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MAXIMAL SUBSETS OF PAIRWISE NONCOMMUTING ELEMENTS OF THREE-DIMENSIONAL GENERAL LINEAR GROUPS

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Abstract

Let G be a group. A subset N of G is a set of pairwise noncommuting elements if $xy \neq yx$ for any two distinct elements x and y in N. If $|N| \geq |M|$ for any other set of pairwise noncommuting elements M in G, then N is said to be a maximal subset of pairwise noncommuting elements. In this paper we determine the cardinality of a maximal subset of pairwise noncommuting elements in a three-dimensional general linear group. Moreover, we show how to modify a given maximal subset of pairwise noncommuting elements into another maximal subset of pairwise noncommuting elements that contains a given 'generating element' from each maximal torus.

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1. Introduction

Let G be a nonabelian group and $Z(G)$ be its centre. We call a subset N of G a *set of pairwise noncommuting elements* if $xy \neq yx$ for any distinct elements x, y in N. If $|N| \geq |M|$ for any other subset of pairwise noncommuting elements M in G, then N is said to be a *maximal subset of pairwise noncommuting elements*. The cardinality of such a subset is denoted by $\omega(G)$. By a famous result of Neumann [9] in answer to a question posed by P. Erdős, the finiteness of $\omega(G)$ in G is equivalent to the finiteness of the factor group $G/Z(G)$. Mason [8] has shown that any finite group G can be covered by at most $[|G|/2] + 1$ abelian subgroups, so we also have $\omega(G) \leq [|G|/2] + 1$. Moreover, $\omega(G)$ is also related to the index of the centre of G: as Pyber [10] has shown, there is some constant c such that $|G : Z(G)| \leq c^{\omega(G)}$. For a prime number p, a finite p-group G is called extra-special if the centre, the Frattini subgroup and the derived subgroup of G all coincide and are cyclic of order p. The cardinalities of maximal subsets of pairwise noncommuting elements of extra-special p-groups are important as they provide combinatorial information which can be used to calculate their cohomology lengths. (The cohomology length of a nonelementary abelian p-group is a cohomology invariant defined as a result of a theorem of Serre [11].) Chin [4] has obtained upper and lower bounds for $\omega(G)$ for extra-special p-groups G, for odd prime numbers p. For $p = 2$, it has been shown by Isaacs (see [2, p. 40]) that $\omega(G) = 2n + 1$ for any extra-special group of order 2^{2n+1}. Also, in [1, Lemma 4.4], it was proved that $\omega(GL(2, q)) = q^2 + q + 1$. In this paper we determine $\omega(GL(3, q))$.

THEOREM 1.1.

$$\omega(GL(3,q)) = \begin{cases} q^6 + q^5 + 3q^4 + 3q^3 + q^2 - q - 1 & \text{if } q \geq 4, \\ 1067 & \text{if } q = 3, \\ 57 & \text{if } q = 2. \end{cases}$$

We believe that a similar result may hold for higher dimensions.

CONJECTURE 1.2. *Let $G = GL(n, q)$, where $q = p^k \geq 4$ and $q \geq n + 1$. Then*

$$\omega(G) \geq q^{2\binom{n}{2}} + \frac{|G|}{q(q-1)^n} + \frac{|G|}{q^{\binom{n}{2}}(q-1)^2}.$$

We show in Section 2 that each maximal subset of pairwise noncommuting elements N of $G = GL(3, q)$ can be modified to contain a given generalized Singer generator or pseudo Singer generator for each maximal torus (see Definition 2.5). This information, together with information about the p-singular elements in N, leads to our determination of $\omega(G)$. We use the usual notation: for example, $C_G(a)$ is the centralizer of an element a in a group G, $N_G(H)$ is the normalizer of a subgroup H in G, $GL(n, q)$ is the general linear group of dimension n over a finite field of order q, and S_n is the symmetric group of degree n.

2. Pairwise noncommuting elements of $GL(3, q)$

In this section we construct a large subset of pairwise noncommuting elements in $GL(3, q)$. For this purpose we introduce Singer generators and pseudo Singer generator elements.

2.1. An exchange lemma In this subsection we first show a connection between subsets of pairwise noncommuting elements and abelian centralizers, and then determine $\omega(GL(3, q))$ for $q \leq 3$.

LEMMA 2.1 (Exchange lemma). *Let N be a set of pairwise noncommuting elements of a group G, and let $g \in G$ be such that $C_G(g)$ is abelian. Then either $N \cup \{g\}$ is a set of pairwise noncommuting elements, or there is an element $x \in N \cap C_G(g)$ such that $(N \setminus \{x\}) \cup \{g\}$ is a set of pairwise noncommuting elements.*

PROOF. Since $C_G(g)$ is abelian, $|N \cap C_G(g)| \leq 1$. If $N \cap C_G(g) = \emptyset$ then, for each $x \in N$, $xg \neq gx$. Thus $N \cup \{g\}$ is a set of pairwise noncommuting elements. So, let $x \in N \cap C_G(g)$. We show that $(N \setminus \{x\}) \cup \{g\}$ is a set of pairwise noncommuting

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elements. Suppose that a, b are distinct elements of $(N \setminus \{x\}) \cup \{g\}$ such that $ab = ba$. Since $N \setminus \{x\}$ consists of pairwise noncommuting elements, we can assume that $a \in N \setminus \{x\}$ and $b = g$. It follows that $a \in C_G(g)$. Thus $N \cap C_G(g)$ contains both a and x, which is a contradiction. □

We note some simple facts about $\omega(G)$ without proof.

LEMMA 2.2. *Let G be a finite group. Then:*

(i) *for any subgroup H of G, $\omega(H) \leq \omega(G)$;*
(ii) *for any normal subgroup N of G, $\omega(G/N) \leq \omega(G)$.*

Next we compute $\omega(GL(3, q))$ for $q = 2, 3$.

