(\eta \cdot, \cdot) + \eta \xi(\cdot, \cdot)$ is a non-degenerate K/\mathbb{Q} -Hermitian form. Define b to be the dimension of the largest $K \otimes \mathbb{R}$ -subspace of $V_{\mathbb{R}} := V_{\mathbb{Z}} \otimes \mathbb{R}$ on which ϕ is negative definite. Set $f = \text{disc}(\phi)$. Observe that (b, f) remains unchanged when η is replaced by a positive rational multiple.

3. Starting with $(V_{\mathbb{Z}}, \xi)$ we may construct a polarized Abelian variety by choosing an ‘admissible complex structure’ on $V_{\mathbb{R}}$; that is by choosing $J \in \text{End}(V_{\mathbb{R}})$ satisfying

- (1) $J^2 = -\text{Id}$,
- (2) J is $K \otimes \mathbb{R}$ -linear,
- (3) $\xi(J, J) = \xi(\ , \)$,
- (4) $\xi(\ , J)$ is positive definite.

Polarized Abelian varieties constructed in this way are called Abelian varieties of Weil type.

4. LEMMA

- (1) *The admissible complex structures on $V_{\mathbb{R}}$ are parametrized by b -dimensional $K \otimes \mathbb{R}$ -subspaces of $V_{\mathbb{R}}$ on which ϕ is negative definite. These in turn are in bijective correspondence with the points of the connected manifold $\mathbf{J} := SU(V, \phi)/S(U(a) \times U(b))$.*
- (2) *The multiplicity with which the character ψ appears in the representation of K^* on the global holomorphic 1 forms of the Abelian variety $(V_{\mathbb{R}}/V_{\mathbb{Z}}, J)$ is $g - b$.*
- (3) *the Hodge level of the Weil cohomology is $|g - 2b|$.*
- (4) *The invariant (b, f) is independent of the choice of field homomorphism $\psi: K \rightarrow \mathbb{C}$ exactly when $g = 2b$.*

Proof. (1) If J is admissible, it is diagonalizable over $K \otimes \mathbb{R}$ and we write $T^{\pm} \subset V_{\mathbb{R}}$ for the subspace where J acts by multiplication by the scalar $\pm\eta \otimes |\eta|^{-1}$. By 3(3) $T^- = (T^+)^{\perp}$ and by 3(4) ϕ is negative definite on T^+ and positive definite on T^- . Conversely, given any b -dimensional $K \otimes \mathbb{R}$ -subspace $T \subset V_{\mathbb{R}}$ on which ϕ is negative definite, define

$$J := \eta \otimes |\eta|^{-1} \cdot \text{Id}|_T \oplus -\eta \otimes |\eta|^{-1} \cdot \text{Id}|_{T^{\perp}}.$$

Then J satisfies 3(1-4). For the second assertion note that the stabilizer of T in $SU(V, \phi)$ is a maximal compact subgroup isomorphic to $S(U(a) \times U(b))$. The action of $SU(V, \phi)$ on the set of all T 's as above is transitive because maximal compact subgroups are conjugate [H–N].

(2) The vector space of global holomorphic 1 forms is canonically identified with $\text{Hom}_{\mathbb{C}}((V_{\mathbb{R}}, J), \mathbb{C})$. The subspace where K^* acts by the character ψ is $\text{Hom}_{\mathbb{C}}(T^{\perp}, \mathbb{C})$.

(3) This follows directly from the explicit basis for the Weil cohomology given in [Sch, p. 24].

(4) Changing ψ amounts to replacing ϕ by $-\phi$ which changes (b, f) to $(g - b, (-1)^g f)$.

5. The manifold \mathbf{J} has a unique complex structure such that the data $(V_{\mathbb{Z}}, \xi, J)$ is a polarized variation of Hodge structure [De, 1.1.14(i)]. This gives rise to a structure of complex manifold on $\mathbf{J} \times V_{\mathbb{R}}$. The action of $V_{\mathbb{Z}}$ by translation is holomorphic.

The quotient is a holomorphic family of Abelian varieties over \mathbf{J} which we refer to as the ‘universal’ family of Weil Abelian varieties associated to the Weil pair $(V_{\mathbb{Z}}, \xi)$.

6. Given two Weil pairs $(V_{\mathbb{Z}}, \xi)$, $(V'_{\mathbb{Z}}, \xi')$ and an isometry of the associated Hermitian inner product spaces $\iota: (V_{\mathbb{Q}}, \phi) \rightarrow (V'_{\mathbb{Q}}, \phi')$, there is an integer $N \neq 0$ such that $N\iota(V_{\mathbb{Z}}) \subset V'_{\mathbb{Z}}$. The map,

$$\mathbf{J} \times V_{\mathbb{R}} \rightarrow \mathbf{J}' \times V'_{\mathbb{R}}, \quad (J, v) \rightarrow (\iota J \iota^{-1}, N\iota v),$$

gives rise to an isogeny between the associated ‘universal’ families. Pulling back with respect to this isogeny gives an isomorphism on Weil cohomology.

