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COMPACTNESS OF THE FLUCTUATIONS ASSOCIATED WITH SOME GENERALIZED NONLINEAR BOLTZMANN EQUATIONS

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ABSTRACT. In this paper, we develop a new approach to obtain the compactness of the fluctuation processes for Boltzmann dynamics. Our method is applicable to Kac's model, already studied by Uchiyama, but it covers many other cases. A novelty worth mentioning is the use of the weak topology of a Hilbert space.

1. **Introduction.** This work is concerned with the fluctuation problem for generalized nonlinear Boltzmann equations. A first successful work in this sense has been done in a very clever way by Uchiyama [27] for Kac's caricature. To get the fluctuation tightness, Uchiyama's main idea was to look for a Hilbert space whose dual contains the fluctuation and to control the norm of the fluctuations using some Fourier transform bounds.

Later, Ferland [7, 8] observed that these bounds were also applicable to the Boltzmann energy equations which are scaling invariant. Unfortunately, it seems that the same approach can not work for the non scaling invariant case.

In this work we modify Uchiyama's approach to cover this last case. To avoid Fourier transforms we directly control the norm of the fluctuation by choosing a more appropriate Hilbert space. After that, instead of considering a Hilbert triple like Uchiyama, we put on the state space the *weak* topology and we obtain the relative compactness using a general criterion for processes with values in a Lusin space.

We will first present our approach for Kac's caricature. Then we will apply it to Boltzmann energy equations both scaling and non scaling invariant showing as such that our method is more generally applicable.

2. **The general set-up.** We are studying real generalized Boltzmann equations with bounded interaction intensity. The interaction between two independent particles is described by a Markov kernel $Q\colon \mathbb{R} \times \mathbb{R} \times \mathcal{B}(\mathbb{R} \times \mathbb{R}) \longrightarrow [0,1]$. The latter is assumed to be *symmetric* in the sense that $Q(x,y\ ;\ A \times B) = Q(y,x\ ;\ B \times A)$ for every $x,y \in \mathbb{R}$ and $A,B \in \mathcal{B}(\mathbb{R})$. The generalized Boltzmann equation associated with Q is given by

$$\begin{cases} \frac{d}{dt}\langle u(t), \phi\rangle = \int_{\mathbb{R}} (A_t\phi)(x)u(t,dx) \\ u(0) = \mu \end{cases}$$

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where

$$(A_t\phi)(x) = \int_{\mathbb{R}} \Big[\int_{\mathbb{R}\times\mathbb{R}} \{\phi(x^\star) - \phi(x)\} Q(x, y\,;\, dx^\star, dy^\star) \Big] u(t, dy),$$

ϕ is a bounded measurable function and $\langle u(t), \phi\rangle$ is the integral of ϕ with respect to the probability measure $u(t)$. Because the interaction intensity is bounded, this equation has a unique solution (see [2, 17, 22, 29] for details).

The n-particles interacting process associated with Q is the Markov pure jump process $\big(X^n(t)\big)_{t\geq 0}$ regulated by the infinitesimal generator:

$$G_n f(x_1, \ldots, x_n) = \frac{1}{n} \sum_{1\leq i<j\leq n} (Q^{i,j}f - f)(x_1, \ldots, x_n)$$

where

$$Q^{i,j}f(x_1, \ldots, x_n) = \int_{\mathbb{R}} \int_{\mathbb{R}} f^{i,j}(x^*, y^*) Q(x_i, x_j\,;\, dx^*\, dy^*)$$

and $f^{i,j}(x^*, y^*)$ is obtained from f by changing the variables x_i and x_j with x^* and y^*. All the processes $\{X^n\}_{n=2}^{\infty}$ are supposed to be defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the paths are right continuous with left hand limits. The probability law induced on $\mathbb{R}^n$ by $X^n(t)$ is denoted by $u_n(t)$. Throughout this work $u_n(0)$ is assumed to be permutation invariant. Since G_n commutes with the permutation of coordinates, the symmetry of $u_n(0)$ is inherited by $u_n(t)$.

The link between $\{X^n\}_{n=2}^{\infty}$ and $\big(u(t)\big)_{t\geq 0}$ is given by the *chaos propagation property* which states that the empirical measures

$$\alpha_t^n = \frac{1}{n} \sum_{j=1}^{n} \delta_{X_j^n(t)}$$

converge weakly to $u(t)$ provided α_0^n converge to $u(0)$ [7, 12, 15, 25, 26].

The *scaled fluctuation* of α_t^n about $u(t)$ is $\eta_t^n = \sqrt{n}\big(\alpha_t^n - u(t)\big)$. The process $(\eta_t^n)_{t\geq 0}$ is a measure-valued temporally inhomogeneous Markov process. We name it the (n-particles) fluctuation process. This work is concerned with the compactness of the fluctuation processes sequence. (See [4, 5, 13, 18, 19, 21, 23, 24, 28] for related works.)

3. **Kac's caricature.**

3.1 *The state space.* From the definition of η_t^n, we see that if the fluctuation processes are to converge in law, the limiting process might not always be measure-valued. It means that we have to weaken the notion of convergence and thereby enlarge the state space.

Let $\mathcal{K}$ be the set of all bounded continuous functions with a bounded continuous derivative in $\mathbf{L}^2(\mathbb{R})$. We define on $\mathcal{K}$ a semi-norm N by

$$N(\phi) = \Big(\int_{\mathbb{R}} \phi'(v)^2\, dv \Big)^{1/2}.$$

This semi-norm is Hilbertian and the corresponding Hilbert space is noted $\mathcal{H}$. The letter N is used to denote both the norm in $\mathcal{H}$ and the dual norm in $\mathcal{H}'$.