LEMMA 2.3.

$$\omega(GL(3, q)) = \begin{cases} 57 & \text{if } q = 2, \\ 1067 & \text{if } q = 3. \end{cases}$$

PROOF. We have $GL(3, 2) \cong PSL(2, 7)$ and, by [1, Lemma 4.4], $\omega(PSL(2, 7)) = 57$. Let $G = GL(3, 3)$. A computation using *GAP* [5] shows that the set of orders of elements of G is $\{1, 2, 3, 4, 6, 8, 13, 26\}$ and if $A = \{C_G(g) \mid g \in G, |C_G(g)| = 12\}$, $B = \{C_G(g) \mid g \in G, |C_G(g)| = 16\}$, $C = \{C_G(g) \mid g \in G, |C_G(g)| = 18\}$ and $D = \{C_G(g) \mid g \in G, \ |C_G(g)| = 26\}$, then $|A| = 468$, $|B| = 351$, $|C| = 104$ and $|D| = 144$. It follows that there exist elements $a_i, b_j, c_k, d_l \in G$ such that $|C_G(a_i)| = 12$ for $1 \leq i \leq 468$, $|C_G(b_j)| = 16$ for $1 \leq j \leq 351$, $|C_G(c_k)| = 18$ for $1 \leq k \leq 104$ and $|C_G(d_l)| = 26$ for $1 \leq l \leq 144$. Set $X = \{a_i, b_j, c_k, d_l \mid 1 \leq i \leq 468, 1 \leq j \leq 351, 1 \leq k \leq 104, 1 \leq l \leq 144\}$. Now each subgroup in $A \cup B \cup C \cup D$ is abelian and $G = \bigcup_{x \in X} C_G(x)$. We show that X is a subset of pairwise noncommuting elements and $\omega(G) = |X|$. Let $x, y \in X$ and $x \neq y$ such that $xy = yx$. Then $x \in C_G(y)$. Since $C_G(y)$ is abelian, it follows that $C_G(y) \subseteq C_G(x)$. Similarly, $C_G(x) \subseteq C_G(y)$. Hence $C_G(x) = C_G(y)$, a contradiction. Thus X is a subset of pairwise noncommuting elements and hence $|X| \leq \omega(G)$. On the other hand, suppose N is a set of pairwise noncommuting elements of G of size $\omega(G)$. Then $N \subseteq G = \bigcup_{x \in X} C_G(x)$. For each $a \in X$, $C_G(a)$ is abelian, and hence, $|N \cap C_G(a)| \leq 1$. It follows that $\omega(G) \leq |X|$. This completes the proof. □

2.2. An audit of the elements of $GL(3, q)$ For larger q we generalize the approach used for the proof when $q = 3$. By considering the actions of elements of $GL(3, q)$ on $V = V(3, q)$, we see that there are five conjugacy classes of abelian element centralizers in $GL(3, q)$.

Let $g \in GL(3, q)$ and $V = \bigoplus_f V_f$ be a primary decomposition of V as $F\langle g\rangle$-module, where the sum is over all monic irreducible polynomials $f \in F[t]$ (see [6, Theorem 7.1 and Lemma 8.10]). Thus each V_f is g-invariant and if $V_f \neq 0$ then the restriction $g\,|_{V_f}$ to V_f has characteristic polynomial f^{a_f} for some $a_f \geq 1$. We enumerate the possibilities:

(i) g is irreducible, $V = V_f$, where $\deg f = 3$, $a_f = 1$.
(ii) $V = V_{f_1} \oplus V_{f_2}$, where $\deg f_1 = 1$, $\deg f_2 = 2$ and $a_{f_1} = a_{f_2} = 1$.
(iii) $V = V_{f_1} \oplus V_{f_2} \oplus V_{f_3}$, where $\deg f_i = 1 = a_{f_i}$ for $i = 1, 2, 3$; in this case $q \geq 4$.
(iv) $V = V_{f_1} \oplus V_{f_2}$, where $\deg f_i = 1$ for $i = 1, 2$, $a_{f_1} = 1$ and $a_{f_2} = 2$. Thus $f_1(t) = t - \mu$ and $f_2(t) = t - \lambda$, where $\lambda \neq \mu$. There are two possible actions of g on V, namely g is conjugate to one of the matrices

$$A_1 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

where $\lambda \neq \mu$, $\lambda \neq 0$ and $\mu \neq 0$:

(a) g is conjugate to A_1 and $C_{GL(3,q)}(g)$ is abelian of order $q(q-1)^2$ consisting of all matrices of the form

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

with $\alpha \neq 0$ and $\gamma \neq 0$.

(b) g is conjugate to A_2 and $C_{GL(3,q)}(g) \cong GL(2, q) \times GL(1, q)$ is nonabelian of order $q(q^2 - 1)(q - 1)^2$; moreover, each of these elements g centralizes an element of type (ii).

(v) $\deg f = 1$, $a_f = 3$ and $f(t) = t - \lambda$, for some $\lambda \neq 0$. There are three possible actions of g on V, namely g is conjugate to one of the matrices

$$B_1 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad B_2 = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad B_3 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}: \tag{2.1}$$

(a) g is conjugate to B_1, and $C_{GL(3,q)}(g)$ is abelian of order $q^2(q - 1)$ (see Lemma 4.6).

(b) g is conjugate to B_2, and $C_{GL(3,q)}(g)$ is nonabelian of order $q^3(q - 1)^2$; moreover, each of these elements centralizes an element of type iv(a), for example B_2 centralizes the matrix A_2.

(c) g is (conjugate to) B_3 with nonabelian centralizer $GL(3, q)$; in particular, g centralizes every element of $GL(3, q)$.

LEMMA 2.4. *Let $G = GL(3, q)$ and $I = \{\text{i, ii, iii, iv(a), v(a)}\}$, and, for $\kappa \in I$, let $S(\kappa) = \{C_{GL(3,q)}(g) \mid g \text{ of type } \kappa\}$. Then:*

(a) $G = \bigcup_{\kappa \in I} (\bigcup_{X \in S(\kappa)} X)$;

(b) $\omega(G) \leq \sum_{\kappa \in I} |S(\kappa)|$.

