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Bi-free entropy with respect to completely positive maps

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Abstract. In this paper, a notion of non-microstate bi-free entropy with respect to completely positive maps is constructed thereby extending the notions of non-microstate bi-free entropy and free entropy with respect to a completely positive map. By extending the operator-valued bi-free structures to allow for more analytical arguments, a notion of conjugate variables is constructed using both moment and cumulant expressions. The notions of free Fisher information and entropy are then extended to this setting and used to show minima of the Fisher information and maxima of the non-microstate bi-free entropy at bi-R-diagonal elements.

1 Introduction

Free entropy was introduced in a series of papers by Voiculescu including [14, 15] that cemented the foundations of free probability and its applications to operator algebras. Of note is the non-microstate approach of [14] that generalized the notions of Fisher information and entropy to the noncommutative random variables studied in free probability by using a conjugate variable system and free Brownian motions. These ideas were further extended to the operator-valued setting by Shlyakhtenko in [10] by modifying the conjugate variable formulae to involve a completely positive map on the algebra of amalgamation. One immediate application was [10, Proposition 7.14], which obtained a formula for the Jones index of a subfactor. Furthermore, free entropy with respect to a completely positive map was essential to the work in [8], which demonstrated that minimal values for the free Fisher information and maximal values for the non-microstate free entropy existed and were obtained at R-diagonal elements.

More recently, in [16], Voiculescu extended the notion of free independence to simultaneously study the left and right actions of algebras on reduced free product spaces. In particular, this permits a notion of independence, called bi-free independence, that contains both free and classical independence (see [11]) and a free probability construction to simultaneously study both a von Neumann algebra and its commutant (see Example 2.3). Significant effort has gone into enhancing results from free probability to the bi-free setting and examining potential applications. In terms

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of entropy, Charlesworth and the second author recently developed [4, 5], thereby extending both the microstate and non-microstate free entropies to the bi-free setting.

The purpose of this paper is to further develop the theory of non-microstate bi-free entropy χ^* to incorporate the existence of a completely positive map and examine applications of said theory. In particular, our main applications are Theorem 9.5, which examines the minimal value of the bi-free Fisher information for collections of pairs of operators with similarities in their distributions, and the following theorem, which cumulates the first author's work on bi-R-diagonal elements from [7].

Theorem 1.1 *Let $(\mathcal{A}, \varphi)$ be a C^*-noncommutative probability space, and let $x, y \in \mathcal{A}$ be such that x^*x and xx^* have the same distribution with respect to φ and y^*y and yy^* have the same distribution with respect to φ. With $\tau_2 : \mathcal{A} \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{op} \to \mathbb{C}$ defined by*

$$\tau_2(T \otimes b_1 \otimes b_2) = \varphi(T)\operatorname{tr}_2(b_1 b_2)$$

and

$$X = x \otimes E_{1,2} \otimes I_2 + x^* \otimes E_{2,1} \otimes I_2 \quad \text{and} \quad Y = y \otimes I_2 \otimes E_{1,2} + y^* \otimes I_2 \otimes E_{2,1},$$

we have that

$$\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) \leq 2\chi^*(X \sqcup Y)$$

and equality holds whenever the pair (x, y) is bi-R-diagonal and alternating adjoint flipping.

This paper is structured as follows. After reviewing some preliminaries and notation pertaining to bi-free probability in Section 2, Section 3 will extend the structures used in operator-valued bi-free probability. This is necessary as expectations in operator-valued bi-free probability need not be positive and thus to perform analytical computations additional structures are required. These structures occur and are modeled based on the left and right actions of a II_1 factor on its L_2-space (see Example 3.5). By adding a tracial state on the algebra of amalgamation that satisfies certain compatibility conditions, the appropriate L_2-spaces can be constructed and used to study operator-valued bi-free probability. It is these structures that the authors believe will be vital to future applications of operator-valued bi-free probability.

In Section 4, the operator-valued bi-free cumulant functions are extended to allow for the last entry to be an element of the corresponding L_2-space. This is thematic in bi-free probability where the last entry corresponds to the location where left and right operators mix and, therefore, it is natural to extend the bi-free moment and cumulant functions to have a mixture of left and right operators in the last entry. These analytical extensions of the operator-valued bi-free cumulants are shown to have the appropriate bi-multiplicative properties by methods similar to [2].

In Section 5, the bi-free conjugate variables with respect to completely positive maps are defined via moment relations. It is then shown that these moment relations transfer to cumulant relations using the results of Section 4 and thus the natural properties of free conjugate variables extend to this setting. One technical detail in developing the bi-free entropy with respect to a completely positive map is to show that if one perturbs operators by operator-valued bi-free central limit distributions,

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then the resulting operators have bi-free conjugate variables. Thus, it is necessary to show that one can always add in the appropriate operator-valued bi-free central limit distributions into the structures from Section 3 and remain within that context. Results along these line sufficient for this paper are developed in Section 6.

In Sections 7 and 8, the bi-free Fisher information and entropy with respect to a completely positive map are defined and shown to have the desired properties from [4, 10, 14] with very similar proofs. Sections 9 and 10 extend the techniques from [8] taking into account the differences in bi-free probability to obtain the minimal value of the bi-free Fisher information of a collection of pairs of operators with similarities in their distributions and the above theorem. Finally, Section 11 outlines some other results that immediately extend from [8] to the bi-free setting using the results and techniques from this paper.

2 Preliminaries

In this section, we will remind the reader on the basic combinatorial and operator-valued structures that have been used in previous papers on bi-free independence. For a more in-depth reminder of these concepts, we refer the readers to the original papers [2, 3].

2.1 Combinatorics on the lattice of bi-non-crossing partitions

For $n \in \mathbb{N}$, the collection of all partitions on $\{1, \ldots, n\}$ is denoted by $\mathcal{P}(n)$. The elements of any $\pi \in \mathcal{P}(n)$ are called the *blocks* of π. A partial ordered on $\mathcal{P}(n)$ is defined via refinement where for $\pi, \sigma \in \mathcal{P}(n)$ we write $\pi \leq \sigma$ if every block of π is contained in a block of σ. The maximal element of $\mathcal{P}(n)$ with respect to this partial order is the partition consisting of one block and is denoted by 1_n, whereas the minimal element is the partition consisting of n-blocks and is denoted by 0_n. Note that $\mathcal{P}(n)$ becomes a lattice under this partial ordering. For $\pi, \sigma \in \mathcal{P}(n)$, the *join* of π and σ, denoted $\pi \vee \sigma$, is the minimum element of the non-empty set $\{\upsilon \in \mathcal{P}(n) \mid \upsilon \geq \pi, \sigma\}$. A partition π on $\{1, \ldots, n\}$ is said to be *non-crossing* if whenever V, W are blocks of π and $v_1, v_2 \in V, w_1, w_2 \in W$ are such that

$$v_1 < w_1 < v_2 < w_2,$$

then $V = W$. The lattice of non-crossing partitions on $\{1, \ldots, n\}$ is denoted $\mathrm{NC}(n)$.

In the bi-free setting, all operators are implicitly imbued with a direction, either left or right. Given a sequence of n operators, we will use a map $\chi \in \{\ell, r\}^n$ to distinguish whether the kth operator is a left or a right operator. Such a map automatically gives rise to a permutation s_χ on $\{1, \ldots, n\}$ defined as follows: if $\chi^{-1}(\{\ell\}) = \{i_1 < \cdots < i_p\}$ and $\chi^{-1}(\{r\}) = \{j_1 < \cdots < j_{n-p}\}$, then

$$s_\chi(k) = \begin{cases} i_k, & \text{if } k \leq p, \\ j_{n+1-k}, & \text{if } k > p. \end{cases}$$

From a combinatorial point of view, the main difference between free and bi-free probabilities arises from dealing with s_χ. The permutation s_χ naturally induces a total

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order $\prec_\chi$ on $\{1,\ldots,n\}$, henceforth referred to as the χ *-order* as follows:

$$i \prec_\chi j \iff s_\chi^{-1}(i) < s_\chi^{-1}(j).$$

Instead of reading $\{1,\ldots,n\}$ in the traditional order, this corresponds to first reading the elements of $\{1,\ldots,n\}$ labeled "ℓ" in increasing order followed by reading the elements labeled "r" in decreasing order. Note that if V is any non-empty subset of $\{1,\ldots,n\}$, the map $\chi|_V$ naturally gives rise to a map $s_{\chi|_V}$, which should be thought of as a permutation on $\{1,\ldots,|V|\}$.

Definition 2.1 Let $n \in \mathbb{N}$ and $\chi \in \{\ell, r\}^n$. A partition $\pi \in \mathcal{P}(n)$ is called *bi-non-crossing with respect to* χ if the partition $s_\chi^{-1} \circ \pi$ (i.e., the partition obtained by applying the permutation s_χ^{-1} to each entry of every block of π) is non-crossing. Equivalently, π is bi-non-crossing with respect to χ if whenever V, W are blocks of π and $v_1, v_2 \in V, w_1, w_2 \in W$ are such that

$$v_1 \prec_\chi w_1 \prec_\chi v_2 \prec_\chi w_2,$$

then $V = W$. The collection of bi-non-crossing partitions with respect to χ is denoted by $\mathrm{BNC}(\chi)$. It is clear that

$$\mathrm{BNC}(\chi) = \left\{\pi \in \mathcal{P}(n) \mid s_\chi^{-1} \circ \pi \in \mathrm{NC}(n)\right\} = \left\{s_\chi \circ \sigma \mid \sigma \in \mathrm{NC}(n)\right\}.$$

In the context when the map χ is clear, we will refer to an element of $\mathrm{BNC}(\chi)$ simply as bi-non-crossing. To each partition $\pi \in BNC(\chi)$, we can associate a "bi-non-crossing diagram" (with respect to χ) by placing nodes along two vertical lines, labeled 1 to n from top to bottom, such that the nodes on the left line correspond to those values for which $\chi(k) = \ell$ (similarly for the right), and connecting those nodes which are in the same block of π in a non-crossing manner. In particular, a partition $\pi \in \mathcal{P}(n)$ is in $\mathrm{BNC}(\chi)$ if and only if it can be drawn in this non-crossing way.