Recall that for Kac's caricature the kernel Q is given by

$$Q(v, w\ ; B) = \int_0^{2\pi} 1_B(v\cos\theta - w\sin\theta, v\sin\theta + w\cos\theta)\,\overline{d}\theta$$

where $\overline{d}\theta = d\theta/2\pi$. Since this kernel obviously preserves the energy, it is natural to take $u(0)$ with a finite variance so that the mean energy $\frac{1}{2}\int v^2 u(t, dv)$ is finite at any time. Under this condition on the moments, we now show that the fluctuation belongs to $\mathcal{H}'$. To this end, let us introduce, for each x, a functional L_x on $\mathcal{K}$ given by the formula $L_x\phi = \phi(x) - \phi(0)$. Schwarz inequality immediately gives the following lemma.

LEMMA A. *L_x is a continuous linear functional on $\mathcal{K}$ of norm at most $|x|^{1/2}$.*

COROLLARY. *$(\eta_t^n)_{t\geq 0}$ is an $\mathcal{H}'$-valued random process.*

PROOF. For any fixed ω, n and t, the fluctuation $\eta_t^n(\omega)$ is in $\mathcal{H}'$ because it is a multiple of the difference of two probability measures with finite moment of order 2 (and therefore also of order 1/2). Furthermore, $\langle \eta_t^n, \phi\rangle$ is a real random variable for all ϕ in $\mathcal{K}$. Since $\mathcal{K}$ generates the Borel σ-field of $\mathcal{H}'$, this is enough to get the conclusion. ∎

We conclude this section with another lemma. We shall use it in the next section to control the norm of the fluctuation, and that is the reason why we chose $\mathcal{H}'$ as the state space for the fluctuation. For any given probability measure μ, the kernel Q defines a natural application $\mathcal{L}(\mu)$ on $\mathcal{K}$ by the formula

$$\mathcal{L}(\mu)\phi(v) = \int_{\mathbb{R}} \mu(dw) \int_0^{2\pi} \varepsilon^\theta\phi(v, w)\,\overline{d}\theta$$

where $\varepsilon^\theta\phi(v, w) = \phi(v\cos\theta - w\sin\theta) + \phi(v\sin\theta + w\cos\theta) - \phi(v) - \phi(w)$.

LEMMA B. *$\mathcal{L}(\mu)$ is a continuous linear operator on $\mathcal{K}$.*

PROOF. It is not hard to see that when ϕ is in $\mathcal{K}$, the function $\mathcal{L}(\mu)\phi$ is bounded, continuous, and has a bounded continuous derivative. Moreover, we have

$$\begin{aligned} N\big(\mathcal{L}(\mu)\phi\big)^2 &= \int_{\mathbb{R}} \Big(\frac{d}{dv}\mathcal{L}(\mu)\phi(v)\Big)^2\,dv \\ &\leq \int_{\mathbb{R}} dv \int_{\mathbb{R}} \mu(dw) \int_0^{2\pi} \Big|\frac{\partial}{\partial v}\varepsilon^\theta\phi(v, w)\Big|^2\,\overline{d}\theta \\ &\leq \int_{\mathbb{R}} \int_0^{2\pi} N\big(\varepsilon^\theta\phi(\cdot, w)\big)^2 \mu(dw)\,\overline{d}\theta. \end{aligned}$$

A simple calculation gives $N\big(\varepsilon^\theta\phi(\cdot, w)\big)^2 \leq 9N(\phi)^2$. The proof is complete. ∎

REMARK. Lemma B is also true if μ is a signed measure instead. The constant 9 would have to be changed to $9|\mu|$ where $|\mu|$ is the total variation norm of μ.

3.2 *Preliminary results.* Let us introduce two conditions for $\{u_n(0)\}_{n=2}^\infty$:

$$\text{(H0)} \qquad \sup_n \mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(0)|^2\Big] < \infty\,;$$

$$\text{(H1)} \qquad \sup_n \mathbf{E}[N(\eta_0^n)^2] < \infty.$$

The condition (H0) will be used to control the quadratic parts of the sequence of fluctuation process generators. To understand the role of (H1), let us note that if the processes $\{\eta^n\}_{n=2}^{\infty}$ are relatively compact, the same must be true for the random variables $\{\eta_0^n\}_{n=2}^{\infty}$. But with the help of (H1), one can choose for any $\varepsilon > 0$ a constant M_ε large enough to get $\mathbb{P}\{N(\eta_0^n) > M_\varepsilon\} < \varepsilon$. Since the set $\{\eta \in \mathcal{H}' \mid N(\eta) \leq M_\varepsilon\}$ is weakly compact and the Borel σ-field of $\mathcal{H}'$ is generated by its weak topology, we see that, under (H1), the sequence $\{\eta_0^n\}_{n=2}^{\infty}$ is relatively compact. It is the purpose of this work to prove that the conditions (H0) and (H1) are sufficient to get the relative compactness of the fluctuation processes sequence.

In order to prove the previous statement, the main step will be to show that the finiteness of (H0) and (H1) propagates in time. This is true for (H0) as a consequence of the conservation of energy. But the case of (H1) is much harder and, in fact, the space $\mathcal{H}$ has been chosen precisely to make this task feasable.