PROOF. Part (a) follows from the discussion above. Let N be a maximal subset of pairwise noncommuting elements of G, so $\omega(G) = |N|$. Let $X \in \bigcup_{\kappa \in I} S(\kappa)$. Since X is abelian, $|N \cap X| \leq 1$ and hence $\omega(G) = |N| \leq \sum_{\kappa \in I} |S(\kappa)|$. □

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2.3. Generalized Singer elements in general linear groups Every element in $GL(3, q)$ has one of the forms as listed in Section 2.2. In this section we introduce Singer generators and pseudo Singer generators, and prove that their centralizers are abelian.

DEFINITION 2.5.

(a) Let $g \in GL(n, q)$ where $q = p^k$, p is prime, and $|g| = q^n - 1$. Then $\langle g \rangle$ is called a *Singer cycle subgroup* of $GL(n, q)$.

(b) Let V be a vector space over a finite field F of dimension 3 and let $\underline{\mathbf{n}} = (n_1, \ldots, n_k)$ be (3), (1, 2) or (1, 1, 1). We call $V = V_{n_1} \oplus \cdots \oplus V_{n_k}$ an $\underline{\mathbf{n}}$-*decomposition* if, for $i = 1, 2, \ldots, k$, V_{n_i} is a subspace of V of dimension n_i.

(c) An element g of $GL(3, q)$ is called an $\underline{\mathbf{n}}$-*Singer generator* if there is an $\underline{\mathbf{n}}$-decomposition $V = V_{n_1} \oplus \cdots \oplus V_{n_k}$ of V such that $g = g_{n_1} g_{n_2} \cdots g_{n_k}$ where, (i) for each i, $\langle g_{n_i} \rangle$ is a Singer cycle subgroup of $GL(V_{n_i})$, or $\underline{\mathbf{n}} = (1, 1, 1)$ and g_{n_1} has eigenvalue 1, and (ii) if $n_i = n_j$ with $i \neq j$, then $c_{g_{n_i}}(t) \neq c_{g_{n_j}}(t)$, where $c_{g_{n_i}}(t)$ is the characteristic polynomial for g_{n_i} on V_{n_i}. We call $\prod_{i=1}^{k} \langle g_{n_i} \rangle$ the $\underline{\mathbf{n}}$-*maximal torus corresponding to* g.

(d) An element g of $GL(3, q)$ is called a (1, 2)-*pseudo Singer generator* if there is a (1, 2)-decomposition $V = V_1 \oplus V_2$ and distinct primitive elements $\alpha, \beta \in F$ such that $g = g_1 g_2$, where $g_1 \in GL(V_1)$ acts as $g_1 : v \mapsto \beta v$ and $g_2 \in GL(V_2)$ is conjugate to a matrix $\left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right)$. We call $\langle g_1 \rangle \times C_{GL(V_2)}(g_2)$ the (1, 2)-*maximal pseudo torus corresponding to* g.

Note that $GL(3, q)$ has no (1, 1, 1)-Singer generator unless $q \geq 4$, and no (1, 2)-pseudo Singer generator unless $q \geq 3$. Recall the definition of $S(\kappa)$ in Lemma 2.4.

LEMMA 2.6. *Let* $G = GL(3, q)$, *where* $q = p^k \geq 4$.

(a) *Suppose that* $g \in G$ *is an* $\underline{\mathbf{n}}$-*Singer generator, where* $\underline{\mathbf{n}} = (n_1, \ldots, n_k)$ *is* (3), (1, 2) *or* (1, 1, 1). *Then* $C_G(g) = \prod_{i=1}^{k} \langle g_{n_i} \rangle \in S(\kappa)$ *is a subgroup of order* $\prod_{i=1}^{k} (q^{n_i} - 1)$, *for* $\kappa =$ (i), (ii) *or* (iii) *respectively. In particular,* p *does not divide* $|C_G(g)|$.

(b) *Suppose that* $g = g_1 g_2 \in G$ *is a* (1, 2)-*pseudo Singer generator relative to* $V = V_1 \oplus V_2$. *Then* $C_G(g) = \langle g_1 \rangle \times B$, *where* $B = C_{GL(V_2)}(g_2) = Z_q \cdot Z_{q-1} \in S(\text{iv(a)})$ *and is conjugate to* $\{\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right) \mid \alpha, \beta \in F, \alpha \neq 0\}$. *Moreover,* $C_G(g)$ *has order* $q(q-1)^2$ *and does not contain an* $\underline{\mathbf{n}}$-*Singer generator for any* $\underline{\mathbf{n}}$.

PROOF. (a) Suppose that V is a three-dimensional vector space over a finite field F with size q. So by Definition 2.5, we have one of the following:

(1) If g is a (3)-Singer generator of G then $g = g_3$. So, by [7, Satz 7.3], $C_G(g) = \langle g \rangle \in S(\text{i})$ of order $q^3 - 1$.

(2) If g is a (1, 2)-Singer generator of G then by Definition 2.5, there is a g-invariant (1, 2)-decomposition $V = V_1 \oplus V_2$ such that $g|_{V_i} = g_i$, for $i = 1, 2$, and $Z_{q-1} \times Z_{q^2-1} \cong \langle g_1 \rangle \times \langle g_2 \rangle \subseteq C_G(g)$. Suppose that $h \in C_G(g)$. Now g leaves invariant a unique decomposition $V = V_1 \oplus V_2$ with $\dim V_1 = 1$, $\dim V_2 = 2$, and moreover,

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$(V_i^h)^g = (V_i^{hgh^{-1}})^h = (V_i^g)^h = V_i^h$, for $i = 1, 2$, and $V = V_1^h \oplus V_2^h$. It follows that, for $i = 1, 2$, $V_i^h = V_i$ and hence there exist $h_1 \in GL(V_1)$ and $h_2 \in GL(V_2)$ such that $h = h_1 h_2$. Now $gh = hg$ if and only if $g_i h_i = h_i g_i$, for $i = 1, 2$. Therefore $h_i \in C_{GL(V_i)}(g_i)$, for $i = 1, 2$. By [7, Satz 7.3], $C_{GL(V_i)}(g_i) = \langle g_i \rangle$, for $i = 1, 2$. Thus $h \in \langle g_1 \rangle \times \langle g_2 \rangle$. Hence $C_G(g) = \prod_{i=1}^{2} \langle g_i \rangle \in S(\text{ii})$ of order $(q-1)(q^2-1)$.