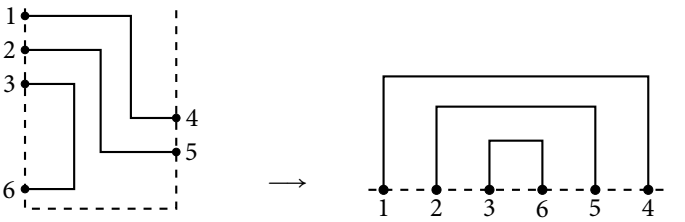
Example 2.1 If $\chi \in \{\ell, r\}^6$ is such that $\chi^{-1}(\{\ell\}) = \{1,2,3,6\}$ and $\chi^{-1}(\{r\}) = \{4,5\}$, then

$$(s_\chi(1),\ldots,s_\chi(6)) = (1,2,3,6,5,4)$$

and the partition given by

$$\pi = \{\{1,4\},\{2,5\},\{3,6\}\}$$

is bi-non-crossing with respect to χ even though $\pi \notin \mathrm{NC}(6)$. This may also be seen via the following diagrams:



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The set of bi-non-crossing partitions with respect to a map $\chi \in \{\ell, r\}^n$ automatically inherits the lattice structure from $\mathcal{P}(n)$ via the partial order of refinement. The minimal and maximal elements of $\mathrm{BNC}(\chi)$ will be denoted by 0_χ and 1_χ, respectively, and note that $0_\chi = s_\chi(0_n) = 0_n$ and $1_\chi = s_\chi(1_n) = 1_n$. For $\varnothing \neq V \subseteq \{1, \ldots, n\}$, we denote by $\min_{\leq} V$ and $\min_{\leq_\chi} V$ the minimum element of V with respect to the natural order and the χ-order of $\{1, \ldots, n\}$, respectively, with similar notation used for the maximum elements. For $\pi, \sigma \in \mathrm{BNC}(\chi)$ with $\sigma \leq \pi$, we will denote by $[\sigma, \pi]$ the interval with respect to the partial order of refinement.

Definition 2.2 The *bi-non-crossing Möbius function* is the map

$$\mu_{\mathrm{BNC}} : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times \mathrm{BNC}(\chi) \to \mathbb{Z}$$

defined recursively by

$$\sum_{\substack{\nu\in\mathrm{BNC}(n)\\ \pi\leq\nu\leq\sigma}} \mu_{\mathrm{BNC}}(\pi, \nu) = \sum_{\substack{\tau\in\mathrm{BNC}(n)\\ \pi\leq\nu\leq\sigma}} \mu_{\mathrm{BNC}}(\nu, \sigma) = \begin{cases} 1, & \text{if } \pi = \sigma \\ 0, & \text{if } \pi < \sigma \end{cases}$$

whenever $\pi \leq \sigma$ while taking the zero value otherwise.

The connection between the bi-non-crossing Möbius function and the Möbius function on the lattice of non-crossing partitions μ_{NC} is given by the formula

$$\mu_{\mathrm{BNC}}(\pi, \sigma) = \mu_{\mathrm{NC}}(s_\chi^{-1} \circ \pi, s_\chi^{-1} \circ \sigma)$$

for all $\pi \leq \sigma \in \mathrm{BNC}(\chi)$. Hence, μ_{BNC} inherits permuted analogs of the multiplicative properties of μ_{NC} (see [3, Section 3]). In particular, if $n \in \mathbb{N}$, if $\chi \in \{\ell, r\}^n$, if $\pi, \sigma \in \mathrm{BNC}(\chi)$ such that $\pi \leq \sigma$, and if $V_1, \ldots, V_m$ are unions of blocks of π that partition $\{1, \ldots, n\}$, then the natural map

$$\begin{aligned} [\pi, \sigma] &\longrightarrow \prod_{k=1}^{m} [\pi|_{V_k}, \sigma|_{V_k}] \\ \tau &\mapsto \quad (\tau|_{V_k})_{k=1}^{m} \end{aligned}$$

is a bijection and

$$\mu_{\mathrm{BNC}}(\pi, \sigma) = \mu_{\mathrm{BNC}}(\pi|_{V_1}, \sigma|_{V_1}) \cdots \mu_{\mathrm{BNC}}(\pi|_{V_m}, \sigma|_{V_m}).$$

2.2 *B-B*-noncommutative probability spaces and bi-freeness

To study operator-valued bi-free independence, certain structures are required. Thus, we shall remind the reader of the general structures as developed in [2] and refer the reader there for more details.

Definition 2.3 Let B be a unital $*$-algebra, and let B^{op} denote the unital $*$-algebra with the same elements as B with the opposite multiplication. A *B-B-noncommutative probability space* consists of a triple (A, E, ε), where A is a unital $*$-algebra, $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ is a unital $*$-homomorphism such that the restrictions $\varepsilon|_{B\otimes 1_B}$ and $\varepsilon|_{1_B\otimes B^{\mathrm{op}}}$ are both injective, and $E : A \to B$ is a unital linear map such that

$$E(\varepsilon(b_1 \otimes b_2)a) = b_1 E(a) b_2 \qquad \text{and} \qquad E(a\varepsilon(b \otimes 1_B)) = E(a\varepsilon(1_B \otimes b)),$$

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for all $b, b_1, b_2 \in B$ and $a \in A$. In addition, consider the unital $*$-subalgebras A_ℓ and A_r of A given by

$$A_\ell = \{a \in A \mid a\varepsilon(1_B \otimes b) = \varepsilon(1_B \otimes b)a \text{ for all } b \in B\}$$

and

$$A_r = \{a \in A \mid a\varepsilon(b \otimes 1_B) = \varepsilon(b \otimes 1_B)a \text{ for all } b \in B\}.$$

We call A_ℓ and A_r the *left and right algebras of* A, respectively.

Note that one can always assume that a B-B-noncommutative probability space is generated as a $*$-algebra by A_ℓ and A_r.

Example 2.2 Let $\mathcal{A}$ and B be unital $*$-algebras, and let $\varphi : \mathcal{A} \to \mathbb{C}$ be a unital, linear map. If $A = \mathcal{A} \otimes B \otimes B^{\text{op}}$, if $\varepsilon : B \otimes B^{\text{op}} \to A$ is defined by $\varepsilon(b_1 \otimes b_2) = 1_{\mathcal{A}} \otimes b_1 \otimes b_2$ for all $b_1, b_2 \in B$, and $E : A \to B$ is defined by

$$E(a \otimes b_1 \otimes b_2) = \varphi(a)b_1b_2$$

for all $a \in A$ and $b_1, b_2 \in B$, then (A, E, ε) is a B-B-noncommutative probability space. Indeed, clearly, ε is a unital injective $*$-homomorphism. Furthermore, note for all $Z \in \mathcal{A}$ and $b, b_1, b_2, b_3, b_4 \in B$ that

$$E((1_{\mathcal{A}} \otimes b_1 \otimes b_2)(Z \otimes b_3 \otimes b_4)) = \varphi(Z)b_1b_3b_4b_2 = b_1E(Z \otimes b_3 \otimes b_4)b_2$$

and

$$E((Z \otimes b_1 \otimes b_2)(1_{\mathcal{A}} \otimes b \otimes 1_B)) = \varphi(Z)b_1bb_2 = E((Z \otimes b_1 \otimes b_2)(1_{\mathcal{A}} \otimes 1_B \otimes b)).$$

Hence, E satisfies the required properties.

For future use, notice that

$$\mathcal{A} \otimes B \otimes 1_B \subseteq \mathcal{A}_\ell \qquad \text{and} \qquad \mathcal{A} \otimes 1_B \otimes B^{\text{op}} \subseteq \mathcal{A}_r.$$

Moreover, in the case $B = \mathbb{C}$, (A, E, ε) efficiently reduces down to $(\mathcal{A}, \varphi)$, the usual notion of a noncommutative probability space.

Example 2.3 Let $\mathfrak{M}$ be a finite von Neumann algebra with a tracial state $\tau : \mathfrak{M} \to \mathbb{C}$, and let $L_2(\mathfrak{M}, \tau)$ be the Gelfand-Naimark-Segal Hilbert space generated by $(\mathfrak{M}, \tau)$. For $T \in \mathfrak{M}$, let L_T denote the left action of T on $L_2(\mathfrak{M}, \tau)$, and let R_T denote the right action of T on $L_2(\mathfrak{M}, \tau)$. Furthermore, let A be the algebra generated by $\{L_T, R_T \mid T \in \mathfrak{M}\}$.

Let B be a unital von Neumann subalgebra of $\mathfrak{M}$, and let $E_B : \mathfrak{M} \to B$ be the conditional expectation of $\mathfrak{M}$ onto B. Recall that if $P : L_2(\mathfrak{M}, \tau) \to L_2(B, \tau)$ is the orthogonal projection of $L_2(\mathfrak{M}, \tau)$ onto $L_2(B, \tau)$, then $E_B(Z) = P(Z1_{\mathfrak{M}})$ for all $Z \in \mathfrak{M}$.

Define $\varepsilon : B \otimes B^{\text{op}} \to A$ by $\varepsilon(b_1 \otimes b_2) = L_{b_1}R_{b_2}$ and define $E : A \to B$ by

$$E(Z) = P(Z1_{\mathfrak{M}})$$

for all $Z \in A$. Elementary von Neumann algebra theory implies that the range of E is indeed contained in B. To see that (A, E, ε) is a B-B-noncommutative probability space, first note that ε is clearly a unital $*$-homomorphism that is injective when

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restricted to $B \otimes 1_B$ and when restricted to $1_B \otimes B^{\mathrm{op}}$. Moreover, note for all $Z \in A$ and $b, b_1, b_2 \in B$ that

$$E(L_{b_1} R_{b_2} Z) = P(b_1(Z1_{\mathfrak{M}})b_2) = b_1 P(Z1_{\mathfrak{M}})b_2 = b_1 E(Z) b_2$$

and

$$E(TL_b) = P(TL_b 1_{\mathfrak{M}}) = P(Tb) = P(TR_b 1_{\mathfrak{M}}) = E(TR_b).$$

Hence, E satisfies the required properties.

The map $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ encodes the left and right elements of B in A. For notational purposes, for each $b \in B$, we will denote $\varepsilon(b \otimes 1_B)$ and $\varepsilon(1_B \otimes b)$ by L_b and R_b, respectively, and we denote

$$B_\ell = \varepsilon(B \otimes 1_B) = \{L_b \mid b \in B\} \quad \text{and} \quad B_r = \varepsilon(1_B \otimes B^{\mathrm{op}}) = \{R_b \mid b \in B\}.$$

To examine bi-free independence with amalgamation over B, it is necessary that left operators are contained in A_ℓ (i.e., commute with the right copy of B) and right operators are contained in A_r (i.e., commute with the left copy of B).

Definition 2.4 [2] Let (A, E, ε) be a B-B-noncommutative probability space.

