LUNAR OCCULTATION THEORY AND TECHNIQUES

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Abstract. The formulae for the monochromatic radiation intensity at the Earth during the lunar occultation of an incoherent source are examined. It is shown that the oscillatory portion of the radiation intensity may be assigned an envelope amplitude and phase a^+ each instant of time. Each complex number thus defined gives the amplitude and phase of one spatial Fourier component of the source brightness distribution. Resolution is limited by the maximum available spatial frequency, and is shown to depend on integration time, seeing, antenna aperture, bandwidth, and the signal-to-noise ratio of the observation. When observations are polychromatic, the monochromatic intensity is modified by a certain function of bandwidth and wavelength. This function is specified and theoretical occultation curves of a point source are given for various bandwidth-wavelength ratios. A new method of estimating a brightness distribution is illustrated by taking the inverse Fourier transform of the spatial Fourier components present in an occultation record of 3C 49. Restored distributions for the quasars 3C 273 and 3C 245 are also given. Source positions and/or lunar limb slopes may be determined from such distributions. When the width of single sources or the angular separation of double sources is all that is required, only the inner part of the envelope of the occultation record need be determined. Theoretical curves and envelopes are presented for use in comparisons with observed data.

1. Introduction

When the amplitude and phase of the oscillatory portion of a record taken during the occultation of an extended source are compared with those of an occulted point source, the spatial Fourier components of the brightness distribution of the source may be specified. Such a comparison defines the same Fourier components as does a variable baseline interferometer, of which one element is on the lunar limb, and the other on the line-of-sight to the occulted source and at the Moon's distance from the observer. Because of various observational constraints, spatial Fourier components are only available up to a maximum spatial frequency which corresponds to a maximum available interferometer baseline. Any estimated brightness distribution must, therefore, be limited in angular resolution. This limit is a function of integration time, seeing, antenna size, observing bandwidth, and the signal-to-noise ratio of the observation. By first evaluating the available Fourier components of the brightness distribution of a source, and then taking the inverse Fourier transform, the brightness distribution may be estimated. Source positions and/or lunar limb slopes may be determined from such distributions. In order to determine the width of single sources, or the angular separation of double sources, it is only required that observed occultation records be compared with theoretical ones.

2. The Earth-Moon System as a Variable Baseline Interferometer

Let us suppose that an incoherent, monochromatic source of wavelength, λ, passes through the reception pattern of a two-element interferometer, one element of which

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is on the limb of the Moon, and the other on the line joining the observer to the source and at a distance θD from the Moon. The power detected by such an interferometer will show an almost periodic rise and fall as the signals from the two elements interfere. When the amplitude and phase of the periodic component of this fringe pattern are compared with those expected from a point source, one spatial Fourier component of the source brightness distribution may be specified. With reference to Figure 1, the relative amplitude, $V(\theta D/\lambda)$, and phase, $\psi(\theta D/\lambda)$, of the fringe pattern define the function

$$B_\lambda(\theta D/\lambda) = V(\theta D/\lambda)\, e^{i\psi(\theta D/\lambda)}, \tag{1}$$

which is the normalized Fourier transform, $B_\lambda(s)$, of the brightness distribution, b_λ, evaluated at the spatial frequency, $s = \theta D/\lambda$. In general, $B_\lambda(s)$ is defined by

$$B_\lambda(s) = S^{-1}\left\{\int_{-\infty}^{+\infty} b_\lambda(\theta)\, e^{-i2\pi s\theta}\, \mathrm{d}\theta\right\}, \tag{2}$$

where $S = \int_{-\infty}^{+\infty} b_\lambda(\tau)\, \mathrm{d}\tau$ is the source flux density.

