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Uniqueness of patterns generated by repetition

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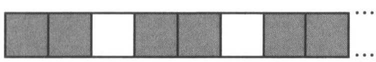
Consider two related tasks:

A) The diagram below displays a fragment of a pattern known to extend indefinitely and to be generated by an endlessly repeating pattern. What kind of shading will the 137th cell have? What number cell will the 137th white cell be in?



B) What is the next term in the sequence 3, 6, 11, 18, 27, ...?

Tasks like the first can provide excellent experience of detecting and expressing generality as one route to algebra, but only if learners are first asked to describe a structural relationship which generates the succeeding items. Tasks like the second feature in intelligence tests, but it is well known that such tasks are mathematically incomplete.

Without an underlying structural constraint it is impossible to predict the 'next term'. For example the numerical sequence of task B can be extended in any way one likes: the terms displayed are generated and extended by the function

$$t(n) = 2 + n^2 + (n - 1)(n - 2)(n - 3)(n - 4)(n - 5)f(n)$$

where f is any function from $\mathbb{Z}$ to the complex numbers.

Similarly, for the string shown in task A, coding the displayed pattern as *SSWSSWSS* where *S* stands for *shaded* and *W* for *white*, the pattern could be generated by repetitions of the block *SSW*, but it could also be generated by repetitions of the block *SSWSSWS* or indeed by any longer block that begins *SSWSSWSS*.

This raises the question of exactly what information is needed about a portion of a sequence or of a string formed by repetitions of a single block of symbols in order to be certain that the pattern is uniquely determined.

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Begin by looking at strings of symbols. A consecutive substring (block) which generates the pattern by being repeated is referred to as a *generating block*. If you know that a generating block has been repeated a specified number of times to generate the displayed string, then reconstructing it is straight forward and so the string extends uniquely. But what if you know only that some parts have appeared at least a certain number of times?

For example, if you are told that in the string of symbols shown above, a generating block appears at least once, then there is ambiguity. However if you are told that a generating block appears at least twice, the string must be generated by repetitions of *SSW*. There is no ambiguity. This applies generally.

Theorem: If in a consecutive substring of a repeating pattern, it is known that a generating block has appeared at least twice, then the string is uniquely specified.

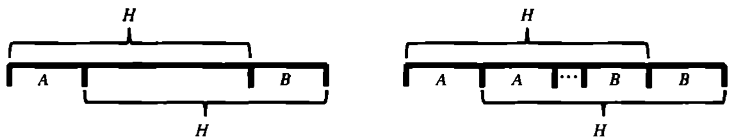
Throughout the reasoning, capital letters such as S are used for (possibly empty) blocks of symbols, superscripts mean repetitions of the corresponding block, and $|S|$ denotes the length of the block of symbols making up S. Sometimes it is convenient to break a block into segments, and for this purpose either brackets or a full stop are used. Thus $AAB.AAB$ or $(AAB)^2$ draws attention to the repetition of the block of symbols AAB formed from the blocks A and B. A non-empty initial substring of S will be called a *head string*, with a *proper head string* of S being not the whole of S. A non-empty final substring of S will be called a *tail string*, with a *proper tail string* being not the whole of S.

Observation: Given a string S with a proper head string H and a proper tail string T, then either there exists M (for 'middle') such that $S = HMT$ or else there exist blocks A, B and C (for 'common') such that $H = AC$, $T = CB$ and $S = ACB$.

The reason is that either H and T do not overlap, or else they overlap in the common block C. The special case in which a head string and a tail string are identical lies at the heart of the reasoning.

Special case: If H is a proper head string of S and also a proper tail string, then either there exists M such that $S = HMH$ or else there exists a block A and a positive integer n such that $S = A^n$.

The reason is that on the one hand, if $2|H| \leqslant |S|$ then, by the observation, $S = HMH$. On the other hand, if $2|H| > |S|$ let $S = AH = HB = ACB$. Here A is the proper head string of S preceding the tail occurrence of H and B is the proper tail string of S following the head occurrence of H. The remaining part of S is C. Since $2|H| > S = |A| + |H|$, $|A| < |H|$. The following diagrams display the first two steps either of an induction argument or direct reasoning to show that H consists of n copies of block A followed by a block C followed by n copies of block B.