(1) A *pair of B-algebras* is a pair (C, D) of unital subalgebras of A such that

$$B_\ell \subseteq C \subseteq A_\ell \quad \text{and} \quad B_r \subseteq D \subseteq A_r.$$

(2) A family $\{(C_k, D_k)\}_{k \in K}$ of pairs of B-algebras in A is called *bi-free with amalgamation over B* if there exist B-B-bimodules with specified B-vector states $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k \in K}$ and unital homomorphisms

$$l_k : C_k \to \mathcal{L}_\ell(\mathcal{X}_k) \quad \text{and} \quad r_k : D_k \to \mathcal{L}_r(\mathcal{X}_k),$$

such that the joint distribution of the family $\{(C_k, D_k)\}_{k \in K}$ with respect to E coincides with the joint distribution of the images

$$\{((\lambda_k \circ l_k)(C_k), (\rho_k \circ r_k)(D_k))\}_{k \in K}$$

in the space $\mathcal{L}(*_{k \in K} \mathcal{X}_k)$ with respect to $E_{\mathcal{L}(*_{k \in K} \mathcal{X}_k)}$, where $*_{k \in K} \mathcal{X}_k$ is the reduced free product of $\{(\mathcal{X}_k, \mathring{\mathcal{X}}_k, p_k)\}_{k \in K}$ with amalgamation over B, λ_k is the left-regular representation of $\mathcal{L}_\ell(\mathcal{X}_k)$ on $*_{k \in K} \mathcal{X}_k$, and ρ_k is the right-regular representation of $\mathcal{L}_r(\mathcal{X}_k)$ on $*_{k \in K} \mathcal{X}_k$.

Remark 2.4 Let $\mathcal{A}$ and B be unital $*$-algebras, and let $\varphi : \mathcal{A} \to \mathbb{C}$ be a unital linear map. Let (A, E, ε) be as in Example 2.2. By [5, 11], if $\{(C_k, D_k)\}$ are $*$-subalgebras of $\mathcal{A}$ that are bi-free with respect to φ, then $\{(C_k \otimes B \otimes 1_B, D_k \otimes 1_B \otimes B^{\mathrm{op}})\}_{k \in K}$ are bi-free with amalgamation over B with respect to E. Thus, Example 2.2 is the correct notion of "inflating $(\mathcal{A}, \varphi)$ by B" in the bi-free setting.

Example 2.5 Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be finite von Neumann algebras with a common von Neumann subalgebra B and tracial states τ_1 and τ_2, respectively, such that $\tau_1|_B = \tau_2|_B$. Let $\mathfrak{M} = \mathfrak{M}_1 *_B \mathfrak{M}_2$ be the reduced free product von Neumann algebra with amalgamation over B, let $E_B : \mathfrak{M} \to B$ be the conditional expectation of $\mathfrak{M}$ onto B,

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and let $\tau = \tau_1 * \tau_2 = \tau_1|_B \circ E_B$ be the tracial state on $\mathfrak{M}$. If E and ε are as in Example 2.3 for $(\mathfrak{M}, \tau)$, then

$$\{(\{L_X \mid X \in \mathfrak{M}_1\}, \{R_Y \mid Y \in \mathfrak{M}_1\})\} \qquad \text{and} \qquad \{(\{L_X \mid X \in \mathfrak{M}_2\}, \{R_Y \mid Y \in \mathfrak{M}_2\})\}$$

are bi-free with amalgamation over B.

In order to study bi-free independence with amalgamation, the operator-valued bi-free moment and cumulant functions are key. These functions have specific properties that are described via the following concept. In what follows and for the remainder of the paper, given an n-tuple $(Z_1, \ldots, Z_n)$ and $V \subseteq \{1, \ldots, n\}$, we will use $(Z_1, \ldots, Z_n)|_V$ to denote the $|V|$-tuple where only the entries Z_k where $k \in V$ remain.

Definition 2.5 [2, Definition 4.2.1] Let (A, E, ε) be a B-B-noncommutative probability space. A map

$$\Phi : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times A_{\chi(2)} \times \cdots \times A_{\chi(n)} \to B$$

is called *bi-multiplicative* if it is $\mathbb{C}$-linear in each of the $A_{\chi(k)}$ entries and for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $b \in B$, and $Z_k \in A_{\chi(k)}$, the following four conditions hold:

(1) Let

$$q = \max_{\leq} \{k \in \{1, \ldots, n\} \mid \chi(k) \neq \chi(n)\}.$$

If $\chi(n) = \ell$, then

$$\Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n L_b) = \begin{cases} \Phi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q R_b, Z_{q+1}, \ldots, Z_n), & \text{if } q \neq -\infty, \\ \Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n)b, & \text{if } q = -\infty. \end{cases}$$

If $\chi(n) = r$, then

$$\Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n R_b) = \begin{cases} \Phi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q L_b, Z_{q+1}, \ldots, Z_n), & \text{if } q \neq -\infty, \\ b\Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n), & \text{if } q = -\infty. \end{cases}$$

(2) Let $p \in \{1, \ldots, n\}$, and let

$$q = \max_{\leq} \{k \in \{1, \ldots, n\} \mid \chi(k) = \chi(p), k < p\}.$$

If $\chi(p) = \ell$, then

$$\Phi_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_b Z_p, Z_{p+1} \ldots, Z_n) = \begin{cases} \Phi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q L_b, Z_{q+1}, \ldots, Z_n), & \text{if } q \neq -\infty, \\ b\Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n), & \text{if } q = -\infty. \end{cases}$$

If $\chi(p) = r$, then

$$\Phi_{1_\chi}(Z_1, \ldots, Z_{p-1}, R_b Z_p, Z_{p+1} \ldots, Z_n) = \begin{cases} \Phi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q R_b, Z_{q+1}, \ldots, Z_n), & \text{if } q \neq -\infty, \\ \Phi_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n)b, & \text{if } q = -\infty. \end{cases}$$

(3) Suppose $V_1, \ldots, V_m$ are unions of blocks of π that partition $\{1, \ldots, n\}$ with each being a χ-interval (i.e., an interval in the χ-ordering) and the sets $V_1, \ldots, V_m$ are ordered by $\preceq_\chi$ (i.e., $(\min_{\preceq_\chi} V_k) \prec_\chi (\min_{\preceq_\chi} V_{k+1})$ for all k). Then,

$$\Phi_\pi(Z_1, \ldots, Z_n) = \Phi_{\pi|_{V_1}}((Z_1, \ldots, Z_n)|_{V_1}) \cdots \Phi_{\pi|_{V_m}}((Z_1, \ldots, Z_n)|_{V_m}).$$

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(4) Suppose that V and W are unions of blocks of π that partition $\{1,\dots,n\}$, V is a χ-interval, and $s_\chi(1), s_\chi(n) \in W$. Let

$$p = \max_{\preceq_\chi}\left\{k \in W \;\middle|\; k \prec_\chi \min_{\preceq_\chi} V\right\} \qquad \text{and} \qquad q = \min_{\preceq_\chi}\left\{k \in W \;\middle|\; \max_{\preceq_\chi} V \prec_\chi k\right\}.$$

Then, we have that

$$\begin{aligned}\Phi_\pi(Z_1,\dots,Z_n) &= \begin{cases}\Phi_{\pi|_W}\left((Z_1,\dots,Z_{p-1},Z_pL_{\Phi_{\pi|_V}((Z_1,\dots,Z_n)|_V)},Z_{p+1},\dots,Z_n)|_W\right), & \text{if } \chi(p)=\ell,\\ \Phi_{\pi|_W}\left((Z_1,\dots,Z_{p-1},R_{\Phi_{\pi|_V}((Z_1,\dots,Z_n)|_V)}Z_p,Z_{p+1},\dots,Z_n)|_W\right), & \text{if } \chi(p)=r,\end{cases}\\ &= \begin{cases}\Phi_{\pi|_W}\left((Z_1,\dots,Z_{q-1},L_{\Phi_{\pi|_V}((Z_1,\dots,Z_n)|_V)}Z_q,Z_{q+1},\dots,Z_n)|_W\right), & \text{if } \chi(q)=\ell,\\ \Phi_{\pi|_W}\left((Z_1,\dots,Z_{q-1},Z_qR_{\Phi_{\pi|_V}((Z_1,\dots,Z_n)|_V)},Z_{q+1},\dots,Z_n)|_W\right), & \text{if } \chi(q)=r.\end{cases}\end{aligned}$$

Given a B-B-noncommutative probability space (A,E,ε), the moment and cumulant functions are well-defined bi-multiplicative functions.

Definition 2.6 Let (A,E,ε) be a B-B-noncommutative probability space.

(1) The *operator-valued bi-free moment function*

$$E : \bigcup_{n\in\mathbb{N}}\bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi)\times A_{\chi(1)}\times\cdots\times A_{\chi(n)} \to B$$

is the bi-multiplicative function (see [2, Theorem 5.1.4]) that satisfies

$$E_{1_\chi}(Z_1,Z_2,\dots,Z_n) = E(Z_1Z_2\cdots Z_n),$$

for all $n\in\mathbb{N}$, $\chi\in\{\ell,r\}^n$, and $Z_k\in A_{\chi(k)}$.

(2) The *operator-valued bi-free cumulant function*

$$\kappa^B : \bigcup_{n\in\mathbb{N}}\bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi)\times A_{\chi(1)}\times\cdots\times A_{\chi(n)} \to B$$

is the bi-multiplicative function (see [2, Corollary 6.2.2]) defined by

$$\kappa^B_\pi(Z_1,\dots,Z_n) = \sum_{\substack{\sigma\in\mathrm{BNC}(\chi)\\ \sigma\le\pi}} E_\sigma(Z_1,\dots,Z_n)\mu_{\mathrm{BNC}}(\sigma,\pi),$$

for each $n\in\mathbb{N}$, $\chi\in\{\ell,r\}^n$, $\pi\in\mathrm{BNC}(\chi)$, and $Z_k\in A_{\chi(k)}$. In the special case when $\pi = 1_\chi$, the map $\kappa^B_{1_\chi}$ is simply denoted by κ^B_χ. An instance of Möbius inversion yields that the equality

$$E_\sigma(Z_1,\dots,Z_n) = \sum_{\substack{\pi\in\mathrm{BNC}(\chi)\\ \pi\le\sigma}} \kappa^B_\pi(Z_1,\dots,Z_n)$$

holds for all $n\in\mathbb{N}$, $\chi\in\{\ell,r\}^n$, $\sigma\in\mathrm{BNC}(\chi)$, and $Z_k\in A_{\chi(k)}$.

The condition of bi-freeness with amalgamation over B for a family of pairs of B-faces is equivalent to the vanishing of their mixed operator-valued bi-free cumulants, as the following result indicates.

Theorem 2.6 [2, Theorem 8.1.1] *Let (A,E,ε) be a B-Bnon-commutative probability space, and let $\{(C_k,D_k)\}_{k\in K}$ be a family of pairs of B-algebras in A. The following are equivalent:*

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(1) *The family* $\{(C_k, D_k)\}_{k\in K}$ *is bi-free with amalgamation over* B.
(2) *For all* $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $Z_1, \ldots, Z_n \in A$, *and nonconstant maps* $\gamma : \{1, \ldots, n\} \to K$ *such that*

$$Z_k \in \begin{cases} C_{\gamma(k)}, & \text{if } \chi(k) = \ell, \\ D_{\gamma(k)}, & \text{if } \chi(k) = r, \end{cases}$$

we have that

$$\kappa_\chi^B(Z_1, \ldots, Z_n) = 0.$$

3 Analytical *B*-*B*-noncommutative probability spaces

The notion of a B-B-noncommutative probability space is purely an algebraic construction. In order to perform the more analytical computations necessary in this paper, additional structure is needed. These structures are analogous to those observed in Example 2.3 and will be seen to be the correct enhancement of a B-B-noncommutative probability space to perform functional analysis.