Before we proceed any further, we need to introduce some ancillary quantities. Define for any ϕ in $\mathcal{K}$,

$$M_t^n(\phi) = \langle \eta_t^n, \phi\rangle - \int_0^t \mathcal{A}_s^n(\phi)\, ds$$
$$S_t^n(\phi) = M_t^n(\phi)^2 - \int_0^t \mathcal{Q}_s^n(\phi)\, ds$$

where

$$\mathcal{A}_t^n(\phi) = \langle \eta_t^n \otimes u(t), \Lambda_1\phi\rangle + \frac{1}{2\sqrt{n}}\{\langle \eta_t^n \otimes \eta_t^n, \Lambda_1\phi\rangle - \langle \alpha_t^n, \Gamma_1\phi\rangle\}$$
$$\mathcal{Q}_t^n(\phi) = \frac{1}{2}\Big\{\langle \alpha_t^n \otimes \alpha_t^n, \Lambda_2\phi\rangle - \frac{1}{n}\langle \alpha_t^n, \Gamma_2\phi\rangle\Big\}$$
$$\Lambda_k\phi(v,w) = \int_0^{2\pi} \{\varepsilon^\theta \phi(v,w)\}^k\, \bar{d}\theta$$

and $\Gamma_k\phi(v) = \Lambda_k\phi(v,v)$. These quantities are related to the fluctuation process generator and for more information we refer the reader to Uchiyama [27].

We know that the space $\mathcal{H}$ is separable and that a complete orthonormal set $\{\phi_k\}_{k=1}^{\infty}$ can be found in $\mathcal{K}$. We will use this basis to express the dual norm of the fluctuation. For any ϕ_k we have

$$\begin{aligned}\mathbf{E}[\langle \eta_t^n, \phi_k\rangle^2] &\leq 2\mathbf{E}[M_t^n(\phi_k)^2] + 2\mathbf{E}\Big[\Big(\int_0^t \mathcal{A}_s^n(\phi_k)\, ds\Big)^2\Big] \\ &= 2\mathbf{E}[S_t^n(\phi_k)] + 2\mathbf{E}\Big[\int_0^t \mathcal{Q}_s^n(\phi_k)\, ds\Big] + 2\mathbf{E}\Big[\Big(\int_0^t \mathcal{A}_s^n(\phi_k)\, ds\Big)^2\Big] \\ &\leq 2\mathbf{E}[S_t^n(\phi_k)] + 2\int_0^t \mathbf{E}[\mathcal{Q}_s^n(\phi_k)]\, ds + 2t\int_0^t \mathbf{E}[\mathcal{A}_s^n(\phi_k)^2]\, ds.\end{aligned}$$

Since $\big(S_t^n(\phi_k)\big)_{t\geq 0}$ is a martingale (see [6, 27]), we also have

$$\mathbf{E}[S_t^n(\phi_k)] = \mathbf{E}[S_0^n(\phi_k)] = \mathbf{E}[|\langle \eta_0^n, \phi_k\rangle|^2].$$

Summing over k we get that $\mathbf{E}[N^2(\eta_t^n)]$ is bounded by

$$\mathbf{E}[N^2(\eta_0^n)] + 2\int_0^t \sum_{k\geq 1} \mathbf{E}[\mathcal{Q}_s^n(\phi_k)]\,ds + 2t\int_0^t \sum_{k\geq 1} \mathbf{E}[\mathcal{A}_s^n(\phi_k)^2]\,ds.$$

Upper-bounds for the last two terms are given in Lemmas 3.1 and 3.2 which are based on Lemmas A and B.

LEMMA 3.1. *For all $t \geq 0$ we have that*

$$\sum_{k\geq 1} \mathbf{E}[\mathcal{Q}_t^n(\phi_k)] \leq 6\mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(0)|^2\Big]^{1/2}.$$

PROOF. When v, w and θ are fixed, we may use Schwarz inequality to show that $\varepsilon^\theta\phi(v,w)$ is a continuous linear functional on $\mathcal{K}$ and the square of its norm in $\mathcal{H}'$ is bounded by $6(|v|+|w|)$. Parseval's identity gives

$$\sum_{k\geq 1} |\varepsilon^\theta\phi_k(v,w)|^2 \leq 6(|v|+|w|)$$

and therefore the following inequality holds:

$$\sum_{k\geq 1} \Lambda_2\phi_k(v,w) = \int_0^{2\pi} \sum_{k\geq 1} |\varepsilon^\theta\phi_k(v,w)|^2\,\overline{d}\theta \leq 6(|v|+|w|).$$

On the other hand, $\mathcal{Q}_t^n(\phi)$ is positive and bounded by $\langle \alpha_t^n \otimes \alpha_t^n, \Lambda_2\phi\rangle/2$. Then we can write:

$$\begin{aligned}
\sum_{k\geq 1} \mathbf{E}[\mathcal{Q}_t^n(\phi_k)] &\leq \frac{1}{2}\sum_{k\geq 1} \mathbf{E}[\langle \alpha_t^n \otimes \alpha_t^n, \Lambda_2\phi_k\rangle] \\
&\leq \frac{1}{2}\mathbf{E}\Big[\Big\langle \alpha_t^n \otimes \alpha_t^n, \sum_{k\geq 1}\Lambda_2\phi_k\Big\rangle\Big] \\
&\leq \frac{1}{2n^2}\sum_{i,j=1}^n \mathbf{E}\Big[\sum_{k\geq 1}\Lambda_2\phi_k\big(X_i^n(t), X_j^n(t)\big)\Big] \\
&\leq \frac{3}{n^2}\sum_{i,j=1}^n \mathbf{E}[|X_i^n(t)| + |X_j^n(t)|] \\
&\leq 6\mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(t)|\Big] \\
&\leq 6\mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(t)|^2\Big]^{1/2} = 6\mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(0)|^2\Big]^{1/2}.
\end{aligned}$$