(3) If g is a $(1, 1, 1)$-Singer generator of G then, by Definition 2.5, there is a g-invariant $(1, 1, 1)$-decomposition $V = V_1 \oplus V_2 \oplus V_3$ such that $g|_{V_i} = g_i$, $\langle g_i \rangle = Z_{q-1}$, and the characteristic polynomials of g_1, g_2 and g_3 are pairwise distinct. So g is conjugate in $GL(3, q)$ to a diagonal matrix with pairwise distinct diagonal entries. It is straightforward to prove that $C_G(g) = \prod_{i=1}^{3} \langle g_i \rangle \in S(\text{iii})$ of order $(q-1)^3$.

(b) Let $g = g_1 g_2$ and $V = V_1 \oplus V_2$ be as in Definition 2.5(d). Then $C_G(g)$ leaves both V_1 and V_2 invariant and hence $C_G(g) = \langle g_1 \rangle \times B \in S(\text{iv(a)})$, where $B = C_{GL(V_2)}(g_2)$ and B is conjugate to $\{ \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right) | \alpha, \beta \in F, \alpha \neq 0\}$. In particular, $|C_G(g)| = q(q-1)^2$ which is not divisible by $q^3 - 1$ or $q^2 - 1$, and so $C_G(g)$ does not contain an $\underline{\mathbf{n}}$-Singer generator for $\underline{\mathbf{n}} = (3)$ or $(1, 2)$. Also each element of $C_G(g)$ has at most two distinct eigenvalues and so $C_G(g)$ does not contain a $(1, 1, 1)$-Singer generator. □

LEMMA 2.7. *Let $G = GL(3, q)$, where $q = p^k \geq 4$. Let x, y, z, u be a (3)-Singer generator, $(1, 2)$-Singer generator, $(1, 1, 1)$-Singer generator and $(1, 2)$-pseudo Singer generator of G, respectively. Then $\{x, y, z, u\}$ is pairwise noncommuting.*

PROOF. If $xw = wx$, where $w \in \{y, z, u\}$, then $x \in C_G(w)$ and hence $q^3 - 1 = |x|$ divides $|C_G(w)|$, which is a contradiction by Lemma 2.6. If $yw = wy$, where $w \in \{z, u\}$ then $y \in C_G(w)$ and hence $q^2 - 1 = |y|$ divides $|C_G(w)|$, which again contradicts Lemma 2.6. Finally, suppose that $zu = uz$. Then $u \in C_G(z)$ with $|u| = p(q-1)$ and $|C_G(z)| = (q-1)^2$, again a contradiction. □

LEMMA 2.8. *Let N be a maximal subset of pairwise noncommuting elements of $G = GL(3, q)$, where $q \geq 4$, let $\underline{\mathbf{n}} = (n_1, \ldots, n_k)$ be (3), $(1, 2)$ or $(1, 1, 1)$ and let g be an $\underline{\mathbf{n}}$-Singer generator or $\underline{\mathbf{n}}$-pseudo Singer generator (if $\underline{\mathbf{n}} = (1, 2)$) relative to the $\underline{\mathbf{n}}$-decomposition $V = V_{n_1} \oplus \cdots \oplus V_{n_k}$, where $\dim V_{n_i} = n_i$. Then N contains an element $x \in C_G(g)$ such that:*

- (i) *$(N \setminus \{x\}) \cup \{g\}$ is also a maximal subset of pairwise noncommuting elements of G;*
- (ii) *if g is an $\underline{\mathbf{n}}$-Singer generator then x acts irreducibly on V_{n_i} for each i;*
- (iii) *if g is a $(1, 2)$-pseudo Singer generator then p divides $|x|$.*

PROOF. By Lemma 2.6, $C_G(g)$ is abelian. By Lemma 2.1, the maximality of N implies that there exists $x \in N \cap C_G(g)$ such that $N' := (N \setminus \{x\}) \cup \{g\}$ is a maximal subset of pairwise noncommuting elements (possibly $x = g$). Suppose first that g is an $\underline{\mathbf{n}}$-Singer generator. We claim that x acts irreducibly on V_{n_i} for each i. Let $g = g_1 \cdots g_k$ so that x lies in $C_G(g) = \prod_{i=1}^{k} \langle g_i \rangle$, say $x = g_1^{a_1} \cdots g_k^{a_k}$. Suppose without loss of generality that $g_1^{a_1}$ acts reducibly on V_{n_1}. Since, by

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Lemma 2.6, the order of $g_1^{a_1}$ is not divisible by p, it follows from Maschke's theorem that $V_{n_1} = U_1 \oplus \cdots \oplus U_t$, where $t \geq 2$, $U_i \neq 0$ and U_i is $g_1^{a_1}$-invariant. Let $\dim U_i = m_i$. Then there exists an $(m_1, \ldots, m_t, n_2, \ldots, n_k)$-Singer generator h for a maximal torus $T = \langle h_1 \rangle \times \cdots \times \langle h_t \rangle \times (\prod_{i=2}^{k} \langle g_i \rangle)$ containing x relative to the $(m_1, \ldots, m_t, n_2, \ldots, n_k)$-decomposition

$$V = (U_1 \oplus \cdots \oplus U_t) \oplus (V_{n_2} \oplus \cdots \oplus V_{n_k}).$$

Note that $x \in C_G(h) = T$ and T is abelian. By Lemma 2.1, there exists $y \in N'$ such that $y \in C_G(h)$ and $(N' \setminus \{y\}) \cup \{h\}$ is maximal pairwise noncommuting. If $y = g$ then g_1 (of order $q^{n_1} - 1$) lies in $C_{GL(V_{n_1})}(h\,|_{V_{n_1}}) = \prod_{i=1}^{t} \langle h_i \rangle$, which is a contradiction. Hence $y \in N \setminus \{x\}$ and as N is noncommuting, $yx \neq xy$. However, it follows from the definitions of h and y that both $x, y \in C_G(h)$ and $C_G(h)$ is abelian. Thus $xy = yx$, which is a contradiction.