Given a unital $*$-algebra A, by a state $\tau : A \to \mathbb{C}$, we will always mean a unital, linear functional with the property that $\tau(a^*a) \geq 0$ for all $a \in A$. If $N_\tau = \{a \in A : \tau(a^*a) = 0\}$, then $L_2(A, \tau)$ will denote the Hilbert space completion of the quotient space A/N_τ with respect to the inner product induced by τ given by

$$\langle a_1 + N_\tau, a_2 + N_\tau \rangle = \tau(a_2^* a_1),$$

for all $a_1, a_2 \in A$ and $\|\cdot\|_\tau$ will denote the Hilbert space norm on $L_2(A, \tau)$.

Definition 3.1 Given a unital $*$-algebra B, an *analytical B-B-noncommutative probability space* consists of a tuple $(A, E, \varepsilon, \tau)$ such that:

(1) (A, E, ε) is a B-B-noncommutative probability space,
(2) $\tau : A \to \mathbb{C}$ is a state that is compatible with E; that is,

$$\tau(a) = \tau\left(L_{E(a)}\right) = \tau\left(R_{E(a)}\right)$$

for all $a \in A$,
(3) the canonical state $\tau_B : B \to \mathbb{C}$ defined by $\tau_B(b) = \tau(L_b)$ for all $b \in B$ is tracial,
(4) left multiplication of A on A/N_τ are bounded linear operators and thus extend to bounded linear operators on $L_2(A, \tau)$, and
(5) E is completely positive when restricted to A_ℓ and when restricted to A_r.

Remark 3.1 Given an analytical B-B-noncommutative probability space $(A, E, \varepsilon, \tau)$, note the following.

(1) The fact that τ_B is a state immediately follows from the fact that τ is a state and ε is a $*$-homomorphism. Specifically, for positivity, notice for all $b \in B$ that

$$\tau_B(b^*b) = \tau(L_{b^*b}) = \tau((L_b)^* L_b) \geq 0.$$

(2) Note for all $b \in B$ that

$$\|b + N_{\tau_B}\|_{\tau_B}^2 = \tau_B(b^*b) = \tau(L_{b^*b}) = \|L_b + N_\tau\|_\tau^2.$$

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Hence, the map from B/N_{τ_B} to $L_2(A,\tau)$ defined by

$$b + N_{\tau_B} \mapsto L_b + N_\tau$$

for all $b \in B$ is a well-defined, linear isometry. Therefore, a standard density argument yields that

$$L_2(B,\tau_B) \cong \overline{\{L_b + N_\tau \mid b \in B\}}^{\|\cdot\|_\tau} \subseteq L_2(A,\tau).$$

Henceforth, we shall only be making reference to the space $L_2(B,\tau_B)$ via this identification.

(3) The state τ naturally extends to a linear functional on $L_2(A,\tau)$ by defining

$$\tau(\xi) = \langle \xi, 1_A + N_\tau \rangle_{L_2(A,\tau)}$$

for all $\xi \in L_2(A,\tau)$. Similarly, the scalar $\tau_B(\zeta) = \tau(\zeta)$ is well defined for any $\zeta \in L_2(B,\tau_B)$.

(4) As left multiplication by A on A/N_τ is bounded, we immediately extend the left multiplication map to obtain a unital $*$-homomorphism from A into $\mathcal{B}(L_2(A))$. Thus, $a\xi$ is a well-defined element of $L_2(A,\tau)$ for all $a \in A$ and $\xi \in L_2(A,\tau)$.

(5) The requirement of the left multiplication inducing bounded operators is immediate in the case when A is a C*-algebra; however, it also holds in more general situations. For instance, when A is a unital $*$-algebra generated by its partial isometries, the left multiplication map is automatically bounded (see [9, Exercise 7.22]).

(6) Since the state τ_B is assumed to be tracial, right multiplication of B on B/N_{τ_B} is also bounded. Thus, for any $b_1, b_2 \in B$ and $\zeta \in L_2(B,\tau_B)$, we have that $b_1\zeta b_2$ is a well-defined element of $L_2(B,\tau_B)$ and, in $L_2(A,\tau)$, $L_{b_1}R_{b_2}\zeta = b_1\zeta b_2$. Furthermore, note that left and right multiplication of B on $L_2(B,\tau_B)$ are commuting $*$-homomorphisms.

(7) For all $a \in A$ and $b \in B$, we automatically have $\tau(aL_b) = \tau(aR_b)$, as τ is compatible with E. Indeed,

$$\tau(aL_b) = \tau(L_{E(aL_b)}) = \tau(L_{E(aR_b)}) = \tau(aR_b),$$

as desired. Hence, $L_b + N_\tau = R_b + N_\tau$ for all $b \in B$.

In some cases, property (v) of Definition 3.1 is redundant.

Lemma 3.2 *Let $(A, E, \varepsilon, \tau)$ satisfy assumptions (i)–(iv) of Definition 3.1. If B is a C*-algebra and τ_B is faithful, then property (v) of Definition 3.1 holds.*

Proof To see that E is completely positive on A_ℓ, let $d \in \mathbb{N}$ and $A = [a_{i,j}] \in M_d(A_\ell)$. To verify that $E_d(A^*A) \geq 0$ in B, as B is a C*-algebra and τ_B is faithful, it suffices to show for all $h = (b_1, \ldots, b_d) \in B^d$ that

$$\langle E_d(A^*A)h, h \rangle_{L_2(B,\tau_B)^{\oplus d}} \geq 0.$$

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Note that

$$\begin{aligned}
\langle E_d(A^*A)h, h\rangle_{L_2(B,\tau_B)^{\oplus d}} &= \sum_{i,j,k=1}^{d} \tau_B\left(b_i^* E(a_{k,i}^* a_{k,j}) b_j\right) \\
&= \sum_{i,j,k=1}^{d} \tau_B\left(E(R_{b_j} L_{b_i^*} a_{k,i}^* a_{k,j})\right) \\
&= \sum_{i,j,k=1}^{d} \tau_B\left(E(L_{b_i^*} a_{k,i}^* a_{k,j} R_{b_j})\right) \\
&= \sum_{i,j,k=1}^{d} \tau_B\left(E(L_{b_i}^* a_{k,i}^* a_{k,j} L_{b_j})\right) \\
&= \sum_{i,j,k=1}^{d} \tau\left(L_{b_i}^* a_{k,i}^* a_{k,j} L_{b_j}\right) \\
&= \sum_{k=1}^{d} \tau\left(c_k^* c_k\right),
\end{aligned}$$

where $c_k = \sum_{j=1}^{d} a_{k,j} L_{b_j}$. Hence, as τ is positive and the computation for A_r is similar, the result follows. ∎

At this point, let us revisit Examples 2.2 and 2.3 to provide the canonical examples of analytical B-B-noncommutative probability spaces.

Example 3.3 Let $\mathcal{A}$ and B be unital C*-algebras, and let $\varphi : \mathcal{A} \to \mathbb{C}$ be a state. Recall from Example 2.2 that (A, E, ε) is a B-B-noncommutative probability space where $A = \mathcal{A} \otimes B \otimes B^{\mathrm{op}}$, $\varepsilon : B \otimes B^{\mathrm{op}} \to A$ is the natural embedding, and $E : A \to B$ is defined by

$$E(Z \otimes b_1 \otimes b_2) = \varphi(Z) b_1 b_2,$$

for all $Z \in \mathcal{A}$ and $b_1, b_2 \in B$.

Let $\tau_B : B \to \mathbb{C}$ be any tracial state. Extend τ_B to a linear map $\tau : A \to \mathbb{C}$ by defining

$$\tau(Z \otimes b_1 \otimes b_2) = \tau_B(E(Z \otimes b_1 \otimes b_2)) = \varphi(Z)\tau_B(b_1 b_2),$$

for all $Z \otimes b_1 \otimes b_2 \in A$. We claim that $(A, E, \varepsilon, \tau)$ is an analytical B-B-noncommutative probability space. To see this, it suffices to prove that τ is a state that is compatible with E, since $\mathcal{A}$ and B being unital C*-algebras automatically imply that left multiplication will be bounded on $L_2(A, \tau)$, and Lemma 3.2 implies that E is completely positive when restricted to A_ℓ or A_r (or one may simply use the fact that states are completely positive).

Clearly, τ is a unital, linear map that is compatible with E. To see that τ is positive, let $(Z_i)_{i=1}^n \subseteq \mathcal{A}$, $(b_k)_{k=1}^n, (c_k)_{k=1}^n \subseteq B$, and

$$a = \sum_{k=1}^{n} Z_k \otimes b_k \otimes c_k \in A.$$

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To see that $\tau(a^*a) \geq 0$, note that

$$\begin{aligned}\tau(a^*a) &= \sum_{i,j=1}^{n} \tau(Z_i^* Z_j \otimes b_i^* b_j \otimes c_j c_i^*) \\ &= \sum_{i,j=1}^{n} \varphi(Z_i^* Z_j)\tau_B(b_i^* b_j c_j c_i^*) \\ &= \sum_{i,j=1}^{n} \varphi(Z_i^* Z_j)\tau_B(c_i^* b_i^* b_j c_j) \\ &= \sum_{i,j=1}^{n} \varphi(Z_i^* Z_j)\tau_B((b_i c_i)^*(b_j c_j)),\end{aligned}$$

with the third equality being due to the fact that τ_B is tracial. Observe that the matrices

$$[Z_i^* Z_j] \qquad \text{and} \qquad [(b_i c_i)^*(b_j c_j)]$$

are positive in $M_n(\mathcal{A})$ and $M_n(B)$, respectively. Therefore, as states on C*-algebras are completely positive, this implies that the matrices

$$[\varphi(Z_i^* Z_j)] \qquad \text{and} \qquad [\tau_B((b_i c_i)^*(b_j c_j))]$$

are positive in $M_n(\mathbb{C})$. Consequently,

$$[\varphi(Z_i^* Z_j)\tau_B((b_i c_i)^*(b_j c_j))]$$

is also positive being the Schur product of positive matrices (see, for instance, [9, Lemma 6.11]). Therefore, as the sum of all entries of a positive matrix equals a positive scalar, we obtain that $\tau(a^*a) \geq 0$. Hence, $(A, E, \varepsilon, \tau)$ is an analytical B-B-noncommutative probability space.

Remark 3.4 Note that Example 3.3 demonstrates that E need not be a positive map on A since the product of two positive matrices need not be positive. Thus, even if $\tau_B : B \to \mathbb{C}$ is defined to be a state, $\tau_B \circ E$ may not be for an arbitrary A.