7. Two g -dimensional Hermitian inner product spaces $(V_{\mathbb{Q}}, \phi)$ and $(V'_{\mathbb{Q}}, \phi')$ are isometric exactly when the signatures and discriminants of ϕ and ϕ' coincide [De-M, p. 50]. Furthermore, given $(b, f) \in \{0, 1, 2, \dots, g\} \times \mathbb{Q}^*/\mathbb{N}_{\mathbb{Q}}^K K^*$ satisfying $\text{sign}(f) = (-1)^b$, there exists a non-degenerate Hermitian inner product space $(V_{\mathbb{Q}}, \phi)$ with signature $(g - b, b)$ and discriminant f [De-M, p. 50].

8. Let $(V_{\mathbb{Z}}, \xi)$ be a Weil pair of rank g . By replacing ξ by $a\xi$ with a a positive integer, we change the discriminant invariant from f to $a^g f$. If g is even, this operation has no effect on f . However if g is odd, any element $h \in \mathbb{Q}^*/\mathbb{N}_{\mathbb{Q}}^K K^*$ with $\text{sign}(h) = \text{sign}(f)$ has the form $a^g f$ for an appropriate choice of a .

9. The Abelian variety A in the statement of the theorem may be given a polarization ξ_A which satisfies (1.1). In fact, if ξ_0 is any polarization, choose a non-zero integer M such that $M\eta \subset R$ and define

$$\xi_A(v_1, v_2) = M^2 |\eta|^2 \xi_0(v_1, v_2) + \xi_0(M\eta v_1, M\eta v_2).$$

The invariant of the Weil pair $(H_1(A, \mathbb{Z}), \xi_A)$ has the form $(2, f_A)$ for some $f_A \in \mathbb{Q}^*/\mathbb{N}_{\mathbb{Q}}^K K^*$ with $\text{sign}(f_A) = 1$ by 4(2).

10. PROPOSITION. *Let (A, ξ_A) be a Weil type Abelian four-fold with polarization invariant $(2, f_A)$ for some $f_A \in \mathbb{Q}^*/\mathbb{N}_{\mathbb{Q}}^K K^*$ with $\text{sign}(f_A) = 1$. Fix $f \in \mathbb{Q}^*/\mathbb{N}_{\mathbb{Q}}^K K^*$ with $\text{sign}(f) = -1$ and a Weil pair of rank 6, $(V_{\mathbb{Z}}, \xi)$, with invariant $(3, f)$. Suppose that for each Abelian variety in the associated ‘universal’ family, the Weil cohomology is generated by classes of codimension 3 algebraic cycles. Then the Weil cohomology of A , W_A , is generated by classes of codimension 2 algebraic cycles.*

Proof. Choose $f' \in \mathbb{Q}^*/\mathbb{N}_{\mathbb{Q}}^K K^*$ such that $f' f_A = f$. By 7 there exists a Weil pair of rank 2, $(V'_{\mathbb{Z}}, \xi')$ with invariant $(1, f')$. The choice of an admissible complex

structure $J' \in \text{End}(V_{\mathbb{R}}')$ gives rise to a Weil type Abelian surface A' , whose Weil cohomology, $W_{A'}$, has Hodge type $(1, 1)$. $W_{A'}$ is generated by cohomology classes of divisors. The Weil pair associated to the product $A \times A'$ is $(H_1(A, \mathbb{Z}) \times V_{\mathbb{Z}}', \xi_A \oplus \xi')$. By 7 there is an isometry of Hermitian inner product spaces

$$\iota: (H_1(A, \mathbb{Q}) \oplus V_{\mathbb{Q}}', \phi_A \oplus \phi') \rightarrow (V_{\mathbb{Q}}, \phi)$$

which leads to an isogeny of Abelian varieties of Weil type

$$A \times A' \xrightarrow{N\iota} (V_{\mathbb{R}}/V_{\mathbb{Z}}, J)$$

for an appropriate choice of admissible complex structure J . It follows from 6 and the hypotheses of the proposition that the Weil cohomology, $W_{A \times A'}$, is generated by cohomology classes of algebraic cycles. We claim that W_A is generated by cohomology classes of algebraic cycles of the form $\text{pr}_{A*}(z \cdot (A \times D))$ where $\text{cl}(z) \in W_{A \times A'}$ and $\text{cl}(D) \in W_{A'}$. To check this, it is only necessary to show that the cup product in the following diagram is surjective:

$$W_{A \times A'} \otimes (H^0(A) \otimes W_{A'}) \xrightarrow{\cup} W_A \otimes H^4(A') \xrightarrow{\text{pr}_{A*}} W_A.$$