Finally let g be a $(1, 2)$-pseudo Singer generator, and suppose that p does not divide $|x|$. By Lemma 2.6, it follows that $x = x_1 x_2$ with $x_1 \in GL(V_1)$ and $x_2 \in Z(GL(V_2))$. Let y_2, y_2' be Singer generators in $GL(V_2)$ such that $\langle y_2 \rangle \neq \langle y_2' \rangle$, and let $y = x_1 y_2$ and $y' = x_1 y_2'$. Then $y, y' \in C_G(x)$ (since x_2 is central in $GL(V_2)$) and $yy' \neq y'y$ (since $\langle y_2 \rangle \neq \langle y_2' \rangle$). The maximality of N implies that $N \cap C_G(x) = \{x\}$ (so that $y, y' \notin N$). Hence, applying Lemma 2.1 twice, we obtain that $(N \setminus \{x\}) \cup \{y\}$ and $(N \setminus \{x\}) \cup \{y'\}$ are both pairwise noncommuting, and it follows that $(N \setminus \{x\}) \cup \{y, y'\}$ is also pairwise noncommuting, contradicting the maximality of N. Thus p divides $|x|$. □

LEMMA 2.9. *Let $G = GL(3, q)$, where $q = p^k > 2$, and let N_3 consist of one* (3)*-Singer generator of G corresponding to each* (3)*-maximal torus of G. Then N_3 is a subset of pairwise noncommuting elements of size $|S(\mathrm{i})| = |G|/(3(q^3 - 1))$.*

PROOF. Let $g, g' \in N_3$ such that $gg' = g'g$. By Lemma 2.6, $C_G(g) = \langle g \rangle$ and hence $g' \in \langle g \rangle$. Similarly, $g \in \langle g' \rangle$. By the definition of N_3, $g = g'$ and so N_3 is a subset of pairwise noncommuting elements. By [7, Satz 7.3], $|N_G(\langle g \rangle)| = 3|g| = 3(q^3 - 1)$, and hence $|N_3| = |G : N_G(\langle g \rangle)| = |G|/(3(q^3 - 1))$. □

LEMMA 2.10. *Let $G = GL(3, q)$, where $q = p^k > 2$. Let N_{12} consist of one* $(1, 2)$*-Singer generator of G corresponding to each* $(1, 2)$*-maximal torus of G. Then N_{12} is a subset of pairwise noncommuting elements of size $|S(\mathrm{ii})| = |G|/(2(q^2 - 1)(q - 1))$.*

PROOF. Let V be a vector space over a finite field F with dimension 3 and $|F| = q$. Let g and g' be $(1, 2)$-Singer generators of G such that $gg' = g'g$. By Definition 2.5, there exist a one-dimensional subspace V_1 and a two-dimensional subspace V_2 of V such that $V = V_1 \oplus V_2$, each V_i is g-invariant, and $g = g_1 g_2$, where, for $i = 1, 2$, $\langle g_i \rangle$ is a Singer cycle subgroup of $GL(V_i)$. Similarly, for g', there exist a one-dimensional subspace V_1' and a two-dimensional subspace V_2' of V such that $V = V_1' \oplus V_2'$, each V_i' is g'-invariant, and $g' = g_1' g_2'$, where, for $i = 1, 2$, $\langle g_i' \rangle$ is a Singer cycle subgroup of $GL(V_i')$. Since $gg' = g'g$, $g' \in C_G(g)$. By Lemma 2.6, $C_G(g)$ and $C_G(g')$ are both abelian so $C_G(g) = C_G(g')$. It follows that $\langle g_1 \rangle \times \langle g_2 \rangle = \langle g_1' \rangle \times \langle g_2' \rangle$ is a

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$(1, 2)$-maximal torus, and $V_i = V_i'$ for $i = 1, 2$. However, N_{12} contains only one generator of each $(1, 2)$-maximal torus of V. Hence $g = g'$. Thus N_{12} is a subset of pairwise noncommuting elements. The number of one-dimensional subspaces of V not contained in V_2 is $(q^3 - 1)/(q - 1) - (q^2 - 1)/(q - 1)$ and the number of two-dimensional subspaces of V is $(q^3 - 1)/(q - 1)$. Also the number of Singer cycle subgroups of $GL(V_2)$ is $|GL(2, q)|/(2(q^2 - 1))$. Consequently,

$$|N_{12}| = \left(\frac{q^3 - 1}{q - 1} - \frac{q^2 - 1}{q - 1}\right) \times \frac{|GL(2, q)|}{2(q^2 - 1)} \times \frac{q^3 - 1}{q - 1} = \frac{|G|}{2(q^2 - 1)(q - 1)}. \qquad \square$$

LEMMA 2.11. *Let $G = GL(3, q)$, where $q = p^k \geq 4$. Let N_{111} consist of one $(1, 1, 1)$-Singer generator of G corresponding to each $(1, 1, 1)$-maximal torus of G. Then N_{111} is a subset of pairwise noncommuting elements of size $|S(\mathrm{iii})| = |G|/(6(q - 1)^3)$.*