The last equality follows from preservation of energy. ∎

LEMMA 3.2. *For all $t \geq 0$ we have that*

$$\sum_{k\geq 1} \mathbf{E}[\mathcal{A}_t^n(\phi_k)^2] \leq 36\mathbf{E}[N(\eta_t^n)^2] + \frac{12}{n}\mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(0)|^2\Big]^{1/2}.$$

PROOF. From the definition of $\mathcal{A}_t^n(\phi)$ we have that

$$\mathcal{A}_t^n(\phi)^2 \leq 2\langle \eta_t^n \otimes u(t), \Lambda_1\phi\rangle^2 + \frac{1}{n}\langle \eta_t^n \otimes \eta_t^n, \Lambda_1\phi\rangle^2 + \frac{1}{n}\langle \alpha_t^n, \Gamma_1\phi\rangle^2.$$

But $\langle \eta_t^n \otimes u(t), \Lambda_1\phi\rangle$ is linear in ϕ and, because of Lemma B, we see that it is continuous. Moreover,

$$|\langle \eta_t^n \otimes u(t), \Lambda_1\phi\rangle| = \left|\langle \eta_t^n, \mathcal{L}\big(u(t)\big)\phi\rangle\right| \leq N(\eta_t^n)N\Big(\mathcal{L}\big(u(t)\big)\phi\Big) \leq 3N(\eta_t^n)N(\phi).$$

Consequently,

$$\sum_{k\geq 1}\langle \eta_t^n \otimes u(t), \Lambda_1\phi_k\rangle^2 \leq 9N(\eta_t^n)^2$$

since the series is just the square of the norm in $\mathcal{H}'$ of the random functional. By the same token,

$$\sum_{k\geq 1}\langle \eta_t^n \otimes \eta_t^n, \Lambda_1\phi_k\rangle^2 \leq 18nN(\eta_t^n)^2.$$

Combining these inequalities we get:

$$\sum_{k\geq 1}\mathbf{E}[\mathcal{A}_t^n(\phi_k)^2] \leq 36\mathbf{E}[N(\eta_t^n)^2] + \frac{1}{n}\mathbf{E}\Big[\sum_{k\geq 1}\langle \alpha_t^n, \Gamma_1\phi_k\rangle^2\Big].$$

Finally, we obtain

$$\begin{aligned}
\mathbf{E}\Big[\sum_{k\geq 1}\langle \alpha_t^n, \Gamma_1\phi_k\rangle^2\Big] &\leq \mathbf{E}\Big[\Big\langle \alpha_t^n, \sum_{k\geq 1}\Gamma_2\phi_k\Big\rangle\Big] \\
&\leq \mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n\sum_{k\geq 1}\Gamma_2\phi_k\big(X_j^n(t)\big)\Big] \\
&\leq 12\mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(t)|\Big] \\
&\leq 12\mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(0)|^2\Big]^{1/2}. \qquad \blacksquare
\end{aligned}$$

We may now prove that the finiteness of (H1) propagates.

PROPOSITION 3.3. *Let us suppose that the conditions* (H0) *and* (H1) *hold. Then there exists a nondecreasing function K_t such that, for all $t \geq 0$,*

$$\sup_n \mathbf{E}[N(\eta_t^n)^2] \leq K_t.$$

PROOF. Set $y^n(t) = \mathbf{E}[N(\eta_t^n)^2]$. Lemmas 3.1 and 3.2 give

$$y^n(t) \leq 2y^n(0) + 12\Big(t + \frac{2t^2}{n}\Big)\mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(0)|^2\Big]^{1/2} + 72t\int_0^t y^n(s)\,ds.$$

By Gronwall's lemma we have

$$y^n(t) \le \Big\{2y^n(0) + 12\Big(t + \frac{2t^2}{n}\Big)\mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^{n}|X_j^n(0)|^2\Big]^{1/2}\Big\}\exp(72t^2),$$

which proves the proposition. ∎

3.3 *Compactness of the fluctuation processes.* In this section, we look at the fluctuation processes on a fixed compact interval $[0,T]$. We first prove that the fluctuation trajectories are almost surely strongly cadlag in $\mathcal{H}'$. Then we show, using recent results in [9], that the fluctuation processes sequence is relatively compact when $\mathcal{H}'$ is endowed with its weak topology. Finally, we obtain that any limiting process is strongly continuous.

PROPOSITION 3.4. $\sup_n \sum_{k\ge 1} \mathbf{E}[\sup_{0\le t\le T}\langle \eta_t^n, \phi_k\rangle^2] < \infty.$

PROOF. Let $M_t^n(\phi)$ be as in the previous section. Then $\big(M_t^n(\phi)\big)_{t\ge 0}$ is a martingale [6]. A martingale inequality gives $\mathbf{E}[\sup_{0\le t\le T} M_t^n(\phi)^2] \le 4\mathbf{E}[M_T^n(\phi)^2]$. Therefore, we can write

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\le t\le T}\langle \eta_t^n, \phi_k\rangle^2\Big] &\le 2\mathbf{E}\Big[\sup_{0\le t\le T} M_t^n(\phi_k)^2\Big] + 2\mathbf{E}\Big[\Big(\int_0^T \mathcal{A}_s^n(\phi_k)\,ds\Big)^2\Big]\\ &\le 16\mathbf{E}[\langle \eta_T^n, \phi_k\rangle^2] + 18T\int_0^T \mathbf{E}[\mathcal{A}_s^n(\phi_k)^2]\,ds.\end{aligned}$$