This follows from an explicit computation with differential forms. Using the notation of [Sch, p. 24], we write down bases for the following vector spaces:

$$\begin{aligned} H^1(A, \mathbb{C})^{\psi_i} &: \{ \omega_{1, \psi_i}, \dots, \omega_{4, \psi_i} \}, & i \in \{1, 2\} \\ H^1(A', \mathbb{C})^{\psi_i} &: \{ \omega_{5, \psi_i}, \omega_{6, \psi_i} \}, & i \in \{1, 2\} \\ H^1(A \times A', \mathbb{C})^{\psi_i} &: \{ \omega_{1, \psi_i}, \dots, \omega_{6, \psi_i} \}, & i \in \{1, 2\} \\ W_A \otimes_{\mathbb{Q}} \mathbb{C} &: \{ \omega_{1, \psi_i} \wedge \dots \wedge \omega_{4, \psi_i} \}_{1 \leq i \leq 2}, \\ W_{A'} \otimes_{\mathbb{Q}} \mathbb{C} &: \{ \omega_{5, \psi_i} \wedge \omega_{6, \psi_i} \}_{1 \leq i \leq 2}, \\ W_{A \times A'} \otimes_{\mathbb{Q}} \mathbb{C} &: \{ \omega_{1, \psi_i} \wedge \dots \wedge \omega_{6, \psi_i} \}_{1 \leq i \leq 2}. \end{aligned}$$

Since $\omega_{5, \psi_1} \wedge \omega_{6, \psi_1} \wedge \omega_{5, \psi_2} \wedge \omega_{6, \psi_2}$ is a basis for $H^4(A', \mathbb{C})$ the desired surjectivity is clear.

11. In order to prove the theorem it remains only to construct the appropriate family of 6 dimensional Abelian varieties and to show that the Weil cohomology is generated by classes of algebraic cycles. This is done by applying the methods of [Sch, Sect. 3] to a family of generalized Prym varieties which we now construct.

Let \mathcal{M} be the moduli space of smooth genus 4 curves. There is an étale morphism $\varsigma: \mathcal{N} \rightarrow \mathcal{M}$ and a family of genus 4 curves $f: \mathcal{X} \rightarrow \mathcal{N}$ satisfying:

- (1) the moduli map associated to f is ς ;
- (2) f has a section;

- (3) there is a finite étale morphism $\rho: \mathcal{C} \rightarrow \mathcal{X}$, which is Galois with group $\mathbb{Z}/3$ and whose fibers over closed points $m \in \mathcal{N}$ are (irreducible) genus 10 curves.

Recall that the generalized Prym, \mathcal{B} , for \mathcal{C}/\mathcal{X} is defined to be the connected component of the identity in the kernel of the norm map

$$\mathrm{Pic}^0(\mathcal{C}/\mathcal{N}) \rightarrow \mathrm{Pic}^0(\mathcal{X}/\mathcal{N}).$$

After replacing \mathcal{N} by a further étale cover if necessary, we may assume that

- (4) all the 5-torsion in $\mathrm{Pic}^0(\mathcal{X}/\mathcal{N})$ and in \mathcal{B} is rational over \mathcal{N} .

The group $\mathrm{Gal}(\mathcal{C}/\mathcal{X})$ acts on \mathcal{B}/\mathcal{N} and respects the polarization which is pulled back from the canonical polarization on $\mathrm{Pic}^0(\mathcal{C}/\mathcal{N})$. Thus \mathcal{B} is a family of Abelian varieties of Weil type. The Weil cohomology of each fiber is generated by classes of codimension 3 algebraic cycles [Sch, 3.1].

12. Associated to the integral homology of this family is a Weil pair $(V_{\mathbb{Z}}, \xi)$ of rank 6. The invariant is $(3, f)$ for some $f \in \mathbb{Q}^*/\mathbb{N}_{\mathbb{Q}}^K K^*$ with $\mathrm{sign}(f) = -1$ [Sch, 1.6a]. Let ϕ denote the Hermitian form (uniquely defined up to scalar multiplication by \mathbb{Q}^*) which was associated to $(V_{\mathbb{Z}}, \xi)$ in **2**. Define

$$\Gamma := \mathrm{GL}(V_{\mathbb{Z}}) \cap \mathrm{SU}(V_{\mathbb{Q}}, \phi), \quad \Gamma_1 := \mathrm{Ker}[\Gamma \rightarrow \mathrm{GL}(V_{\mathbb{Z}}/5V_{\mathbb{Z}})].$$