PROOF. Suppose that $g, g' \in N_{111}$ such that $gg' = g'g$. By Definition 2.5, there exist g_1, g_2, g_3 of G such that $g = g_1 g_2 g_3$ where, for $i = 1, 2, 3$, g_i is a generator of a Singer cycle subgroup of $GL(V_i)$ and $V = V_1 \oplus V_2 \oplus V_3$. Let $t - \lambda_i$ be the characteristic polynomial of g_i, for $i = 1, 2, 3$. By Definition 2.5, $\lambda_1, \lambda_2, \lambda_3$ are pairwise distinct eigenvalues of g. Similarly, there exist g_1', g_2', g_3' such that $g' = g_1' g_2' g_3'$ and g' has three distinct eigenvalues $\lambda_1', \lambda_2', \lambda_3'$. According to Lemma 2.6, $C_G(g)$ and $C_G(g')$ are both abelian, and since $gg' = g'g$, then $C_G(g) = C_G(g')$. So $\prod_{i=1}^3 \langle g_i \rangle = \prod_{i=1}^3 \langle g_i' \rangle$. By the definition of N_{111} this implies that $g = g'$. Hence N_{111} is a subset of pairwise noncommuting elements. Now we determine $|N_{111}|$, which is the number of decompositions $V_1 \oplus V_2 \oplus V_3$. We count ordered triples (V_1, V_2, V_3) of one-dimensional subspaces such that $V = V_1 \oplus V_2 \oplus V_3$. The number of one-dimensional subspaces V_1 of V is $(q^3 - 1)/(q - 1)$ and the number of one-dimensional subspaces V_2 of V, where $V_2 \neq V_1$, is $((q^3 - 1)/(q - 1)) - 1$. Also, the number of one-dimensional subspaces V_3 of V which are not contained in $V_1 \oplus V_2$ is $(q^3 - 1)/(q - 1) - (q^2 - 1)/(q - 1)$. Thus the number of ordered triples

$$(V_1, V_2, V_3) \text{ is } \left(\frac{q^3 - 1}{q - 1}\right) \cdot \left(\frac{q^3 - 1}{q - 1} - 1\right) \cdot \left(\frac{q^3 - 1}{q - 1} - \frac{q^2 - 1}{q - 1}\right) = \frac{|G|}{(q - 1)^3}.$$

And as each decomposition has been counted 6 times it follows that $|N_{111}| = |G|/(6(q - 1)^3)$. $\square$

LEMMA 2.12. *Let $G = GL(3, q)$, where $q = p^k \geq 4$. Let N_{12}^* consist of one $(1, 2)$-pseudo Singer generator of G corresponding to each $(1, 2)$-maximal pseudo torus of G. Then N_{12}^* is a subset of pairwise noncommuting elements of size $|S(\mathrm{iv(a)})| = |G|/(q(q - 1)^3)$. Moreover, $N_3 \cup N_{12} \cup N_{111} \cup N_{12}^*$ is a subset of pairwise noncommuting elements with N_3, N_{12}, N_{111} as in Lemmas 2.9, 2.10 and 2.11, respectively.*

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PROOF. Let V be a vector space over a finite field F with dimension 3 and $|F| = q$. Let g and g' be $(1, 2)$-pseudo Singer generators of G such that $gg' = g'g$. By Definition 2.5, there exist a one-dimensional subspace V_1 and a two-dimensional subspace V_2 of V such that $V = V_1 \oplus V_2$ and $g = g_1 g_2$, where $\langle g_1 \rangle$ is a Singer cycle subgroup of $GL(V_1)$ and g_2 is conjugate to the matrix $b = \left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right)$, where α is a primitive element of F. We may assume that $g_2 = b$. Similarly, for g', there exist a one-dimensional subspace V_1' and a two-dimensional subspace V_2' of V such that $V = V_1' \oplus V_2'$ and $g' = g_1' g_2'$, where $\langle g_1' \rangle$ is a Singer cycle subgroup of $GL(V_1')$ and g_2' is conjugate to the matrix $b' = \left(\begin{smallmatrix} \alpha' & 1 \\ 0 & \alpha' \end{smallmatrix}\right)$ with α' a primitive element of F. Since $gg' = g'g$, $g' \in C_G(g)$. By Lemma 2.6, $C_G(g)$ and $C_G(g')$ are both abelian, so $C_G(g) = C_G(g')$. Thus g and g' determine the same $(1, 2)$-maximal pseudo torus. However, N_{12}^* contains only one element of each $(1, 2)$-maximal pseudo torus of V. Hence $g = g'$. Thus N_{12}^* is a subset of pairwise noncommuting elements. The number of one-dimensional subspaces of V not contained in V_2 is $(q^3 - 1)/(q - 1) - (q^2 - 1)/(q - 1)$ and the number of two-dimensional subspaces of V is $(q^3 - 1)/(q - 1)$. An easy computation shows that the number of conjugates of $C_{GL(V_2)}(g_2)$ in $GL(V_2)$ is $|GL(2, q)|/(q(q - 1)^2)$. Consequently,

$$|S(\mathrm{iv(a)})| = |N_{12}^*| = \left(\frac{q^3 - 1}{q - 1} - \frac{q^2 - 1}{q - 1}\right) \times \frac{|GL(2, q)|}{q(q - 1)^2} \times \frac{q^3 - 1}{q - 1} = \frac{|G|}{q(q - 1)^3}.$$

Finally, according to Lemmas 2.7, 2.9, 2.10, 2.11, we have that $N_3 \cup N_{12} \cup N_{111} \cup N_{12}^*$ is a subset of pairwise noncommuting elements. □

COROLLARY 2.13. *Let $G = GL(3, q)$, where $q = p^k \geq 4$. Then*

$$\omega(G) \geq |S(\mathrm{i})| + |S(\mathrm{ii})| + |S(\mathrm{iii})| + |S(\mathrm{iv(a)})| = q^6 + q^5 + 2q^4 + 2q^3 + q^2.$$

PROOF. By Lemmas 2.9, 2.10, 2.11 and 2.12, $N_3 \cup N_{12} \cup N_{111} \cup N_{12}^*$ is a subset of pairwise noncommuting elements of size

$$\begin{aligned} &|S(\mathrm{i})| + |S(\mathrm{ii})| + |S(\mathrm{iii})| + |S(\mathrm{iv(a)})| \\ &\quad = \frac{|G|}{3(q^3 - 1)} + \frac{|G|}{2(q^2 - 1)(q - 1)} + \frac{|G|}{6(q - 1)^3} + \frac{|G|}{q(q - 1)^3} \\ &\quad = q^6 + q^5 + 2q^4 + 2q^3 + q^2. \end{aligned}$$

□

3. Noncommuting subsets of p-elements in finite groups

In this section we prove a general result about subsets of pairwise noncommuting elements consisting of p-elements (p a prime) in arbitrary finite groups. It is used later in the paper. We denote the number of Sylow p-subgroups of a finite group G by $\nu_p(G)$.