It means that $\sup_n \sum_{k\ge 1}\mathbf{E}[\sup_{0\le t\le T}\langle \eta_t^n, \phi_k\rangle^2]$ is bounded by

$$16\sup_n \mathbf{E}[N(\eta_T^n)^2] + 18T\int_0^T \sup_n \mathbf{E}\Big[\sum_{k\ge 1}\mathcal{A}_s^n(\phi_k)^2\Big]\,ds.$$

The first term is finite by Proposition 3.3. The second is also finite because the integrand is bounded on $[0,T]$ by Lemma 3.2 and Proposition 3.3. ∎

PROPOSITION 3.5. *The trajectories of the fluctuation processes are almost surely strongly cadlag in $\mathcal{H}'$.*

PROOF. Using Proposition 3.4, one can find a set Ω_o such that $\mathbf{P}(\Omega_o) = 1$, and

$$\forall \omega \in \Omega_o, \quad \sum_{k\ge 1}\sup_{0\le t\le T}\langle \eta_t^n(\omega), \phi_k\rangle^2 < \infty.$$

Fix $\omega \in \Omega_o$ and let $t_m \downarrow t$ in $[0,T]$. We must show that, for any $\varepsilon > 0$, there exists $m(\varepsilon)$ such that

$$m \ge m(\varepsilon) \Rightarrow N\big(\eta_{t_m}^n(\omega) - \eta_t^n(\omega)\big) < \varepsilon.$$

To do so, first choose N large enough to get

$$\sum_{k>N}\sup_{0\le t\le T}\langle \eta_t^n(\omega), \phi_k\rangle^2 < \varepsilon^2/6$$

then take $m(k,\varepsilon)$ such that $m \geq m(k,\varepsilon) \Rightarrow |\langle \eta^n_{t_m}, \phi_k\rangle - \langle \eta^n_t, \phi_k\rangle|^2 < \varepsilon^2/3N$. This is possible since $t \longmapsto \langle \eta^n_t(\omega), \phi_k\rangle$ is right-continuous. Let $m(\varepsilon) = \max_{1\leq k\leq N} m(k,\varepsilon)$. Then, for $m \geq m(\varepsilon)$, we have

$$\begin{aligned} N\big(\eta^n_{t_m}(\omega) - \eta^n_t(\omega)\big)^2 &= \sum_{k\geq 1} \langle \eta^n_{t_m}(\omega) - \eta^n_t(\omega), \phi_k\rangle^2 \\ &\leq \sum_{k=1}^{N} \langle \eta^n_{t_m}(\omega) - \eta^n_t(\omega), \phi_k\rangle^2 + 2\sum_{k>N}\{\langle \eta^n_{t_m}(\omega), \phi_k\rangle^2 + \langle \eta^n_t(\omega), \phi_k\rangle^2\} \\ &< \sum_{k=1}^{N} \varepsilon^2/3N + 2\varepsilon^2/6 + 2\varepsilon^2/6 = \varepsilon^2. \end{aligned}$$

Thus the function $t \longmapsto \eta^n_t(\omega)$ is right-continuous. Now, if $t_m \uparrow t$, a similar argument shows that $\{\eta^n_{t_m}(\omega)\}_{m=1}^\infty$ is Cauchy in $\mathcal{H}'$. So $t \longmapsto \eta^n_t(\omega)$ has left-hand limits and the proof is complete. ∎

When the space $\mathcal{H}'$ is endowed with its *weak* topology it is no longer Polish but Lusin instead. However, because $\mathcal{H}'$ is then the weak dual of a separable Fréchet space, the space $\mathcal{D}([0,T],\mathcal{H}')$ (with the Skohorod topology, when we take the *weak* topology of $\mathcal{H}'$) turns out to be a Lusin space also [9, Théorème 3.2.1]. Any probability measure on this path space is tight. This enables us to use a general compactness criterion for processes with values in a Lusin space (see [9, Théorème 4.4]). In our context, it says that the fluctuation processes sequence is relatively compact provided that the two following conditions are true:

a) There exists a sequence $(K_m)_{m\geq 1}$ of weakly compact subsets of $\mathcal{H}'$ such that

$$\forall m \geq 1,\ \forall n \geq 2, \quad \mathbf{P}\{\exists t \in [0,T] \mid \eta^n_t \notin K_m\} \leq 2^{-m}.$$

b) For all $\phi \in \mathcal{K}$, the real processes $\{\langle \eta^n, \phi\rangle\}_{n=2}^\infty$ are relatively compact.

But Proposition 3.4 shows immediately that

$$M = \sup_n \mathbf{E}\Big[\sup_{0\leq t\leq T} N(\eta^n_t)^2\Big] < \infty \tag{3.1}$$

and therefore, the sets $K_m = \{\eta \in \mathcal{H}' \mid N(\eta)^2 \leq M2^m\}$ give the desired sequence for condition a). Condition b) follows from the next proposition [3, Theorem 15.5].