Now Γ_1 acts freely on \mathbf{J} and the quotient is canonically a smooth quasi-projective variety [Ba–Bo]. The holomorphic family of polarized Abelian varieties

$$\Gamma_1 \backslash V_{\mathbb{Z}} \backslash \mathbf{J} \times V_{\mathbb{R}} \rightarrow \Gamma_1 \backslash \mathbf{J} := \mathcal{W} \tag{12.1}$$

is canonically a projective morphism [Chai, 2.4]. Fix a $\mathbb{Z}[\mu_3]/5$ -isomorphism between the 5-torsion sections of \mathcal{B}/\mathcal{N} and $\frac{1}{5}V_{\mathbb{Z}}/V_{\mathbb{Z}}$ which respects the symplectic pairings. Then there is a holomorphic period map $\varphi: \mathcal{N} \rightarrow \mathcal{W}$ such that \mathcal{B} is pulled back from (12.1). This map comes from a morphism of varieties [Bo, Sect. 10]. To show that the Weil cohomology of every fiber of the ‘universal’ family, (12.1) is generated by classes of algebraic cycles, we need only verify that φ is dominant and apply the specialization argument given in [Sch, p. 30].

13. LEMMA. φ is dominant.

Proof. The lemma will follow if the tangent map $T_m\varphi$ can be shown to be surjective at one point $m \in \mathcal{N}$. Because the deformation functors have been rigidified by the imposition of level structure, the tangent spaces have their usual cohomological descriptions. In fact, if $C \rightarrow X$ is the $\mathbb{Z}/3$ -cover corresponding to m , the map dual to $T_m\varphi$ may be identified with the multiplication map [Sch, p. 29–30]

$$H^0(C, \omega_C)^{\chi} \otimes H^0(C, \omega_C)^{\chi^{-1}} \rightarrow H^0(C, \omega_C^{\otimes 2})^{\mathrm{Gal}(C/X)}, \tag{13.1}$$

where $\chi: \text{Gal}(C/X) \rightarrow \mathbb{C}^*$ is a non-trivial character.

We will check that (13.1) is injective in the special case that C is the maximal Abelian, exponent 3 cover of \mathbb{P}^1 which is unramified outside the set of four distinct points $\{a_1, a_2, a_3, \infty\}$. The function field of this genus 10 curve is given by

$$\mathbb{C}(x)[t_1, t_2, t_3]/(t_1^3 - (x - a_1), t_2^3 - (x - a_2), t_3^3 - (x - a_3)).$$

Clearly $\text{Gal}(C/\mathbb{P}^1) \simeq (\mathbb{Z}/3)^3$ and

$$\gamma_i(t_j) = \begin{cases} \exp(2\pi i/3)t_j, & \text{if } j = i \\ t_j, & \text{if } j \neq i, \end{cases} \quad i \in \{1, 2, 3\}$$

is a minimal set of generators. For X we take the quotient of C by the subgroup generated by $\gamma_1\gamma_2$. The inertia groups above the four branch points of C/\mathbb{P}^1 are easily seen to be generated by $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_0 := (\gamma_1\gamma_2\gamma_3)^{-1}$. Since $\gamma_1\gamma_2$ is not contained in any of these inertia groups, C/X is unramified.

The map (13.1) is $\text{Gal}(C/\mathbb{P}^1)$ -equivariant. The left hand side may be decomposed into irreducible $\text{Gal}(C/\mathbb{P}^1)$ -modules with the help of the Chevalley–Weil formula [C–W]. This says that the multiplicity with which a non-trivial character $\kappa: \text{Gal}(C/\mathbb{P}^1) \rightarrow \mathbb{C}^*$ appears in the representation $H^0(C, \omega_C)$ is given by

$$\nu_\kappa = -1 + \sum_{j=0}^3 \left\langle \frac{-\alpha_j}{3} \right\rangle,$$

where $\kappa(\gamma_j) = \exp(2\pi i\alpha_j/3)$ and $\langle u \rangle = u - [u]$ is the fractional part of u . It is now straightforward to check that the left hand side of (13.1) decomposes as a sum of one dimensional spaces

$$\bigoplus_{1 \leq i, j \leq 3} H^0(C, \omega_C)^{\kappa_i} \otimes H^0(C, \omega_C)^{\kappa'_j}, \tag{13.2}$$

with the property that the nine characters $\{\kappa_i \cdot \kappa'_j\}_{1 \leq i, j \leq 3}$ are distinct. Since each of the factors in the tensor product (13.2) is one dimensional, the multiplication map (13.1) restricted to each summand in (13.2) is injective. The distinctness of the characters now implies that (13.1) itself is injective.

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