Example 3.5 For a finite von Neumann algebra $\mathfrak{M}$ with a unital von Neumann subalgebra B and tracial state τ, let (A, E, ε) be the B-B-noncommutative probability space as in Example 2.3. Note that τ extends to a unital linear map $\tau_A : A \to \mathbb{C}$ defined by

$$\tau_A(T) = \langle T1_{\mathfrak{M}}, 1_{\mathfrak{M}} \rangle_{L_2(\mathfrak{M},\tau)}$$

for all $T \in A$. Clearly, τ_A is a state as $A \subseteq \mathcal{B}(L_2(\mathfrak{M}, \tau))$ and τ_A is a vector state. Furthermore, notice that

$$\tau_A(T) = \langle P(T1_{\mathfrak{M}}), 1_{\mathfrak{M}} \rangle_{L_2(\mathfrak{M},\tau)} = \langle L_{E(T)}1_{\mathfrak{M}}, 1_{\mathfrak{M}} \rangle_{L_2(\mathfrak{M},\tau)} = \tau_A(L_{E(T)})$$

for all $T \in A$ and $\tau_A(T) = \tau_A(R_{E(T)})$ by a similar computation. Finally, as

$$\tau_A(L_b) = \langle b, 1_{\mathfrak{M}} \rangle_{L_2(\mathfrak{M},\tau)} = \tau(b)$$

for all $b \in B$, we see that $\tau_B = \tau_A \circ E$ is tracial on B as τ is tracial. Again, we automatically have that left multiplication will be bounded on $L_2(A, \tau)$ and that E is completely

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positive when restricted (as they are the conditional expectations of a copy of $\mathfrak{M}$ onto B). Hence, $(A, E, \varepsilon, \tau_A)$ is an analytical B-B-noncommutative probability space.

As motivated by Example 3.5, it is natural in an analytical B-B-noncommutative probability space to extend the expectation $E : A \to B$ to a map from $L_2(A, \tau)$ to $L_2(B, \tau_B)$ via orthogonal projection. From this point onward, for $a \in A$, we will often denote the coset $a + N_\tau$ simply by a and, for $b \in B$, we will often denote the coset $b + N_{\tau_b}$ by $\widehat{b}$. Note that if τ_B is faithful, then the map $b \mapsto \widehat{b}$ is a bijection.

Proposition 3.6 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space. If $\widetilde{E} : L_2(A, \tau) \to L_2(B, \tau_B)$ denotes the orthogonal projection, then*

$$\widetilde{E}(a) = \widehat{E(a)}$$

for all $a \in A$. In particular, when τ_B is faithful, $\widetilde{E}$ extends E.

Proof Notice for all $a \in A$ and $b \in B$ that

$$\begin{aligned}\left\langle a - \widehat{E(a)}, L_b \right\rangle_{L_2(B,\tau_B)} &= \langle L_{b^*}(a - L_{E(a)})1_A, 1_A \rangle_{L_2(A,\tau)} \\ &= \tau\left(L_{b^*}(a - L_{E(a)})\right) \\ &= \tau(L_{b^*}a) - \tau\left(L_{b^*}L_{E(a)}\right) \\ &= \tau\left(L_{E(L_{b^*}a)}\right) - \tau\left(L_{b^*E(a)}\right) = 0.\end{aligned}$$

Since b was arbitrary, the element $a - \widehat{E(a)}$ is orthogonal to $L_2(B, \tau_B)$ and hence $\widetilde{E}(a) = \widehat{E(a)}$. ∎

Remark 3.7 Notice in Proposition 3.6 that if B is finite-dimensional and the trace $\tau_B : B \to \mathbb{C}$ is faithful, then $L_2(B, \tau_B) \cong B$, so $E : A \to B$ extends to a map from $L_2(A, \tau)$ into B.

Of course, $\widetilde{E}$ inherits many properties that E is required to have.

Proposition 3.8 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, and let $\widetilde{E} : L_2(A, \tau) \to L_2(B, \tau_B)$ denote the orthogonal projection. For $a \in A$, $b, b_1, b_2 \in B$, $\xi, \xi_1, \xi_2 \in L_2(A, \tau)$, and $\zeta \in L_2(B, \tau_B)$, the following hold:*

(1) $\tau(\xi) = \tau_B\left(\widetilde{E}(\xi)\right)$.
(2) $\widetilde{E}(aL_b) = \widetilde{E}(aR_b)$.
(3) $\widetilde{E}(L_{b_1}R_{b_2}\xi) = b_1\widetilde{E}(\xi)b_2$.
(4) *If $a \in A_\ell$, then* $\widetilde{E}(a\zeta) = E(a)\zeta$.
(5) *If $a \in A_r$, then* $\widetilde{E}(a\zeta) = \zeta E(a)$.
(6) *If $\tau(L_b\xi_1) = \tau(L_b\xi_2)$ for all $b \in B$, then* $\widetilde{E}(\xi_1) = \widetilde{E}(\xi_2)$.
(7) *If $\tau(R_b\xi_1) = \tau(R_b\xi_2)$ for all $b \in B$, then* $\widetilde{E}(\xi_1) = \widetilde{E}(\xi_2)$.

Proof For (i), since $L_{1_B} = 1_A$ as ε is unital, note that

$$\tau_B\left(\widetilde{E}(\xi)\right) = \left\langle \widetilde{E}(\xi), 1_B \right\rangle_{L_2(B,\tau_B)} = \left\langle \xi, \widetilde{E}(1_B) \right\rangle_{L_2(A,\tau)} = \langle \xi, 1_A \rangle_{L_2(A,\tau)} = \tau(\xi),$$

as desired.

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For (ii), note for all $b' \in B$ that

$$\left\langle \widetilde{E}(aL_b), \widehat{b'} \right\rangle_{L_2(B,\tau_B)} = \langle aL_b, L_{b'} \rangle_{L_2(A,\tau)} = \tau\left(L_{(b')^*} aL_b\right) = \tau\left(L_{(b')^*} aR_b\right) = \left\langle \widetilde{E}(aR_b), \widehat{b'} \right\rangle_{L_2(B,\tau_B)}.$$

Hence, $\widetilde{E}(aL_b) = \widetilde{E}(aR_b)$.

For (iii), let $(a_n)_{n\geq 1}$ be a sequence of elements of A that converge to ξ in $L_2(A,\tau)$. Since left multiplication in $L_2(A,\tau)$ by elements of A are bounded and thus continuous, and since left and right multiplication in $L_2(B,\tau_B)$ by elements of B are bounded and thus continuous, we obtain that

$$\widetilde{E}(L_{b_1} R_{b_2} \xi) = \lim_{n\to\infty} E(L_{b_1} R_{b_2} a_n) + N_{\tau_B} = \lim_{n\to\infty} b_1 E(a_n) b_2 + N_{\tau_B} = b_1 \widetilde{E}(\xi) b_2,$$

as desired.

For (iv) and (v), let $(c_n)_{n\geq 1}$ be a sequence of elements of B that converge to ζ in $L_2(B,\tau_B)$. Thus, by the inclusion of $L_2(B,\tau_B)$ into $L_2(A,\tau)$, we have that $(L_{c_n})_{n\geq 1}$ is a sequence of elements of $L_2(A,\tau)$ that converge to ζ in $L_2(A,\tau)$. Thus, if $a \in A_\ell$, then

$$\begin{aligned}
\widetilde{E}(a\zeta) &= \lim_{n\to\infty} E(aL_{c_n}) + N_{\tau_B} \\
&= \lim_{n\to\infty} E(aR_{c_n}) + N_{\tau_B} \\
&= \lim_{n\to\infty} E(R_{c_n} a) + N_{\tau_B} \\
&= \lim_{n\to\infty} E(a)c_n + N_{\tau_B} = E(a)\zeta,
\end{aligned}$$

thereby proving (iv). Note that (v) is similar using $(R_{c_n})_{n\geq 1}$ in place of $(L_{c_n})_{n\geq 1}$.

As (vi) and (vii) are similar, we prove (vii). Note by (iii) and the fact that τ_B is tracial that

$$\begin{aligned}
\left\langle \widetilde{E}(\xi_1) - \widetilde{E}(\xi_2), \widehat{b} \right\rangle_{L_2(B,\tau_B)} &= \left\langle \widetilde{E}(\xi_1) b^*, \widetilde{E}(1_A) \right\rangle_{L_2(B,\tau_B)} - \left\langle \widetilde{E}(\xi_2) b^*, \widetilde{E}(1_A) \right\rangle_{L_2(B,\tau_B)} \\
&= \left\langle \widetilde{E}(R_{b^*}\xi_1), \widetilde{E}(1_A) \right\rangle_{L_2(B,\tau_B)} - \left\langle \widetilde{E}(R_{b^*}\xi_2), \widetilde{E}(1_A) \right\rangle_{L_2(B,\tau_B)} \\
&= \langle R_{b^*}\xi_1, 1_A \rangle_{L_2(A,\tau)} - \langle R_{b^*}\xi_2, 1_A \rangle_{L_2(A,\tau)} \\
&= \tau(R_{b^*}\xi_1) - \tau(R_{b^*}\xi_2) = 0.
\end{aligned}$$

As the above holds for all $b \in B$, (vii) follows. ∎

4 Analytical bi-multiplicative functions

In this section, we extend the notion of bi-multiplicative functions on analytical B-B-noncommutative probability spaces in order to permit the last entry to be an element of $L_2(A,\tau)$. This is possible as the last entry can be treated as a left or right operator as [12] shows, or can be treated as a mixture of left and right operators as [4] shows. Extending the operator-valued bi-free cumulant function to permit the last entry to be an element of $L_2(A,\tau)$ is necessary in order to permit the simple development of conjugate variable systems in the next section.

We advise the reader that familiarity with specifics of bi-multiplicative functions, the construction of the operator-valued bi-free moment function, and the construction of the operator-valued bi-free cumulant function from [2] would be of great aid

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in comprehension of this section. As the proofs are nearly identical, to avoid clutter, we will focus on that which is different and why the results of [2] extend.

Definition 4.1 Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, and let Φ be a bi-multiplicative function on (A, E, ε). A function

$$\widetilde{\Phi} : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \cdots \times A_{\chi(n-1)} \times L_2(A,\tau) \to L_2(B,\tau_B)$$

is said to be *analytical extension of* Φ if $\widetilde{\Phi}_\pi$ is $\mathbb{C}$-multilinear function that does not change values if the last entry of χ is changed from an ℓ to an r and satisfies the following three properties: for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $\xi \in L_2(A, \tau)$, $\zeta \in L_2(B, \tau_B)$, $b \in B$, and $Z_k \in A_{\chi(k)}$:

(1) If $\chi(k) = \ell$ for all $k \in \{1, 2, \ldots, n\}$, then

$$\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n\zeta) = \Phi_{1_\chi}(Z_1, \ldots, Z_n)\zeta,$$

and if $\chi(k) = r$ for all $k \in \{1, 2, \ldots, n\}$, then

$$\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{n-1}, Z_n\zeta) = \zeta\Phi_{1_\chi}(Z_1, \ldots, Z_n).$$

In particular, by setting $\zeta = 1_B = 1_A$, we see that $\widetilde{\Phi}$ does extend Φ.