PROPOSITION 3.6. *For any $\phi \in \mathcal{K}$ we have*

$$\lim_{M\uparrow\infty} \sup_n \mathbf{P}\Big\{\sup_{0\leq t\leq T} |\langle \eta^n_t, \phi\rangle| > M\Big\} = 0 \tag{3.2}$$

and for all $\varepsilon > 0$ one can find $\delta > 0$ and $N \geq 2$ such that

$$\sup_{n\geq N} \mathbf{P}\Big\{\sup_{\substack{s,t\in[0,T] \\ |t-s|<\delta}} |\langle \eta^n_t, \phi\rangle - \langle \eta^n_s, \phi\rangle| \geq \varepsilon\Big\} \leq \varepsilon. \tag{3.3}$$

PROOF. Again, (3.2) is an easy consequence of (3.1). Now, in order to show (3.3), we are defining for each function f in $\mathcal{D}([0,T],\mathbb{R})$ a "modulus" V'' as follows :

$$V''(f,\delta) = \sup\{|f(t) - f(r)| \wedge |f(r) - f(s)|\ ;\ 0 \leq s \leq r \leq t \leq T, t - s < \delta\}.$$

It is well-known [20, Lemma 6.4] that

$$\sup_{\substack{s,t\in[0,T]\\ |t-s|<\delta}} |f(t)-f(s)| \leq 2V''(f,\delta) + \sup_{0\leq t\leq T} |f(t)-f(t-)|.$$

Thus, in order to prove (3.3), all we have to do is to choose N_o such that $\frac{4}{\sqrt{n}}\|\phi\|_\infty < \varepsilon/2$ whenever $n \geq N_0$, and then show that

$$\lim_{\delta\downarrow 0}\sup_{n\geq N_0} \mathbf{P}\{V''(\langle \eta^n_\cdot, \phi\rangle, \delta) > \varepsilon/4\} = 0. \tag{3.4}$$

This suffices since from the fact that the probability that more than two components of $X^n(t)$ change at the same time is zero, we have

$$\mathbf{P}\Big\{\sup_{0\leq t\leq T} |\langle \eta^n_t, \phi\rangle - \langle \eta^n_{t-}, \phi\rangle| \leq \frac{4}{\sqrt{n}}\|\phi\|_\infty\Big\} = 1.$$

Let $\tau^n_M = \inf\{t \geq 0\,;\, |\mathcal{A}^n_t(\phi)| > M\}$ and $Y^n_t = \langle \eta^n_{t\wedge\tau^n_M}, \phi\rangle$. We have that

$$\lim_{M\uparrow\infty}\sup_n \mathbf{P}\{\tau^n_M \leq T\} = 0 \tag{3.5}$$

since for $\phi \in \mathcal{K}$

$$\sup_n \mathbf{E}\Big[\sup_{0\leq t\leq T} |\mathcal{A}^n_t(\phi)|\Big] < \infty. \tag{3.6}$$

Indeed, the boundedness of ϕ implies that $|\Gamma_1\phi| \leq 4\|\phi\|_\infty$ and

$$|\langle \alpha^n_t, \Gamma_1\phi\rangle| \leq 4\|\phi\|_\infty.$$

Moreover, in the proof of Lemma 3.2, we noted that

$$|\langle \eta^n_t \otimes u(t), \Lambda_1\phi\rangle| \leq 3N(\eta^n_t)N(\phi)$$

and

$$|\langle \eta^n_t \otimes \eta^n_t, \Lambda_1\phi\rangle| \leq \sqrt{18n}N(\eta^n_t)N(\phi).$$

Since $|\mathcal{A}^n_t(\phi)|$ is bounded by

$$|\langle \eta^n_t \otimes u(t), \Lambda_1\phi\rangle| + \frac{1}{2\sqrt{n}}|\langle \eta^n_t \otimes \eta^n_t, \Lambda_1\phi\rangle| + \frac{1}{2\sqrt{n}}|\langle \alpha^n_t, \Gamma_1\phi\rangle|$$

the result (3.6) follows easily from Proposition 3.3. Furthermore, since τ^n_M is a stopping time, the processes

$$M^n_t = Y^n_t - \int_0^{t\wedge\tau^n_M} \mathcal{A}^n_s(\phi)\,ds$$

$$S^n_t = (M^n_t)^2 - \int_0^{t\wedge\tau^n_M} Q^n_s(\phi)\,ds$$

are martingales; hence, it is routine to see that:

$$\sup_n \mathbf{E}[(Y^n_t - Y^n_r)^2(Y^n_r - Y^n_s)^2] \leq \text{constant} \times (t-s)^2$$

for $0 \leq s \leq r \leq t \leq T$. This implies (see [3, Theorem 15.6]) that:

$$\lim_{\delta\downarrow 0}\sup_{n\geq N_o} \mathbf{P}\{V''(Y^n, \delta) > \varepsilon/4\} = 0$$

which is enough to get (3.4) in view of (3.5). ∎

THEOREM 3.7. *Under* (H0) *and* (H1) *the fluctuation processes are relatively compact on* $\mathcal{D}([0,T],\mathcal{H}')$ *and any limiting process has strongly continuous paths.*

PROOF. Only the last part need to be proved. Let $\mathbb{P}$ be the law on $\mathcal{D}([0,T],\mathcal{H}')$ of any limiting process and let $\mathbb{E}$ stand for the corresponding expectation. By Proposition 3.4, we have $\mathbb{E}[\sum_{k\geq 1}\sup_{0\leq t\leq T}\langle N_t,\phi_k\rangle^2] < \infty$, where N_t is the canonical projection at time t. Moreover, by Proposition 3.6, $t\longmapsto\langle N_t,\phi_k\rangle$ is continuous on a $\mathbb{P}$-null set, for any k. This is enough to show the continuity exactly as in the proof of Proposition 3.5. ∎

4. **Boltzmann energy equations.** Boltzmann energy equations model the time evolution of the energy distribution for spatially homogeneous isotropic systems of identical particles [1, 14]. The energy distribution is a probability measure on $\mathbb{R}_+$ or $\mathbb{N}$ depending on whether the energy is supposed continuous or quantified.