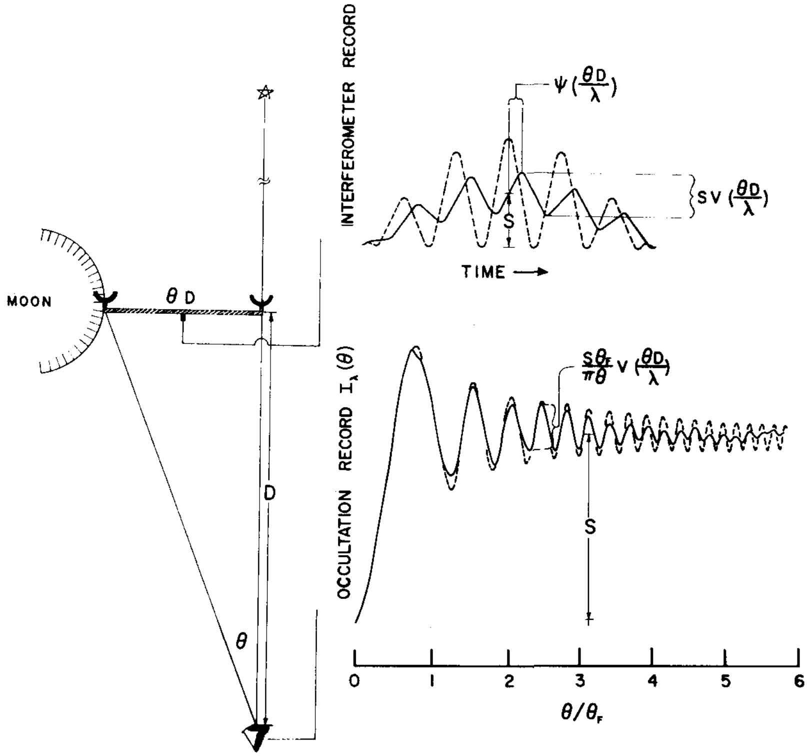

Fig. 1. Interferometer and occultation records for a point source (dashed lines) and an extended source (solid lines). At any given θ, the relative amplitude and phase of the oscillatory portion of the occultation record gives the same information as an interferometer with a baseline θD. The constant $\theta_F = (\lambda/D)^{1/2}$ is the angular radius of the first Fresnel zone.

A record of the power detected during an occultation is very similar to an interferometer record, in that it consists of oscillations the amplitude and phase of which are a function of the extent and asymmetry of the source brightness distribution. It is well known, for example, that the intensity, $I_\lambda(\theta)$, present at the Earth during the occultation of an extended source is the convolution of $p_\lambda(\theta)$, the intensity present at the Earth during the occultation of a point source, with b_λ. One would therefore expect that b_λ might be specified by comparing $I_\lambda(\theta)$ with $p_\lambda(\theta)$. In fact, at any given θ the amplitude and phase of $I_\lambda(\theta)$ differ from that expected from a point source by $V(\theta D/\lambda)$ and $\psi(\theta D/\lambda)$, respectively (cf. Lang, 1969). In particular, an occulted extended source gives rise to the intensity

$$I_\lambda(\theta) = S\left\{1 + \frac{\theta_F}{\pi\theta} V\left(\frac{\theta D}{\lambda}\right)\cos\left[2\pi\left(\frac{\theta^2 D}{2\lambda} + \frac{5}{8}\right) + \psi\left(\frac{\theta D}{\lambda}\right)\right]\right\}, \tag{3}$$

whereas the intensity from an occulted point source is specified by

$$p_\lambda(\theta) = S\left\{1 + \frac{\theta_F}{\pi\theta}\cos\left[2\pi\left(\frac{\theta^2 D}{2\lambda} + \frac{5}{8}\right)\right]\right\}, \tag{4}$$

where $\theta_F = (\lambda/D)^{1/2}$ is the angular radius of the first Fresnel zone. At any given θ, the amplitude and phase of the occultation pattern gives us the same information as that of an interferometer of baseline θD. The occultation system has the advantage that all possible baselines, and therefore all spatial Fourier components of a line-integrated brightness distribution, are present in the occultation record.

3. Resolution Limits

Because only a finite length, θ_{max}, of $I_\lambda(\theta)$ may be reliably extracted from any observation, values of $B_\lambda(s)$ will not be available for spatial frequencies larger than $s_{max} = \theta_{max} D/\lambda = \theta_{max}/\theta_F^2$. When estimating a brightness distribution by taking the inverse Fourier transform of $B_\lambda(s)$, the estimated distribution will be the convolution of b_λ with a sinc function which has a full width to half maximum of $0.3(s_{max}^{-1})$. Whether one obtains source structure by using a restoration process or by a comparison of observed and theoretical occultation curves, resolution must be limited to about s_{max}^{-1}. Various observational constraints upon the maximum detectable spatial frequency include:

A. INTEGRATING TIME

For data sampled at a rate of τ_0 sec, the fastest detectable oscillation will have a period of $2\,\tau_0$. Assuming that the rate of motion of the Moon with respect to the occulted source is $\approx 0.35''$/sec, the sampling rate limits s_{max}^{-1} to $\approx 0.7\,\tau_0''$. As an example, stars smaller than $\approx 0.7 \times 10^{-3}{}''$ cannot be resolved by occultations with a photoelectric system which has a 1 msec sampling rate.