Having isolated A as a head string of S (left-hand diagram), it is also a head string of H (right-hand diagram), and the same applies to B as a tail. This reasoning iterates.

However more must be the case. If A and B are different then as the diagrams show, the two occurrences of H would consist of different numbers of repetitions of A in front of the first occurrence of block B unless H, and hence S, is in fact a number n of repetitions of A.

Lemma 1: If $S = XY = YX$ then there exists a block W and positive integers s and t such that $X = W^s$, $Y = W^t$ and $S = W^{s+t}$.

Proof: If either X or Y is empty, the claim is trivially true. If $|X| = |Y|$ then $XY = YX$ forces $Y = X$. The claim is then satisfied with $W = X = Y$ and $s = t = 1$.

Let $S = XY = YX$ be a counter-example of minimal length. Without loss of generality assume $|X| < |Y|$. Since X and Y start and finish the same way, by the special case of the initial observation either $S = XMX$, or else there exists a block W such that X, Y and S are all repetitions of some block W, in which case the claim is true, contradicting the fact that S is a counter-example.

If $S = XMX$ then $Y = MX = XM$, and so by the minimality of S there exists W for which $Y = W^s$, $M = W^t$ and so $S = W^{s+t}$ contradicting the counter-example assumption.

Lemma 1 can actually be generalised, and provides the core of the proof of the theorem.

Lemma 2: If $S = XY = YZ$ with X and Y proper head strings of S and Y and Z proper tail strings, then there exist blocks P and Q and positive integers s and t such that

$$S = (PQ)^{s+t}P \text{ with } X = (PQ)^s, \quad Y = (PQ)^t P \text{ and } Z = (QP)^s.$$

Proof: Let S be a counter-example of minimal length. Clearly $|X| = |Z|$.

Since $S = XY = YZ$, if $X = Y$ then S is not a counter-example. Otherwise Y has a proper head string in common with X and a proper tail string in common with Z. Thus either $Y = XMZ$ or else $X = AC$, $Z = CB$ and $Y = ACB$ for appropriate choices of A, B and C.

In the first case, $Y = XMZ$ so $S = XXMZ = XMZZ$. This means that $XM = MZ$ and so putting $T = XM = MZ$ gives a string shorter than S but meeting the same criteria. Thus, by minimality, there exist P and Q such that $X = (PQ)^s$, $M = (PQ)^t P$ and $Z = (QP)^s$. Then $Y = (PQ)^{2s+t}P$ and $S = (PQ)^{3s+t}P$ and so the result holds.

In the second case, $X = AC$, $Z = CB$ and $Y = ACB$, so $S = AC.ACB = ACB.CB$. Removing common head and tail strings reveals that $A = B$ and so $X = AC$, $Y = ACA$, $Z = CA$ and $S = ACACA = (AC)^2A$ again contradicting the counter-example assumption. Therefore S is as claimed.

Proof of the Theorem

Suppose that there is a string in which some block is repeated at least twice but that S is ambiguously extendable. Choose such a counter-example string S with minimal length amongst all counter-examples. Then there exist blocks W and X such that $S = (WX)^{\lambda} W$ for some $\lambda \geqslant 2$. However, since S is ambiguously extendable, it is also the case that $S = (YZ)^{\mu} Y$ for some $\mu \geqslant 2$ which extends S differently. This is in fact impossible, as is now demonstrated by showing that S has a unique presentation.

Suppose first that both W and Y are empty strings, so $S = X^{\lambda} = Z^{\mu}$ with λ and μ both at least 2 and $X \neq Z$. Assume that $|X|$ is minimal amongst all such presentations as exact repetitions of some block. Then Z has X as both a proper head string and a proper tail string. By the observation, either $Z = XMX$ for some block M, or else $X = AC = CB$ and $Z = ACB$, for some blocks A, B and C.