(ii) Let $p \in \{1, \ldots, n\}$, and let

$$q = \max_{\leq} \{k \in \{1, \ldots, n\} \mid \chi(k) = \chi(p), k < p\}.$$

If $\chi(p) = \ell$, then

$$\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_bZ_p, Z_{p+1}, \ldots, Z_{n-1}, \xi) = \begin{cases} \widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_qL_b, Z_{q+1}, \ldots, Z_{n-1}, \xi), & \text{if } q \neq -\infty, \\ b\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{n-1}, \xi), & \text{if } q = -\infty, \end{cases}$$

and if $\chi(p) = r$, then

$$\widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{p-1}, R_bZ_p, Z_{p+1}, \ldots, Z_{n-1}, \xi) = \begin{cases} \widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_qR_b, Z_{q+1}, \ldots, Z_{n-1}, \xi), & \text{if } q \neq -\infty, \\ \widetilde{\Phi}_{1_\chi}(Z_1, \ldots, Z_{n-1}, \xi)b, & \text{if } q = -\infty. \end{cases}$$

(iii) Suppose $V_1, \ldots, V_m$ are unions of blocks of π that partition $\{1, \ldots, n\}$, with each being a χ-interval. Moreover, assume that the sets $V_1, \ldots, V_m$ are ordered by $\preceq_\chi$ (i.e., $(\min_{\preceq_\chi} V_k) \prec_\chi (\min_{\preceq_\chi} V_{k+1})$). Let $q \in \{1, \ldots, m\}$ be such that $n \in V_q$, and for each $k \neq q$, let

$$b_k = \Phi_{\pi|_{V_k}}\left((Z_1, \ldots, Z_{n-1}, \xi)|_{V_k}\right).$$

Then, $b_k \in B$ for $k \neq q$ and

$$\widetilde{\Phi}_\pi(Z_1, \ldots, Z_{n-1}, \xi) = b_1b_2\cdots b_{q-1}\widetilde{\Phi}_{\pi|_{V_q}}((Z_1, \ldots, Z_{n-1}, \xi)|_{V_q})b_{q+1}\cdots b_m.$$

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(iv) Suppose that V and W are unions of blocks of π that partition $\{1,\dots,n\}$ such that V is a χ-interval and $s_\chi(1), s_\chi(n) \in W$. Let

$$p = \max_{\preceq_\chi}\left\{k \in W \;\middle|\; k \preceq_\chi \min_{\preceq_\chi} V\right\} \qquad \text{and} \qquad q = \min_{\preceq_\chi}\left\{k \in W \;\middle|\; \max_{\preceq_\chi} V \preceq_\chi k\right\}.$$

Then, one of the following four cases hold:

a) If $n \in V$ and $k = \max_{\preceq} W$, then

$$\widetilde{\Phi}_\pi(Z_1,\dots,Z_{n-1},\xi) = \widetilde{\Phi}_{\pi|_W}\left((Z_1,\dots,Z_{k-1},Z_k\widetilde{\Phi}_{\pi|_V}((Z_1,\dots,Z_{n-1},\xi)|_V),\dots,Z_{n-1},\xi)|_W\right).$$

b) If $n \in W$, then

$$\widetilde{\Phi}_\pi(Z_1,\dots,Z_n) = \begin{cases}\widetilde{\Phi}_{\pi|_W}\left((Z_1,\dots,Z_{p-1},Z_pL_{\Phi_{\pi|_V}((Z_1,\dots,Z_n)|_V)},Z_{p+1},\dots,Z_n)|_W\right) & \text{if } \chi(p)=\ell,\\ \widetilde{\Phi}_{\pi|_W}\left((Z_1,\dots,Z_{p-1},R_{\Phi_{\pi|_V}((Z_1,\dots,Z_n)|_V)}Z_p,Z_{p+1},\dots,Z_n)|_W\right) & \text{if } \chi(p)=r,\end{cases}$$
$$= \begin{cases}\widetilde{\Phi}_{\pi|_W}\left((Z_1,\dots,Z_{q-1},L_{\Phi_{\pi|_V}((Z_1,\dots,Z_n)|_V)}Z_q,Z_{q+1},\dots,Z_n)|_W\right) & \text{if } \chi(q)=\ell,\\ \widetilde{\Phi}_{\pi|_W}\left((Z_1,\dots,Z_{q-1},Z_qR_{\Phi_{\pi|_V}((Z_1,\dots,Z_n)|_V)},Z_{q+1},\dots,Z_n)|_W\right) & \text{if } \chi(q)=r.\end{cases}$$

(Recall that we can set $\chi(n) = \ell$ or $\chi(n) = r$.)

Remark 4.1 Note that the pair of a bi-multiplicative function and its extension are very reminiscent of the two expectation extensions of bi-multiplicative functions used for operator-valued conditional bi-free independence from [6]. The main difference is that the notion in [6] looks at interior versus exterior blocks of the partition, whereas Definition 4.1 looks at the blocks containing the last entry. This is due to the fact that the $L_2(A,\tau)$ element is always the last entry and must be treated differently being a generalization of a mixture of left and right operators.

It is worth pointing out that treating the last entry as an element of $L_2(A,\tau)$ is no issue. In particular, the properties in Definition 4.1 are well defined. Indeed, properties (i) and (ii) of Definition 4.1 are clearly well defined and properties (iii) and (iv) in Definition 4.1 are well defined as all terms where Φ is used over $\widetilde{\Phi}$ never involve an element of $L_2(A,\tau)$ and as elements from B have left and right actions on $L_2(B,\tau_B)$.

Remark 4.2 Note that property (i) of Definition 4.1 is clearly the correct generalization of property (i) from Definition 2.5, as an element of $L_2(B,\tau_B)$ is playing the role of L_b and R_b in this generalization and thus should be able to escape these expressions if only left operators or right operators are present. The absence of the full property (i) from Definition 2.5 causes no issues when attempting to reduce or rearrange the value of $\widetilde{\Phi}_\pi$ to an expression involving only $\widetilde{\Phi}_{1_\chi}$'s, as the last entry of any sequence input into $\widetilde{\Phi}$ is always in $L_2(A,\tau)$, is reduced to an element of $L_2(B,\tau)$, and an element of A_ℓ or A_r then acts on it via the left action of A on $L_2(A,\tau)$. Thus, there is never any need to move the $L_2(A,\tau)$ entry to another position.

If property (i) is ever used, we note that if $\zeta \in L_2(B,\tau_B)$ is viewed as an element of $L_2(A,\tau)$, then $L_b\zeta$ is simply the element $b\zeta \in L_2(B,\tau_B)$ and $R_b\zeta$ is simply the element $\zeta b \in L_2(B,\tau_B)$. Thus, using (i) does not pose problems when trying to "move around L_b and R_b elements" in proofs when trying to show the equivalence of any reductions as the following example demonstrates.

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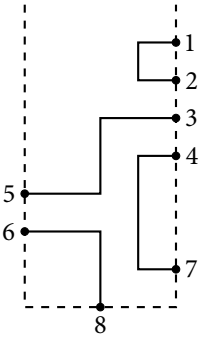
Example 4.3 Let $\chi \in \{\ell, r\}^8$ be such that $\chi^{-1}(\{\ell\}) = \{5, 6\}$, let $\xi \in L_2(A, \tau)$, let $Z_k \in A_{\chi(k)}$, and let $\pi \in \mathrm{BNC}(\chi)$ be the partition

$$\pi = \{\{1, 2\}, \{3, 5\}, \{4, 7\}, \{6, 8\}\}.$$

Note that the bi-non-crossing diagram of π can be represented as the following (with the convention now that the last entry is at the bottom instead of on its respective side):



When reducing $\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_7, \xi)$, we can clearly use property (iii) of Definition 4.1 first to obtain with $U = \{3, 4, \ldots, 8\}$ that

$$\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_7, \xi) = \widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi)\Phi_{1_{(r,r)}}(Z_1, Z_2).$$

To reduce the expression fully, we have to simply reduce $\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi)$ using property (iv) of Definition 4.1 of which there are three ways to do so.

The first way to reduce is to use $V = \{4, 6, 7, 8\}$ and $W = \{3, 5\}$. By applying property (iv) of Definition 4.1, we obtain that

$$\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) = \widetilde{\Phi}_{\pi|_W}\left(Z_3, Z_5\widetilde{\Phi}_{\pi|_V}(Z_4, Z_6, Z_7, \xi)\right).$$

Finally, by applying property (iii) of Definition 4.1 to $\widetilde{\Phi}_{\pi|_V}(Z_4, Z_6, Z_7, \xi)$, we obtain that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5\left(\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\Phi_{1_{(r,r)}}(Z_4, Z_7)\right)\right)\\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right).\end{aligned}$$

The second way to reduce is to use $V = \{6, 8\}$ and $W = \{3, 4, 5, 7\}$. By applying property (iv) of Definition 4.1, we obtain that

$$\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) = \widetilde{\Phi}_{\pi|_W}\left(Z_3, Z_4, Z_5, Z_7\widetilde{\Phi}_{\pi|_V}(Z_6, \xi)\right).$$

By applying property (iv) of Definition 4.1 again as $\{4, 7\}$ is now a $\chi|_W$-interval), we obtain that

$$\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) = \widetilde{\Phi}_{1_{r,\ell}}\left(Z_3, Z_5\left(\widetilde{\Phi}_{1_{(r,r)}}(Z_4, Z_7\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi))\right)\right).$$

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However, as $Z_4, Z_7 \in A_r$, we obtain by property (i) that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5\left(\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\Phi_{1_{(r,r)}}(Z_4, Z_7)\right)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right),\end{aligned}$$

thereby agreeing with the above expression.

The third way to reduce is to use $V = \{4, 7\}$ and $W = \{3, 5, 6, 8\}$. By applying property (iv) of Definition 4.1, we obtain that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{\pi|_W}\left(Z_3, Z_5, Z_6, R_{\Phi_{\pi|_V}(Z_4,Z_7)}\xi\right) \\ &= \widetilde{\Phi}_{\pi|_W}\left(Z_3 R_{\Phi_{\pi|_V}(Z_4,Z_7)}, Z_5, Z_6, \xi\right),\end{aligned}$$

by using the two expressions in property (iv). Using either expression, we will now obtain again property (iv) of Definition 4.1 as $\{6, 8\}$ is now $\chi|_W$-interval. For the first, we obtain that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5\widetilde{\Phi}_{1_{(\ell,\ell)}}\left(Z_6, R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\xi\right)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5\left(\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\Phi_{1_{(r,r)}}(Z_4, Z_7)\right)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right)\end{aligned}$$

where the second equality follows from applying property (ii) of Definition 4.1, as $Z_6 \in A_\ell$. For the second expression, we obtain that

$$\begin{aligned}\widetilde{\Phi}_{\pi|_U}(Z_3, Z_4, \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}, Z_5\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}Z_5\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \\ &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)}\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right),\end{aligned}$$

where the second equality follows from applying property (ii) of Definition 4.1 as the last entry is now the $L_2(A, \tau)$ entry, and the third equality holds as $Z_5 \in A_\ell$ and thus commutes with R_b.