B. SEEING

Atmospheric scintillations may cause fluctuations in the intensity of light from occulted stars. If these fluctuations have a magnitude larger than 10% of the signal level, and a period of 0.1 sec, $\theta_{max} \approx (0.025\text{ sec})\ (0.35''/\text{sec}) \approx 0.9 \times 10^{-2}{}''$. Assuming that $D \approx 3.84 \times 10^8$ m and $\lambda \approx 5000$ Å, $\theta_F \approx 0.7 \times 10^{-2}{}''$, and seeing limits angular resolution to $s_{max}^{-1} \approx 0.6 \times 10^{-2}{}''$.

C. APERTURE SIZE

The displacement of the Moon as viewed from different portions of an antenna aperture of diameter, d, will cause spatial frequencies larger than $s_{max} \approx D/d$ to be blurred in reception by the antenna. Using $D \approx 3.84 \times 10^8$ m, a 100 in. telescope has an aperture limit to resolution of $\approx 10^{-3}{}''$. The Arecibo telescope, which has a 1000-ft diameter, is similarly limited in resolution to $\approx 10^{-1}{}''$.

D. BANDWIDTH

When an occultation is observed over a narrow band of frequencies, the intensity, $p_R(\theta)$, from a point source will be given by $p_R(\theta) = S\{1 + [(p_\lambda(\theta)/S) - 1]\ X(\theta)\}$, where $X(\theta)$ is the cosine transform of $R(\lambda)$, the power response of the receiver (cf. Lang, 1969). If, for example, the receiver response around the nominal wavelength,

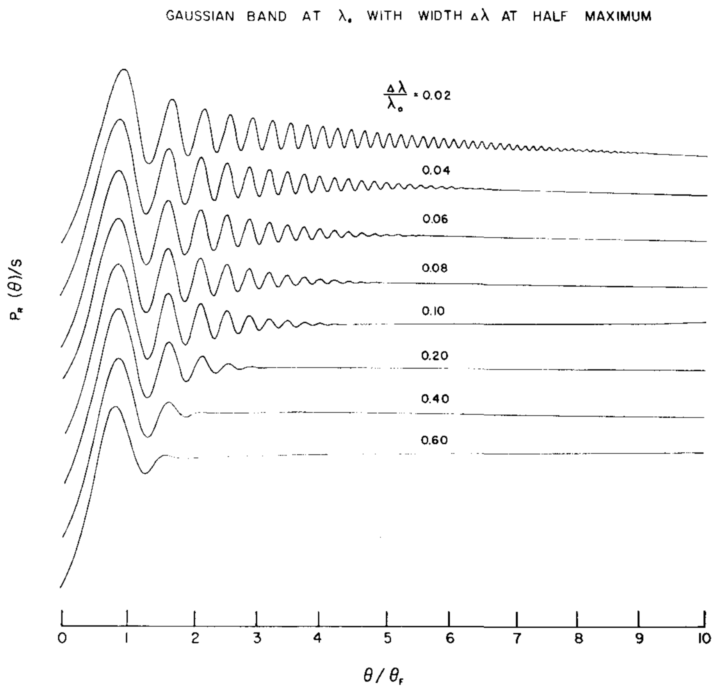

Fig. 2. The relative power, $p_R(\theta)/S$, detected when the occultation of a point source of flux density S is observed with a receiver whose power response about the nominal wavelength λ_0 is a Gaussian function of full width to half maximum, $\Delta\lambda$.