In the first case, $S = X^{\lambda} = (XMX)^{\mu}$ and so $X.X^{\lambda-1} = (XMX)^{\mu-1}XM.X$. Applying Lemma 2 with the common element X, it follows that there exist blocks P and Q such that $X = (PQ)^t P$, $X^{\lambda-1} = (QP)^s$ and $(XMX)^{\mu-1}XM = (PQ)^s$. But this forces $QP = PQ$ since they both present the beginning of X. By Lemma 1 this means that there exists W for which $P = W^n$ and $Q = W^m$. Consequently X is a repetition of W. But this means that S is a repetition of W in both presentations, so they are actually the same. Thus S is actually extended in the same way, contradicting the initial assumption of ambiguity.

In the second case, $X = AC = CB$ and so by Lemma 2, $X = (PQ)^{s+t}P$, $C = (PQ)^t P$ and $B = (QP)^s$.

But then $Z = (PQ)^{s+t}P(QP)^s$ and $S = \left((PQ)^{s+t}P\right)^{\lambda} = \left((PQ)^{s+t}P(QP)^s\right)^{\mu}$. However since $\lambda \geqslant 2$, removing the head string X from both presentations of S and looking at the head string QP of the result shows $PQ = QP$, being of the same length. By Lemma 1, there exists V such that $P = V^n$ and $Q = V^m$ which means that Z and X and hence also S are all repetitions of V and so S is in fact extended the same way using either description, contradicting the initial assumption of ambiguity.

Suppose now that $S = (WX)^{\lambda} W = Y^{\mu}$ where $\lambda \geqslant \mu \geqslant 2$, with $|W| > 0$, and with WX chosen to be the shortest such repeating block. Here

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too W and Y have common head and tail strings with Y no shorter than W. By Lemma 2, either there exists a block M such that $Y = WMW$ or else there exist blocks A and C such that $= A^w$ and $Y = A^y$, which means that S is a repetition of the block A and so is uniquely extended by either presentation, which is a contradiction.

In the first case, $S = (WX)^\lambda W = (WMW)^\mu$, which means that $(WX)^\lambda.W = W.(MWW)^{\mu-1}MW$. By Lemma 2 with the common element W, there exists P and Q such that $W = (PQ)^t P$, while $(WX)^\lambda = (PQ)^s$ and $(MWW)^{\mu-1}MW = (QP)^s$. But this implies that $S = (PQ)^{s+t}P$ from either presentation and so extends uniquely, which is a contradiction.

Suppose now that $S = (WX)^\lambda W = (YZ)^\mu Y$ with neither W nor Y empty. If $|W| = |Y|$ then clearly $W = Y$ in which case $T = (XW)^\lambda X$ is shorter than S and meets the conditions so T is uniquely extendable, which then means that S is also uniquely extendable, a contradiction. Otherwise assume without loss of generality that $|W| < |Y|$. Again W and Y have a common proper head and tail, so either $Y = WMW$ or else W and Y are repetitions of some block A which means again that S is uniquely extended, a contradiction.

In the first case, $S = (WX)^\lambda W = (WMWZ)^\mu WMW$ which means that $(WX)^\lambda.W = W.(MWZW)^\mu MW$. Applying Lemma 2 with W as the common element, there exist P and Q such that $W = (PQ)^t P$, while $(WX)^\lambda = (PQ)^s$ and $(MWZW)^m MW = (QP)^s$. From this it follows again that $S = (PQ)^{s+t}P$ from either presentation, making S uniquely extended and providing a contradiction.

Consequently the assumption that a counter-example exists is contradicted and so the result holds.

When at least ρ of a fundamental block appears at least twice

Suppose you are told that 'at least a certain fraction r of a generating block appears at least twice'. How big does r have to be to guarantee uniqueness? The condition means that a generating block appears fully once and at least r of it appears a second time.