Hence, Definition 4.1 is consistent in this example (and will be in all examples due to similar computations).

Using similar reductions for arbitrary expressions, one can prove the following.

Lemma 4.4 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let Φ be a bi-multiplicative function on (A, E, ε), and let $\widetilde{\Phi}$ be an analytic extension of Φ. Then, properties (i) and (ii) of Definition 4.1 hold when 1_χ is replaced with any $\pi \in \mathrm{BNC}(\chi)$.*

Proof The proof is essentially the same as the proof that properties (i) and (ii) of Definition 2.5 hold for Φ when 1_χ is replaced with any $\pi \in \mathrm{BNC}(\chi)$ as in [2, Proposition 4.2.5]. To see that property (i) of Definition 4.1 extends, note when using (iii) and (iv) to reduce the expression for $\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_{n-1}, Z_n\zeta)$ that one is effectively using the bi-multiplicative properties of Φ and including ζ in the appropriate spot. To see that property (ii) of Definition 4.1 extends, indices that are always adjacent in

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the χ-ordering will remain in the correct ordering so that when L_b or R_b operators are considered, we can always move them outside the Φ- and $\widetilde{\Phi}$-expressions on the correct side to move them to the next operator (that is, things will always move around as they do in the free multiplicative functions from [13] after reordering by the χ-order). For example, in Example 4.3, we showed that

$$\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_7, \xi) = \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2).$$

If Z_3 were replaced with $R_b Z_3$, we would have

$$\begin{aligned}
\widetilde{\Phi}_\pi(Z_1, Z_2, R_b Z_3, Z_4 \ldots, Z_7, \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(R_b Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) b\Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2 R_b) \\
&= \widetilde{\Phi}_\pi(Z_1, Z_2 R_b, Z_3, Z_4 \ldots, Z_7, \xi).
\end{aligned}$$

If ξ were replaced with $L_b \xi$, then clearly the L_b can be moved to give $Z_6 L_b$ via (ii) with a 1_χ as the expression $\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)$ is present. If ξ were replaced with $R_b \xi$, then

$$\begin{aligned}
\widetilde{\Phi}_\pi(Z_1, Z_2, \ldots, Z_7, R_b \xi) &= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, R_b \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} \left(\widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi) b\right)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7)} R_b \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{b\Phi_{1_{(r,r)}}(Z_4,Z_7)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_{1_{(r,\ell)}}\left(Z_3, Z_5 R_{\Phi_{1_{(r,r)}}(Z_4,Z_7 R_b)} \widetilde{\Phi}_{1_{(\ell,\ell)}}(Z_6, \xi)\right) \Phi_{1_{(r,r)}}(Z_1, Z_2) \\
&= \widetilde{\Phi}_\pi(Z_1, \ldots, Z_6, Z_7 R_b, \xi),
\end{aligned}$$

as desired. Thus, the result follows. ∎

4.1 Analytical operator-valued bi-moment function

We will now construct the analytical extension of the operator-valued bi-moment function via recursion and the map $\widetilde{E} : L_2(A, \tau) \to L_2(B, \tau_B)$ from Section 3. Note that the recursive process in the following definition is different than that from [2, Definition 5.1] and [6, Definition 4.4], in order to facilitate the introduction of the $L_2(A, \tau)$ element. The same recursive process could have been used in [2, Definition 5.1] and [6, Definition 4.4], as these processes are equivalent in those settings. Note that we use Ψ in the following to avoid confusion with $\widetilde{E}$ in Section 3, although Ψ is a multi-entry extension of $\widetilde{E}$.

Definition 4.2 Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space. The *analytical bi-moment function*

$$\Psi : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \cdots \times A_{\chi(n-1)} \times L_2(A, \tau) \to L_2(B, \tau_B)$$

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is defined recursively as follows: let $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $\xi \in L_2(A, \tau)$, and $Z_k \in A_{\chi(k)}$.

- If $\pi = 1_\chi$, then

$$\Psi_{1_\chi}(Z_1, Z_2, \ldots, Z_{n-1}, \xi) = \widetilde{E}(Z_1 Z_2 \cdots Z_{n-1}\xi).$$

- If $\pi \neq 1_\chi$, let V be the block in π such that $n \in V$. We divide discussion into two cases:
  - Suppose that $\min_{\preceq_\chi} V = s_\chi(1)$ and $\max_{\preceq_\chi} V = s_\chi(n)$, and let

$$p = \min_{\preceq_\chi}\{i \in \{1, \ldots, n\} \mid i \notin V\}, \quad q = \min_{\preceq_\chi}\{j \in V \mid p \prec_\chi j\}, \quad \text{and}$$
$$m = \max_{\preceq_\chi}\{i \in V \mid i \prec_\chi p\}.$$

    Set

$$W = \{i \in \{1, \ldots, n\} \mid p \preceq_\chi i \prec_\chi q\}.$$

    Note by construction and the fact that $\pi \in \mathrm{BNC}(\chi)$ that W is equal to a union of blocks of π and $\chi(p) = \chi(j)$ for all $j \in W$. Thus, we define

$$\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi) = \begin{cases} \Psi_{\pi|_{W^c}}\left((Z_1, \ldots, Z_{p-1}, Z_m L_{E_{\pi|_W}((Z_1, \ldots, Z_{n-1}, \xi)|_W)}, \ldots, Z_{n-1}, \xi)|_{W^c}\right), & \text{if } \chi(p) = \ell, \\ \Psi_{\pi|_{W^c}}\left((Z_1, \ldots, Z_{q-1}, Z_q R_{E_{\pi|_W}((Z_1, \ldots, Z_{n-1}, \xi)|_W)}, \ldots, Z_{n-1}, \xi)|_{W^c}\right), & \text{if } \chi(p) = r. \end{cases}$$

    (Note that as $n \notin W$, the quantity $E_{\pi|_W}((Z_1, \ldots, Z_{n-1}, \xi)|_W)$ is always a well-defined element of B in this case. Note in the case that $\chi(p) = \ell$ that $m \prec_\chi p \prec_\chi n$ and thus $Z_m \neq \xi$, so $\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi)$ is well defined. Also, in the case when $\chi(p) = r$, note that $n \prec_\chi q$ and thus $Z_q \neq \xi$, so $\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi)$ is well defined.)
  - Otherwise, set

$$\widetilde{V} = \left\{ i \in \{1, \ldots, n\} \;\middle|\; \min_{\preceq_\chi} V \preceq_\chi i \preceq_\chi \max_{\preceq_\chi} V \right\}.$$

    Note that $\widetilde{V}$ is a proper subset of $\{1, \ldots, n\}$ that is a union of blocks of π and is such that $n \in V \subseteq \widetilde{V}$. For $q = \max_{\preceq} \widetilde{V}^c$, define

$$\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi) = \Psi_{\pi|_{\widetilde{V}^c}}\left((Z_1, \ldots, Z_{q-1}, Z_q \Psi_{\pi|_{\widetilde{V}}}((Z_1, \ldots, Z_{n-1}, \xi)|_{\widetilde{V}}), \ldots, Z_{n-1}, \xi)|_{\widetilde{V}^c}\right).$$

    (Note that the quantity $\Psi_{\pi|_{\widetilde{V}}}((Z_1, \ldots, Z_{n-1}, \xi)|_{\widetilde{V}})$ is a well-defined element of $L_2(B, \tau_B)$ due to the recursive nature of our definition. Moreover, the last element of the sequence

$$(Z_1, \ldots, Z_{q-1}, Z_q \Psi_{\pi|_{\widetilde{V}}}((Z_1, \ldots, Z_{n-1}, \xi)|_{\widetilde{V}}), \ldots, Z_{n-1}, \xi)|_{\widetilde{V}^c}$$

    is equal to $Z_q \Psi_{\pi|_{\widetilde{V}}}((Z_1, \ldots, Z_{n-1}, \xi)|_{\widetilde{V}})$, which is an element of $L_2(A, \tau)$, so $\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi)$ is well defined.)

To aid in the comprehension of Definition 4.2, we provide an example using bi-non-crossing diagrams to show the recursive construction. We note that ξ will always

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appear last in a sequence of operators and is an element of $L_2(A,\tau)$ and thus neither a left nor a right operator. As such, we treat it as neither. This is reminiscent of [12, Lemma 2.17], where it does not matter whether we treat the last operator in a list as a left or as a right operator, and of [4, Lemma 2.29 and Proposition 2.30], where the last entry can be a mixture of left and right operators.

Example 4.5 Let $\chi \in \{\ell, r\}^{12}$ be such that $\chi^{-1}(\{\ell\}) = \{1,5,8,9,11,12\}$, let $\xi \in L_2(A,\tau)$, let $Z_k \in A_{\chi(k)}$, and let $\pi \in \mathrm{BNC}(\chi)$ be the partition with blocks

$$V_1 = \{1,3\}, \quad V_2 = \{2\}, \quad V_3 = \{4,5,11,12\}, \quad V_4 = \{6,10\}, \quad V_5 = \{7\}, \quad \text{and} \quad V_6 = \{8,9\}.$$

To compute $\Psi_\pi(Z_1,\ldots,Z_{11},\xi)$, we note that the second part of the second step of the recursive definition from Definition 4.2 applies first. In particular,

$$\widetilde{V} = \bigcup_{k=3}^{6} V_k.$$

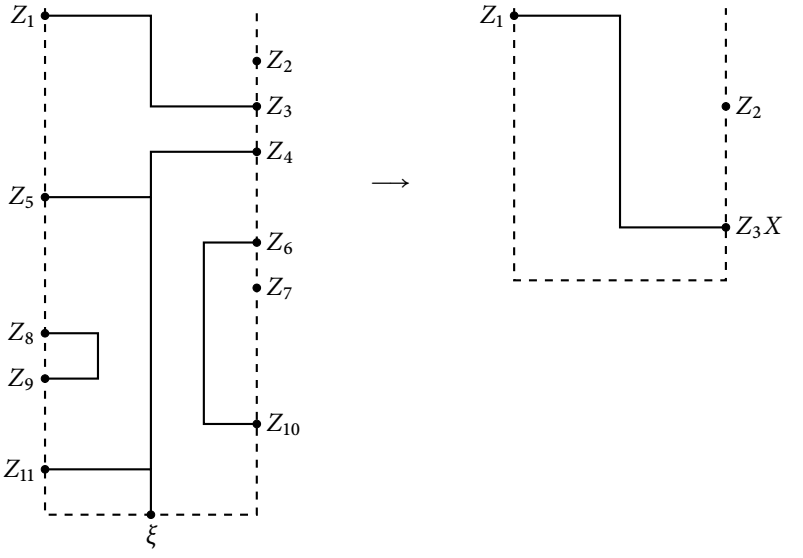
Thus, if

$$X = \Psi_{\pi|_{\widetilde{V}}}(Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11}, \xi),$$

then

$$\Psi_\pi(Z_1,\ldots,Z_{11},\xi) = \Psi_{\pi|_{\widetilde{V}^c}}(Z_1, Z_2, Z_3X).$$

Diagrammatically, this first reduction is seen as follows:



Note that

$$\Psi_{\pi|_{\widetilde{V}^c}}(Z_1, Z_2, Z_3X) = \Psi_{\pi|_{V_2}}(Z_2\Psi_{\pi|_{V_1}}(Z_1, Z_3X)) = \widetilde{E}\left(Z_2\widetilde{E}(Z_1Z_3X)\right),$$

where the first equality holds by the same recursive idea, whereas the second equality holds by the first step of Definition 4.2.