λ_0, is a Gaussian function with a full width to half maximum of $\Delta\lambda$, then $X(\theta) = \exp[-\pi^2\theta^4 D^2 (\Delta\lambda)^2/16\lambda_0^4 \ln 2] \approx 1 - \theta_F^{-4} (\Delta\lambda/\lambda_0)^2 \theta^4$. Plots of the corresponding $p_R(\theta)$ are given in Figure 2 for various values of $\Delta\lambda/\lambda_0$. These figures indicate $\theta_{\max} \approx \approx 1.4 (\Delta\lambda/\lambda_0)^{-1/2} \theta_F$ for any practical observation. Consequently, resolution will be limited to $s_{\max}^{-1} \approx 0.7 (\Delta\lambda/\lambda_0)^{1/2} \theta_F = 0.7 (\Delta\lambda/D)^{1/2}$. Using $D \approx 3.84 \times 10^8$ m, a filter with a width of 1000 Å will limit resolution to about 2×10^{-3}″. An 8 MHz filter centered at 318 MHz will limit resolution to 1.6″. These deductions presume that the source intensity is roughly the same for all wavelengths passed by the receiver. If the source only radiates in a narrow band of wavelengths, then the effective $\Delta\lambda$ will be determined by the source. In addition, a given bandwidth in MHz allows greater resolution at higher observing frequencies, which indicates that observations should be carried out at high frequencies if the intrinsic spectrum of the source is independent of frequency.

E. SIGNAL-TO-NOISE

It is apparent from Equation (3) that the maximum amplitude of $I_\lambda(\theta)$ decreases as θ increases. It becomes very difficult to detect $I_\lambda(\theta)$ when $I_\lambda(\theta) - S$ is less than five times the rms noise fluctuation, N, of the observed record. For a point source, this condition is met when $\theta_F S/\pi\theta = 5N$. The noise limit to the angular resolution of even a point source is therefore $s_{\max}^{-1} \approx 5\pi\theta_F N/S$. Because more extended sources cause $I_\lambda(\theta)$ to drop off even more rapidly with θ, the point source limit is a liberal one. In this context, narrow sources allow a higher angular resolution. Fluctuations in the signal level of recent star occultations (Evans, 1970; Nather and Evans, 1970) have a $S/N \approx 25$, which means these observations are noise limited to a resolution of about 0.4×10^{-2}″.

4. A New Method for Obtaining a Brightness Distribution

It was shown in Section 2 that the power detected during an occultation will be proportional to an $I_\lambda(\theta)$ whose oscillating component has an amplitude and phase which differ from those of a point source by the factors $V(\theta D/\lambda)$ and $\psi(\theta D/\lambda)$, respectively. In order to evaluate these factors, a new abscissa is defined as

$$X = \theta^2 D/2\lambda + \tfrac{5}{8}, \tag{5}$$

and the new function

$$I_\lambda^t(X) = [I_\lambda(\theta)/S] - 1$$

is formed. As illustrated in Figure 3, $I_\lambda^t(X)$ oscillates at nearly unity frequency. The data shown in Figure 3 compare an occultation of 3C 49 taken at Arecibo Observatory with the theoretical $p_R(\theta)/S$ of an 8 MHz bandwidth centered at 318 MHz.