LEMMA 3.1. *Suppose that G is a finite group and p is a prime number dividing $|G|$. Let $P = P_1, P_2, \ldots, P_{\nu_p(G)}$ be the Sylow p-subgroups and for each i choose*

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$x_i \in G$ such that $P^{x_i} = P_i$. If S is a subset of pairwise noncommuting elements of $P \setminus \bigcup_{i=2}^{\nu_p(G)} P_i$ then $\nu_p(G) \times |S| \le \omega(G)$.

PROOF. Let $S = \{a_1, \ldots, a_k\}$ be a subset of pairwise noncommuting elements of $P \setminus \bigcup_{i=2}^{\nu_p(G)} P_i$. For each $a_i \in S$, P is the unique Sylow p-subgroup containing a_i. Then it is easy to see that, for all i, $S^{x_i} = \{a_1^{x_i}, \ldots, a_k^{x_i}\}$ is a subset of pairwise noncommuting elements of $P_i \setminus (P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_{\nu_p(G)})$. Set

$$X = \bigcup_{i=1}^{\nu_p(G)} S^{x_i} = \bigcup_{i=1}^{\nu_p(G)} \{a_1^{x_i}, a_2^{x_i}, \ldots, a_k^{x_i}\}.$$

We claim that X is a subset of pairwise noncommuting elements of G. Suppose to the contrary that $a_i^{x_k} a_j^{x_l} = a_j^{x_l} a_i^{x_k}$, with $a_i^{x_k} \neq a_j^{x_l}$. If $k = l$ this is not possible since S^{x_k} is noncommuting. It follows that $\langle a_i^{x_k}, a_j^{x_l} \rangle$ is an abelian p-subgroup of G, and so there exists a Sylow p-subgroup P^{x_t} of G such that $\langle a_i^{x_k}, a_j^{x_l} \rangle \subseteq P^{x_t}$. By our remark above, P^{x_k} is the unique Sylow p-subgroup containing $a_i^{a_k}$, and so $t = k$. Similarly, $t = l$, and this is a contradiction. Therefore $|X| = \nu_p(G) \times |S| \le \omega(G)$. □

COROLLARY 3.2. *Let G be a finite group and let p be a prime number dividing $|G|$. Suppose that if P_i, P_j are distinct Sylow p-subgroups of G, then $P_i \cap P_j = 1$. Then $\nu_p(G) \le \omega(G)$.*

PROOF. By Lemma 3.1, the proof is straightforward. □

As an application of Corollary 3.2, we have the following result that was proved by a different method in [3, Theorem 1, p. 294] for symmetric groups S_n for arbitrary n.

COROLLARY 3.3. *Let p be a prime number. Then $\omega(S_p) \ge (p-2)!$.*

PROOF. Since $\nu_p(S_p) = (p-2)!$ and any Sylow p-subgroup of S_p is of size p, the assertion follows from Corollary 3.2. □

4. Proof of Theorem 1.1

In this section we construct a subset of pairwise noncommuting elements of $GL(n, q)$ consisting of unipotent elements. We begin this section with the following definition.

DEFINITION 4.1. Let V be a finite-dimensional vector space over F. An endomorphism x of V is called *semisimple* if the minimal polynomial of x has distinct roots, and is called *unipotent* whenever it is the sum of the identity and a nilpotent endomorphism.

REMARK 4.2. If char $F = p > 0$, and V is a finite-dimensional vector space over F, then $x \in GL(V)$ is unipotent if and only if $x^{p^t} = 1$ for some $t \ge 0$. Also x is semisimple if p does not divide the order of x.

PROPOSITION 4.3. *Let $x \in GL(V)$.*

(a) *There exist unique $x_s, x_u \in GL(V)$ satisfying the conditions $x = x_s x_u$, x_s is semisimple, x_u is unipotent, $x_s x_u = x_u x_s$.*
(b) *x_s, x_u commute with any endomorphism of V which commutes with x.*
(c) *If A is an x-invariant subspace of V, then A is invariant under x_s and x_u.*
(d) *If $xy = yx$ ($y \in GL(V)$), then $(xy)_s = x_s y_s$, $(xy)_u = x_u y_u$.*

PROOF. See [6, Ch. VI, Lemma B]. □

We call x_s the *semisimple part* and x_u the *unipotent part* of x. Note that if x is both semisimple and unipotent, then $x = 1$.

DEFINITION 4.4. Let $G = GL(n, q)$, where $q = p^k > 2$ and $n \geq 3$. Let P be the subgroup of G of (upper) unitriangular matrices, that is, matrices with 1 on the diagonal and 0 below it. By [7, Satz 7.1], P is a Sylow p-subgroup of G. Let $F^* = \langle \alpha \rangle$, and, for $j = 2, \ldots, n-1$, let $i_j \in \{1, \ldots, q-1\}$. Set

$$A_{(i_2,\ldots,i_{n-1})} = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & 1 & \alpha^{i_2} & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \alpha^{i_{n-1}} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Let $S = \{A_{(i_2,\ldots,i_{n-1})} \mid i_j \in \{1, \ldots, q-1\}\}$ and $N_U = \bigcup_{g \in G} S^g$.

We note that, in the case $n = 3$, S is a subset of elements of $GL(3, q)$ of type v(a), as described in Section 2.2.