4.1 *The scaling invariant case.* Boltzmann energy equations on $\mathbb{R}_+$ are usually scaling invariant. By this we mean that the kernel Q may be written as

$$Q(x,y\,;B) = \int_0^1\int_0^1 1_B\big(z_1x+z_2y,(1-z_1)x+(1-z_2)y\big)\nu(dz_1,dz_2)$$

where $x,y\in\mathbb{R}_+$, $B\in\mathcal{B}(\mathbb{R}_+\times\mathbb{R}_+)$, and ν is a probability measure on $[0,1]\times[0,1]$. The name "scaling invariant" comes from the fact that ν does not depend on x and y. A famous example of such a kernel is given by the so-called continuous p-q model of Futcher and Hoare [10,11].

It is clear that scaling invariant kernels carry a strong resemblance with Kac's caricature kernel. As a consequence, we will get Theorem 3.7 for those kernels in a very similar fashion. Indeed, we take the same Hilbert space but on $\mathbb{R}_+$ instead. Obviously, Lemma A is still true. Since the probability measure ν is usually such that the energy is preserved, the solution of the Boltzmann energy equation will have its first moment finite at any time provided that it is true initially. Under this condition, the fluctuation processes are $\mathcal{H}'$-valued processes, as before.

On the other hand, it is not very hard to see that Lemma B is also true. Essentially, we just look at the previous proof and replace everywhere the uniform measure on $[0,2\pi]$ by ν and the function $\varepsilon^\theta\phi(x,y)$ by the function

$$\varepsilon^{z_1,z_2}\phi(x,y) = \phi(z_1x+z_2y)+\phi\big((1-z_1)x+(1-z_2)y\big)-\phi(x)-\phi(y).$$

Therefore, if we assume that the following conditions hold:

$$\text{(H0)}\qquad \sup_n \mathbf{E}\Big[\frac{1}{n}\sum_{j=1}^n |X_j^n(0)|\Big] < \infty$$

$$\text{(H1)}\qquad \sup_n \mathbf{E}[N(\eta_0^n)^2] < \infty$$

we can easily get the estimations of Section 3.2. Then, Theorem 3.7 for scaling invariant Boltzmann energy equations will follow.

4.2 *The non scaling invariant case.* Boltzmann energy equations on $\mathbb{N}$ are not scaling invariant because the kernels act on the integers. Nevertheless, we want to indicate that our approach still works. In view of what we said above, all we have to do is to find a discrete analogue for $\mathcal{K}$ and prove Lemma A and B.

Consider the space

$$\mathcal{K} = \Big\{\phi \colon \mathbb{N} \longmapsto \mathbb{R} \Bigm| \sum_{k\geq 0} |\phi(k+1) - \phi(k)|^2 < \infty\Big\}$$

with the hilbertian semi-norm

$$N(\phi) = \Big(\sum_{k\geq 0} |\phi(k+1) - \phi(k)|^2\Big)^{1/2}.$$

Again, Lemma A is true and proved in the same way. The problem is with Lemma B. We need some conditions on the kernel Q. Recall that when the energy is quantified, the kernel is on $\mathbb{N} \times \mathbb{N} \times \mathcal{P}(\mathbb{N} \times \mathbb{N})$ and the symmetry property reduces to $Q\big(i,j\ ;\ \{(\ell, i+j-\ell)\}\big) = Q\big(j,i\ ;\ \{(i+j-\ell, \ell)\}\big)$. As before, we will suppose (and this is natural from a physical point of view) that energy is conserved, that is to say $Q\big(i,j\ ;\ \{(0,i+j),(1,i+j-1),\ldots,(i+j,0)\}\big) = 1$. Additional conditions will be given in the statement of the lemma.

In what follows we will simply write $Q(i,j\ ;\ \ell)$ for $Q\big(i,j\ ;\ \{(\ell, i+j-\ell)\}\big)$ and for any function ϕ in $\mathcal{K}$ we put

$$\Lambda\phi(i,j) = \sum_{\ell\geq 0} \{\phi(\ell) - \phi(i) + \phi(i+j-\ell) - \phi(j)\} Q(i,j\ ;\ \ell).$$

LEMMA B. *Suppose that for all i, j and ℓ we have that:*

a) $\sum_{k=0}^{\ell} Q(i,j\ ;\ k) \geq \sum_{k=0}^{\ell} Q(i+1,j\ ;\ k)$, *and*

b) $Q(i,j\ ;\ \ell) = Q(i,j\ ;\ i+j-\ell)$.

Then, for any probability measure μ on $\mathbb{N}$, the mapping

$$\mathcal{L}(\mu) \colon \phi \longmapsto \sum_{j\geq 0} \Lambda\phi(\cdot,j)\mu(j)$$

is a continuous linear operator on $\mathcal{K}$.