In a small neighborhood of width X_L, centered at X, the in-phase and quadrature

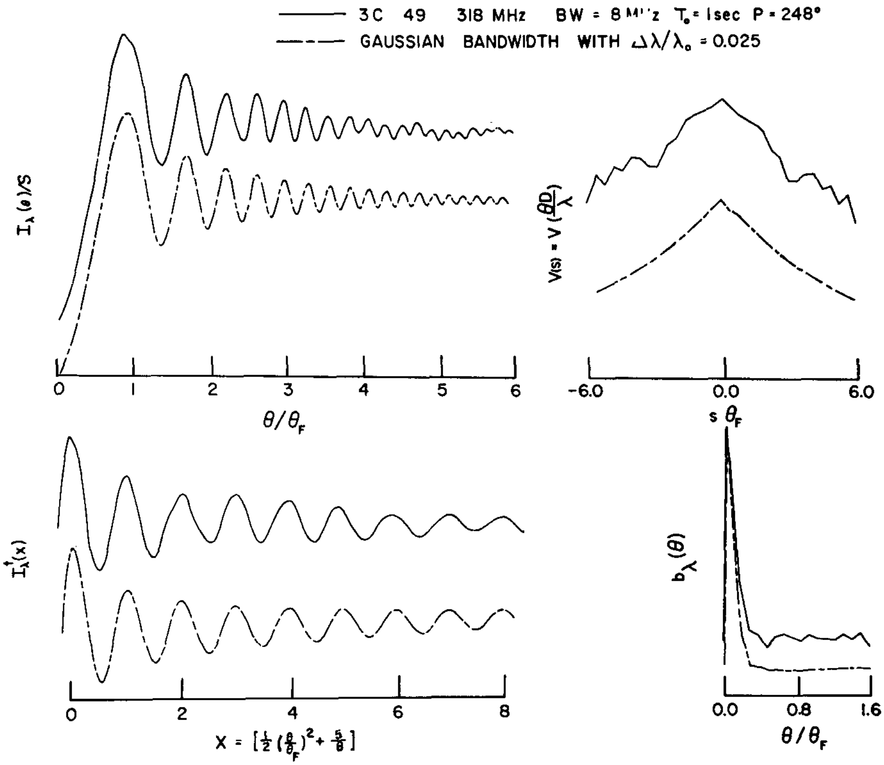

Fig. 3. Theoretical point source data for $\Delta\lambda/\lambda_0 = 0.025$ (dashed lines) compared with observed data for 3C 49 (solid lines). The actual observation, $I_\lambda(\theta)/S$, was changed to the single frequency function, $I^{t}_{\lambda}(X)$, and its envelope amplitude, $V(\theta D/\lambda)$ was evaluated at intervals of $X_L = 1$. The estimated brightness distribution, $b_\lambda(\theta)$, is the Fourier transform of $V(\theta D/\lambda)$. The $b_\lambda(\theta)$ represents the line-integrated brightness distribution along the position angle, P.

components of $I_\lambda^+(X)$ at the nominal frequency can be computed from

$$R(X) = \sum_{X - X_L/2}^{X + X_L/2} I_\lambda^t(X') \cos(2\pi X')\, \Delta X'$$

and (6)

$$I(X) = \sum_{X - X_L/2}^{X + X_L/2} I_\lambda^t(X') \sin(2\pi X')\, \Delta X'.$$

For the data shown in Figure 3, X_L was chosen to be unity and $\Delta X'$ to be 0.1. The amplitude and phase of the Fourier transform of the brightness distribution were then calculated from

$$V(\theta D/\lambda) = k\theta \{[R(X)]^2 + [I(X)]^2\}^{1/2}$$

and (7)

$$\psi(\theta D/\lambda) = \tan^{-1}[I(X)/R(X)],$$

where the constant k was chosen to make $V(\theta D/\lambda)=1$ for all θ when $I_\lambda(\theta)$ becomes $p_\lambda(\theta)$, the intensity at the Earth during the occultation of a point source. Because b_λ must be real, it is known that $V(\theta D/\lambda)=V(-\theta D/\lambda)$, and $\psi(\theta D/\lambda)=-\psi(-\theta D/\lambda)$. It is also known that $B_\lambda(\theta D/\lambda)=1$ at $\theta=0$. Because the phase was found to be constant, only $V(\theta D/\lambda)$ is shown in Figure 3.

Finally, the line-integrated brightness distribution, $b_\lambda(\theta)$, along the position angle, P, was estimated by taking the inverse Fourier transform of $B_\lambda(\theta D/\lambda)$ using the Cooley-Tukey algorithm. Figure 3 indicates that $b_\lambda(\theta)$ for 3C 49 is unresolved at the bandwidth resolution limit of 1.6″.

When converting from the time scale of an observed occultation to the angular scale of $I_\lambda(\theta)$, the wrong scaling factor may be chosen. The restored distributions will then have a negative sidelobe just as in the case of partial restorations (cf. Scheuer, 1962, and Hazard *et al.*, 1967). These sidelobes, which are very pronounced in a grazing occultation of a narrow source, indicate that either the wrong source position has been used or that the lunar limb has a slope at the occultation point. For example, restorations of radio frequency sources with resolutions narrower than 1″ will give positions accurate to 10″ when the angle of occultation is larger than 20°. A positional error of 10″ will cause the same scaling error as that caused by a limb slope of about 1°. For the occultation of radio sources, Watt's profiles (1963) give limb slopes accurate to one degree, and accurate positions may be deduced from single phase occultations. For optical occultations, limb slopes over angular distances of a few θ_F are not available, but the source positions are very accurate. In this case, limb slopes may be deduced from restorations. The correct position and/or slope is given by that restoration which is the narrowest and which exhibits no sidelobes.