LEMMA 4.5. *Let $G = GL(n, q)$, where $q = p^k > 2$. Then N_U is a subset of pairwise noncommuting unipotent elements of size $|G|/(q^{\binom{n}{2}}(q-1)^2)$.*

PROOF. Set $B = A_{(i_2,\ldots,i_{n-1})} - I$, where I is the identity matrix. We shall show that $A_{(i_2,\ldots,i_{n-1})} \in P \setminus \bigcup_g P^g$, where $g \in G \setminus N_G(P)$. Suppose, for a contradiction, that there exists $g \in G \setminus N_G(P)$ such that $A_{(i_2,\ldots,i_{n-1})} \in P^g$. So $gA_{(i_2,\ldots,i_{n-1})} = Cg$, for some $C \in P$. Let

$$C = \begin{pmatrix} 1 & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & 1 & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{n-1n} \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

and set $D = C - I$. Thus $g(I + B) = (I + D)g$, and so $gB = Dg$. Since the last row of D is zero, the last row of Dg is zero, that is, $(Dg)_{ni} = 0$ for $1 \leq i \leq n$. On the other hand, for $1 \leq k \leq n$, $(gB)_{nk} = \sum_{j=1}^{n} g_{nj}(B)_{jk}$. It follows that $g_{n1} = 0$ and, for $2 \leq k \leq n-1$, $g_{nk}\alpha^{i_k} = 0$. Hence $g_{nk} = 0$, for $1 \leq k \leq n-1$. Similarly,

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$g_{ij} = 0$ for $j < i$. Thus g is an upper triangular matrix and hence is in $N_G(P)$, which is a contradiction. Hence $A_{(i_2,\ldots,i_{n-1})}$ lies in $P \setminus \bigcup_g P^g$. Also, it is easy to see that $A_{(i_2,\ldots,i_{n-1})} \times A_{(j_2,\ldots,j_{n-1})} = A_{(j_2,\ldots,j_{n-1})} \times A_{(i_2,\ldots,i_{n-1})}$ if and only if $i_k = j_k$ for $k = 2, \ldots, n-1$. Therefore $S = \{A_{(i_2,\ldots,i_{n-1})} \mid i_k \in \{1, \ldots, q-1\}\}$ is a subset of pairwise noncommuting elements of $P \setminus \bigcup_g P^g$, and $|S| = (q-1)^{n-2}$. Since the number of Sylow p-subgroups is

$$\nu_p(G) = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q-1)}{(q-1)^n},$$

we obtain by Lemma 3.1, a subset of pairwise noncommuting elements of size

$$\nu_p(G) \cdot |S| = \frac{|G|}{q^{\binom{n}{2}}(q-1)^2}. \qquad \square$$

LEMMA 4.6. *Let $G = GL(3, q)$, where $q = p^k \geq 4$. If $u \in N_U$ then $C_G(u)$ is abelian of order $q^2(q-1)$ and $|N_U| = |S(\mathrm{v(a)})|$. Moreover, if $g \in G$ is an $(n_1, \ldots, n_k)$-Singer generator, where $\sum n_i = 3$, and x is a $(1, 2)$-pseudo Singer generator, then $ug \neq gu$ and $ux \neq xu$.*

PROOF. Let u be as in the statement. So there exists $g \in G$ such that $u \in S^g$. Hence there exists $s \in S$ such that $u = s^g$. Let

$$s = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \alpha^i \\ 0 & 0 & 1 \end{pmatrix},$$

where $\langle \alpha \rangle = F^*$ and $i \in \{1, 2, \ldots, q-1\}$. It follows easily that

$$C_G(s) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b\alpha^i \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a, b, c \in F, a \neq 0 \right\}.$$

It is clear that $C_G(s)$ is abelian of order $q^2(q-1)$. Since $C_G(u)$ is conjugate to $C_G(s)$ in G, $C_G(u)$ is abelian of order $q^2(q-1)$.

Thus each element u of N_U is of type v(a), as defined in Section 2.2, and $C_G(u)$ is abelian. Then, since N_U is pairwise noncommuting, it follows that $|N_U| \leq |S(\mathrm{v(a)})|$. Conversely, if $X = C_G(g) \in S(\mathrm{v(a)})$, with g of type v(a), then $g = B_1^h$ for some $h \in G$ and some λ, with B_1 as defined in (2.1) in Section 2.2. Now $C_G(B_1) = C_G(A_{(q-1)})$ and hence $X = C_G(B_1)^h = C_G(A^h_{(q-1)})$ with $A^h_{(q-1)} \in N_U$. Since X is abelian and N_U is noncommuting, distinct subgroups in $S(\mathrm{v(a)})$ are centralizers of distinct elements of N_U, and hence $|S(\mathrm{v(a)})| \leq |N_U|$. It follows that $|S(\mathrm{v(a)})| = |N_U|$.

Let $g \in G$ be an $(n_1, \ldots, n_k)$-Singer generator, where $\sum n_i = 3$, and suppose $ug = gu$, so $u \in C_G(g)$. By Remark 4.2, p divides the order of u and hence divides $|C_G(g)|$, contradicting Lemma 2.6.

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Now, let x be a (1, 2)-pseudo Singer generator such that $ux = xu$. So $x \in C_G(u)$. Since $C_G(u)$ is abelian, $C_G(u) \subseteq C_G(x)$. Similarly, by Lemma 2.6, $C_G(x)$ is abelian of order $q(q-1)^2$. It follows that $C_G(u) = C_G(x)$, a contradiction. This completes the proof. □

Finally we prove the main theorem.

PROOF OF THEOREM 1.1. Let $N = N_3 \cup N_{12} \cup N_{111} \cup N_{12}^* \cup N_U$. If $q \geq 4$ then, by Corollary 2.13 and Lemma 4.6, N is a subset of pairwise noncommuting elements of G and

$$\begin{aligned}|N| = \sum_{\kappa \in I} |S(\kappa)| &= q^6 + q^5 + 2q^4 + 2q^3 + q^2 + \frac{|G|}{q^3(q-1)^2} \\ &= q^6 + q^5 + 3q^4 + 3q^3 + q^2 - q - 1.\end{aligned}$$

Moreover, $\omega(G) \geq |N| = \sum_{\kappa \in I} |S(\kappa)|$. On the other hand, we observed in Lemma 2.4 that $\omega(G) \leq \sum_{\kappa \in I} |S(\kappa)|$. Thus equality holds. If $q = 2$ or 3, the result follows from Lemma 2.3.

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