PROOF. $\mathcal{L}(\mu)$ is clearly linear. We will show that $N\big(\mathcal{L}(\mu)\phi\big)^2 \leq 6N(\phi)^2$ which means that $\mathcal{L}(\mu)$ is both well-defined and continuous. We have the inequalities:

$$\begin{aligned}
N^2\big(\mathcal{L}(\mu)\phi\big) &= \sum_{i\geq 0} |\mathcal{L}(\mu)\phi(i+1) - \mathcal{L}(\mu)\phi(i)|^2 \\
&= \sum_{i\geq 0} \Big|\sum_{j\geq 0} \mu(j)\big(\Lambda\phi(i+1,j) - \Lambda\phi(i,j)\big)\Big|^2 \\
&\leq \sum_{i\geq 0} \Big[\sum_{j\geq 0} \mu(j)|\Lambda\phi(i+1,j) - \Lambda\phi(i,j))|\Big]^2 \\
&\leq \sum_{i\geq 0}\sum_{j\geq 0} \mu(j)|\Lambda\phi(i+1,j) - \Lambda\phi(i,j))|^2 \\
&\leq \sum_{j\geq 0} \mu(j) N\big(\Lambda\phi(\cdot,j)\big)^2.
\end{aligned}$$

We now prove that $N\big(\Lambda\phi(\cdot,j)\big)^2 \leq 6N(\phi)^2$. To this end, let us fix i and j for the moment and look at the difference:

$$
\begin{aligned}
|\Lambda\phi(i+1,j) - \Lambda\phi(i,j)| \\
&\leq \Big|\sum_{\ell=0}^{i+1+j} \phi(\ell)Q(i+1,j\,;\,\ell) - \sum_{\ell=0}^{i+j} \phi(\ell)Q(i,j\,;\,\ell)\Big| \\
&\quad + \Big|\sum_{\ell=0}^{i+1+j} \phi(i+1+j-\ell)Q(i+1,j\,;\,\ell) - \sum_{\ell=0}^{i+j} \phi(i+j-\ell)Q(i,j\,;\,\ell)\Big| \\
&\quad + \big|\phi(i+1) - \phi(i)\big|.
\end{aligned}
$$

For the first term of the right hand side we can write:

$$
\begin{aligned}
\sum_{\ell=0}^{i+1+j} \phi(\ell)Q(i+1,j\,;\,\ell) - \sum_{\ell=0}^{i+j}\phi(\ell)Q(i,j\,;\,\ell) \\
&= \sum_{\ell=0}^{i+j}[\phi(\ell+1) - \phi(\ell)]\Big[\sum_{k=0}^{\ell} Q(i,j\,;\,k) - \sum_{k=0}^{\ell} Q(i+1,j\,;\,k)\Big].
\end{aligned}
$$

Because of a), the term

$$
\tau_{i,j}^{Q}(\ell) = \sum_{k=0}^{\ell} Q(i,j\,;\,k) - \sum_{k=0}^{\ell} Q(i+1,j\,;\,k)
$$

is positive for any $\ell = 0, 1, \ldots, i+j$. This implies that $\tau_{i,j}^{Q} = \sum_{\ell=0}^{i+j} \tau_{i,j}^{Q}(\ell)$ is precisely equal to the difference between the first moment of $Q(i+1,j,\cdot)$ and that of $Q(i,j,\cdot)$. Moreover, because of b), the kernel $Q(i,j,\cdot)$ is symmetric on $\{0,1,\ldots,i+j\}$ and therefore, $\tau_{i,j}^{Q} = 1/2$. Thus, we can write:

$$
\begin{aligned}
\sum_{i\geq 0}\Big|\sum_{\ell=0}^{i+j+1} \phi(\ell)Q(i+1,j\,;\,\ell) - \sum_{\ell=0}^{i+j} \phi(\ell)Q(i,j\,;\,\ell)\Big|^2 \\
&= \sum_{i\geq 0}\Big|\sum_{\ell=0}^{i+j}[\phi(\ell+1) - \phi(\ell)]\tau_{i,j}^{Q}(\ell)\Big|^2 \\
&\leq \sum_{i\geq 0}(\tau_{i,j}^{Q})^2\Big|\sum_{\ell=0}^{i+j}[\phi(\ell+1) - \phi(\ell)]\tau_{i,j}^{Q}(\ell)/\tau_{i,j}^{Q}\Big|^2 \\
&\leq \sum_{i\geq 0}(\tau_{i,j}^{Q})^2 \sum_{\ell=0}^{i+j} |\phi(\ell+1) - \phi(\ell)|^2\tau_{i,j}^{Q}(\ell)/\tau_{i,j}^{Q} \\
&\leq \frac{1}{2}\sum_{i\geq 0}\sum_{\ell=0}^{i+j} |\phi(\ell+1) - \phi(\ell)|^2\tau_{i,j}^{Q}(\ell) \\
&\leq \frac{1}{2}\sum_{\ell\geq 0} |\phi(\ell+1) - \phi(\ell)|^2 \sum_{i\geq 0}\tau_{i,j}^{Q}(\ell) \\
&\leq \frac{1}{2}N(\phi)^2.
\end{aligned}
$$

Because of b), a similar argument gives

$$\left|\sum_{\ell=0}^{i+1+j} \phi(i+1+j-\ell)Q(i+1,j\,;\,\ell) - \sum_{\ell=0}^{i+j} \phi(i+j-\ell)Q(i,j\,;\,\ell)\right| \leq \frac{1}{2}N(\phi)^2$$

and finally we have:

$$\begin{aligned} N\big(\Lambda\phi(\cdot,j)\big)^2 &= \sum_{i\geq 0} |\Lambda\phi(i+1,j) - \Lambda\phi(i,j)|^2 \\ &\leq 3\Big(\frac{1}{2}N(\phi)^2 + \frac{1}{2}N(\phi)^2 + N(\phi)^2\Big) \\ &= 6N(\phi)^2. \end{aligned}$$

The proof is complete. ∎

REMARK. Our approach works also for the discrete p-q model [7, 11]. (The conclusion of Lemma B is still true since the the negative hypergeometric laws can be described by the Polya urn scheme.)

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