5. Double Source Occultations and the Angular Separation of the Source Components

Because many star systems are binary, and because many radio sources are double, it is useful to calculate occultation curves for double sources with components of various angular separations, θ_S. The curves shown in Figure 4 assume that one component of the double source has half the flux of the other. One distinguishing feature of these curves is the sinusoidal oscillation of period $\approx 2\theta_F^2/\theta_S$. This oscillation specifies θ_S. It will not be caused by a lunar limb with an obstacle on it. Consequently, when narrow bandwidths are used, the ambiguity between double star occultations and lunar limb effects (Evans, 1970) may be resolved.

Two well-known quasi-stellar sources, 3C 273 and 3C 245, have been occulted at Arecibo Observatory. The occultation curves of both of these sources show a characteristic double structure which is also shown in the restorations (Figure 5). In both cases, a small angular size object coincides, within the positional errors, with the optical object. From this component a narrow jet-like feature extends into another narrow source. The restorations of 3C 273 compare favorably with partial restorations of the same data (Hazard *et al.*, 1966).

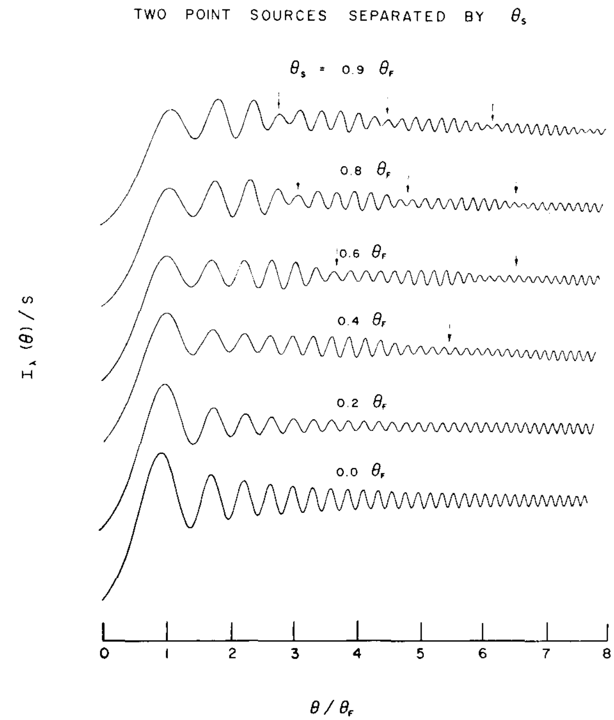

Fig. 4. The relative intensity, $I_\lambda(\theta)/S$, seen at the Earth during the occultation of two point sources with an angular spacing, θ_S, and with flux densities of 0.66 and 0.33 S. Each curve is modulated by a sinusoid whose minima are denoted by arrows and whose period is roughly $2\,\theta_F^2/\theta_S$.