For example, the string A^sBA^s can be extended ambiguously as repetitions of A^sB and as repetitions of A^sBA. In the first case $\dfrac{s}{s+1}$ of the generating block appears the second time, while in the second, $\dfrac{s-1}{s+2}$ of the generating block appears at least twice. Note that $\dfrac{s}{s+1} > \dfrac{s-1}{s+2}$. For this string therefore, there is ambiguity if all you are told is that at least $\dfrac{s-1}{s+2}$ of the fundamental block appears at least twice.

Thus to avoid ambiguity in all possible cases, ρ must be greater than this fraction no matter what the value of s, which means that r must be at least the least upper bound, namely 1. Consequently it is necessary in general to be told that the whole of a fundamental block appears at least twice, although for specific strings, smaller ratios may be sufficient.

Piecemeal repetition

Suppose in a string S of which only some symbols are shown, it is asserted that each symbol in a generating block appears at least twice in its correct position. For example, if a string is generated as $ABAA$ repeated, you might be shown $S = AB\_A\_\_A\_A\_\_A\_BA\_\_\_$ and told that each symbol in each position in the fundamental block appears in its correct position at least twice. Despite this information you would not be able to distinguish between the generating blocks ABA and $ABAA$. You would know that the maximum length of a generating block would be 4, since you are told 8 letters and the block has to repeat at least twice. But longer and longer repetitions of S would increase the number of times each position is presented, yet preserve the ambiguity. Consequently uniqueness depends on being shown consecutive symbols.

Even if there is at least one complete occurrence of a generating block, together with sporadic symbols amongst which each part of the generating block is repeated, ambiguity remains. Consider for example the fragment

$$ABA\_\_\_ABA\ldots.$$

Even if you know that a generating block appears at least twice, each time as a full consecutive block, you cannot distinguish between being generated by AB and being generated by ABA.

Number of ambiguities

The examples offered at the beginning to indicate potential ambiguities might be interpreted as suggesting that even where there is ambiguity the choices are rather limited. However, the string $A^sBA^{t+s}BA^u$ where $u = \min(s, t)$ has the property that it can be extended in $\min(s, t)$ different ways by using as a generating block any one of the strings $A^sBA^{t+s}BA^v$ where $0 \leqslant v < u = \min(s, t)$, because $s + v \geqslant u$ and so the displayed sequence can be augmented with sufficient A s to start off the second copy of the generating block.

Increasing strings

The same idea can be carried into strings that are growing in some regular manner. Consider for example the string $ABAABAAABAAAAB$. Define the first difference string to be the string of increases in the number of occurrences of consecutive A s and consecutive B s. The first difference string would be 1, 0, 1, 0, 1, 0 in this case.

If you are told that in the original string there is a growing pattern for which the first difference string has a generating block that appears at least twice, then the original string is uniquely specified. The reason is that by the theorem, the first difference string is unambiguously determined as repetitions of 1, 0. It follows than that B can only occur once in each generating block. This forces the initial 'generating block' to be AB with the number of A s growing by one each time.

Specify a d-difference-constant string to be a string generated from a block of letters F according to a rule that the k th block is constructed by replacing the j th letter of F by $f_j(k)$ copies of that letter, for $j = 1, \ldots, |F|$ where each f_j is a polynomial function of degree at most d from the positive integers to the positive integers, and at least one of the f_j is of degree d.

For example, suppose $F = AB$ and that $f_1(k) = k^2$ and $f_2(k) = 2k - 1$. Then the string generated would be

$$S = AB.A^4B^3.A^9B^5.A^{16}B^7.A^{25}B^9\ldots$$

(where the dots are displayed only to clarify the specification of the string).

Being told there are at least $d + 2$ blocks present which are formed from an ultimately repeating pattern in a difference sequence determines the string uniquely.

Sequences

Given a finite number of terms $t_1, t_2, \ldots, t_{d+2}$ of a sequence, and asked to predict the next term, the answer is unambiguous if, and only if, you are told that there are at least $d + 2$ terms displayed where the d th differences of the displayed sequence are constant. The string reasoning can be used by considering the string AB as the foundation block F, with $f_1(k) = t_k$ and $f_2(k) = k^d$, which generates a string corresponding to the given number sequence. Applying the theorem recursively to the first difference sequence guarantees uniqueness.

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