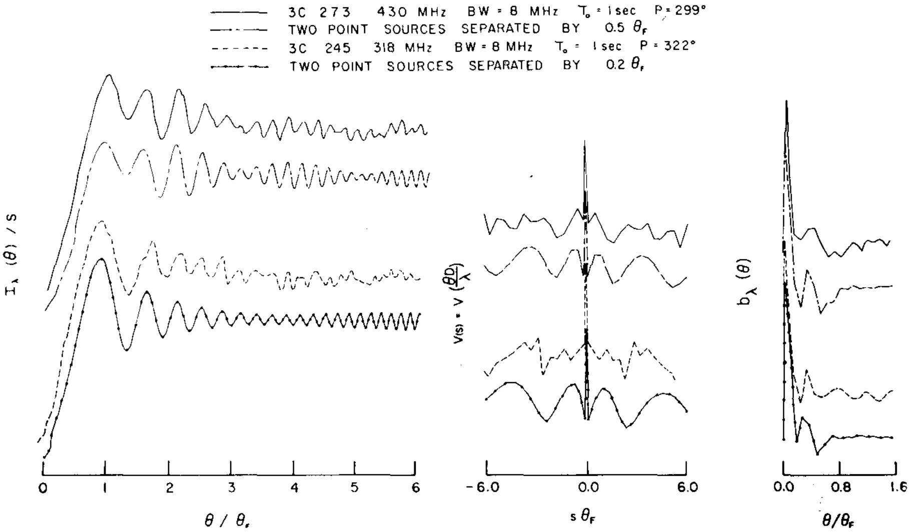

Fig. 5. Theoretical double point source data compared with observed data for 3C 273 and 3C 245. The actual observations, $I_\lambda(\theta)/S$, were changed to the single frequency X scale and the envelope amplitude, $V(\theta D/\lambda)$, was evaluated at intervals of $X_L = 1$. The estimated distributions, $b_\lambda(\theta)$, are the Fourier transforms of the $V(\theta D/\lambda)$. These $b_\lambda(\theta)$ represent the line-integrated brightness distribution along the position angle, P.

6. Methods of Estimating Brightness Distribution Widths

If we assume a model for b_λ, only the amplitude of the envelope of $I_\lambda(\theta)$ need be specified in order to fully specify b_λ. In addition, because the behavior of $b_\lambda(\theta)$ at large θ is reflected in the behavior of its transform near the origin, a measurement of the amplitude of $I_\lambda(\theta)$ near the origin of θ will specify the width of b_λ. Consequently, $I_\lambda(\theta)$ and $V(\theta D/\lambda)$ are given in Figure 6 for Gaussian sources of full width to half

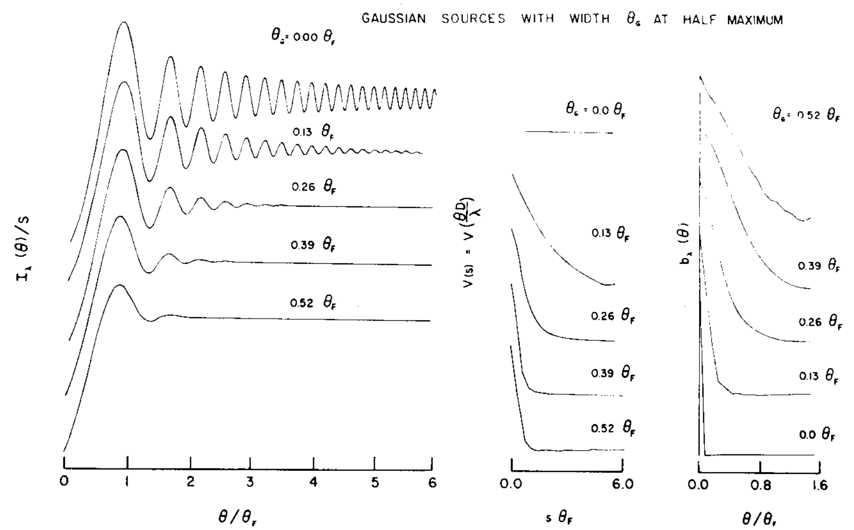

Fig. 6. The relative intensity $I_\lambda(\theta)/S$ seen at the Earth during the occultation of Gaussian sources with flux density S and full widths to half maxima, θ_G. Note that no substantial oscillations are present beyond $\theta_{\max} = \theta_F{}^2/\theta_G$. The envelope amplitudes, $V(\theta D/\lambda)$, and the estimated distributions, $b_\lambda(\theta)$, are also shown.

maximum θ_G. Once $V(\theta D/\lambda)$ is evaluated from observed data by the procedures given in Section 4, a direct comparison with the $V(\theta D/\lambda)$ of Figure 6 will determine a width. A direct comparison of observed and theoretical occultation curves will probably give the same result. For example, the theoretical curves show no substantial oscillations beyond $\theta_{\max} = \theta_F^2/\theta_G$, which means $s_{\max}^{-1} \approx \theta_G$ as expected. The measured width must be compared with the resolution limits given in Section 3, in order to determine if the width is an upper limit to the source width or if the source is actually resolved.

Acknowledgments

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