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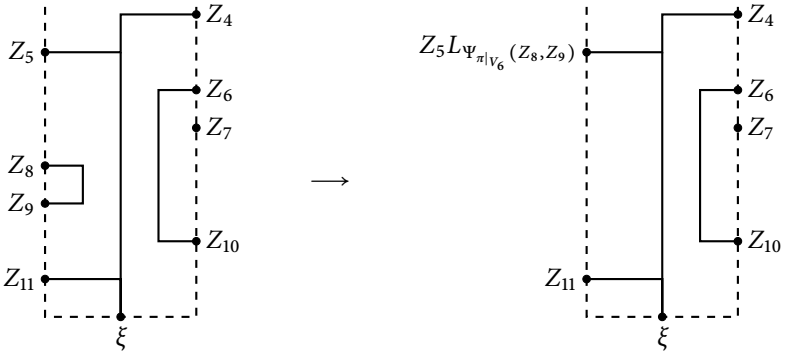
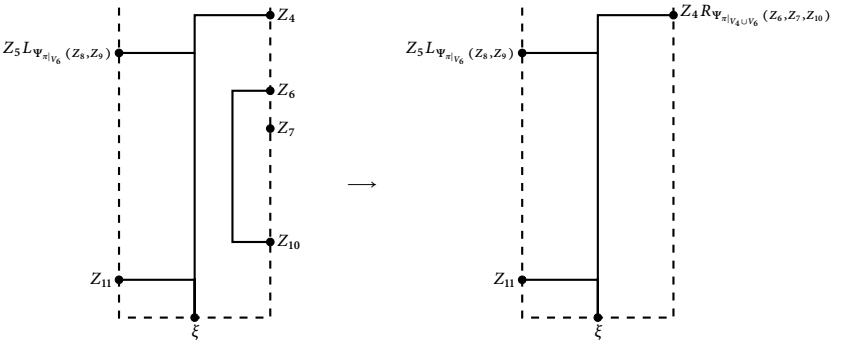
When computing the value of X, the minimal and maximal elements of $\{4, 5, \ldots, 11, 12\}$ in the $\chi|_{\widetilde{V}}$-order are 5 and 4, respectively, and the block that contains the index corresponding to ξ contains both 5 and 4. Thus, the first part of the second step of Definition 4.2 should be used. The algorithm in Definition 4.2 then calculates the value of X by "stripping out" the χ-intervals V_6 and $V_4 \cup V_5$ successively and this is seen via the following two diagrammatic reductions:



and



It is readily verified using the fact that the operator-valued bi-free moment function is bi-multiplicative that $E_{\pi|_{V_6}}(Z_8, Z_9) = E(Z_8 Z_9)$ and $E_{\pi|_{V_4\cup V_6}}(Z_6, Z_7, Z_{10}) = E(Z_6 R_{E(Z_7)} Z_{10})$. Thus, using the fact that $\pi|_{V_3} = 1_{\chi|_{V_3}}$, the first step in Definition 4.2 yields

$$X = \widetilde{E}\left(Z_4 R_{E(Z_6 R_{E(Z_7)} Z_{10})} Z_5 L_{E(Z_8 Z_9)} Z_{11} \xi\right).$$

Hence,

$$\begin{aligned}\Psi_\pi(Z_1, \ldots, Z_{11}, \xi) &= \widetilde{E}\left(Z_2 \widetilde{E}\left(Z_1 Z_3 \widetilde{E}\left(Z_4 R_{E(Z_6 R_{E(Z_7)} Z_{10})} Z_5 L_{E(Z_8 Z_9)} Z_{11} \xi\right)\right)\right) \\ &= \widetilde{E}\left(Z_1 Z_3 \widetilde{E}\left(Z_4 R_{E(Z_6 R_{E(Z_7)} Z_{10})} Z_5 L_{E(Z_8 Z_9)} Z_{11} \xi\right)\right) E(Z_2),\end{aligned}$$

with the last equality following from Proposition 3.8.

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Before investigating the bi-multiplicative properties inherited by the analytical bi-moment function, we note that it is truly an extension of the operator-valued bi-moment function.

Theorem 4.6 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space. For any $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, and $Z_k \in A_{\chi(k)}$,*

$$\Psi_\pi(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau) = E_\pi(Z_1, \ldots, Z_{n-1}, Z_n) + N_{\tau_B}.$$

Proof Note that each step in the recursive definition of Definition 4.2 is a step that can be performed to the operator-valued bi-free moment function as the operator-valued bi-free moment function is bi-multiplicative (see Definition 2.5). Therefore, as Proposition 3.6 implies that

$$\widetilde{E}(a) = E(a) + N_{\tau_B}$$

for all $a \in A$, by applying the same recursive properties to $E_\pi(Z_1, \ldots, Z_{n-1}, Z_n)$ as used to compute $\Psi_\pi(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau)$, the result follows. ∎

Like with the construction of the operator-valued bi-free moment function in [2], although the construction of the analytical bi-moment function is done using specific rules from the operator-valued bi-free moment function in a specific order, we desire more flexibility in the reductions that can be done and the order they can be done in. In particular, we desire to show that the analytical bi-moment function is an analytic extension of the operator-valued bi-free moment function.

The main ideas used to prove this are similar to those utilized in the proof of [2, Theorem 5.1.4], and hence we shall be concerned with demonstrating that the inclusion of the $L_2(A, \tau)$ term and the slightly modified recursive definition are not issue and pose next to no changes. In particular, it may appear that Ψ behaves differently than the operator-valued bi-free moment function as entries in $L_2(A, \tau)$ can also act as mixtures of left and right operators, which was not dealt with in [2]. However, using the properties of $\widetilde{E}$ as developed in Proposition 3.8, one familiar with [2] can easily see that the desired results will hold with simple adaptations. We note that similar adaptations were done in [6] without issue.

When examining the proof of the following, Example 4.5 serves as a good example to keep in mind, just as Example 4.3 aided in comprehending why analytic extensions of bi-multiplicative functions work.

Theorem 4.7 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, and let*

$$\Psi : \bigcup_{n \in \mathbb{N}} \bigcup_{\chi \in \{\ell, r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \cdots \times A_{\chi(n-1)} \times L_2(A, \tau) \to L_2(B, \tau_B)$$

be the analytical bi-moment function. Then, Ψ is an analytically extension of the operator-valued bi-free moment function.

Clearly, the map Ψ_{1_χ} is $\mathbb{C}$-multilinear, and it does not matter whether $\chi(n) = \ell$ or $\chi(n) = r$. A straightforward induction argument using the definition of Ψ shows that the map Ψ_π will be $\mathbb{C}$-multilinear. Thus, we focus on the remaining four properties.

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Proof of Theorem 4.7 property (i) This immediately follows from parts (ii) and (iii) of Proposition 3.8. ∎

Proof of Theorem 4.7 property (ii) To see (ii), fix $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $\xi \in L_2(A, \tau)$, $b \in B$, and $Z_k \in A_{\chi(k)}$, and let p and q be as in the statement of (ii). In the case that $\chi(p) = \ell$, note that

$$\Psi_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_b Z_p, Z_{p+1}, \ldots, Z_{n-1}, \xi) = \widetilde{E}(Z_1 \cdots Z_{p-1} L_b Z_p Z_{p+1} \cdots Z_{n-1} \xi).$$

If $q \neq \infty$, then $Z_{q+1}, \ldots, Z_{p-1} \in A_r$ and thus commute with L_b. Hence,

$$\begin{aligned}\Psi_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_b Z_p, Z_{p+1}, \ldots, Z_{n-1}, \xi) &= \widetilde{E}(Z_1 \cdots Z_{q-1} Z_q L_b Z_{q+1} \cdots Z_{n-1} \xi) \\ &= \Psi_{1_\chi}(Z_1, \ldots, Z_{q-1}, Z_q L_b, Z_{q+1}, \ldots, Z_{n-1}, \xi)\end{aligned}$$

(and note $Z_q L_b \in A_\ell$).

If $q = \infty$, then $Z_1, \ldots, Z_{p-1} \in A_r$ and thus commute with L_b. Hence,

$$\begin{aligned}\Psi_{1_\chi}(Z_1, \ldots, Z_{p-1}, L_b Z_p, Z_{p+1}, \ldots, Z_{n-1}, \xi) &= \widetilde{E}(L_b Z_1 \cdots Z_{n-1} \xi) \\ &= b\widetilde{E}(Z_1 \cdots Z_{n-1} \xi) \\ &= b\Psi_{1_\chi}(Z_1, \ldots, Z_{n-1}, \xi)\end{aligned}$$

by Proposition 3.8. The case $\chi(p) = r$ is similar. ∎

Proof of Theorem 4.7 property (iii) To highlight how the proof works, we begin with the case that π consists of exactly three blocks that are χ-intervals and n is contained in the middle block under the χ-ordering. Suppose that $\pi = \{V_1, V_2, V_3\}$ and thus there exists $i \in \{2, \ldots, n-1\}$ and $j \in \{0, \ldots, n-2\}$ such that

$$\begin{aligned}V_1 &= \{s_\chi(1), s_\chi(2), \ldots, s_\chi(i-1)\}, \\ V_2 &= \{s_\chi(i), s_\chi(i+1), \ldots, s_\chi(i+j)\}, \text{ and} \\ V_3 &= \{s_\chi(i+j+1), s_\chi(i+j+2), \ldots, s_\chi(n)\}.\end{aligned}$$

Thus, $n = s_\chi(k)$ for some $i \leq k \leq i+j$. This implies that $\chi(p) = \ell$ for all $p \in V_1$ and $\chi(p) = r$ for all $p \in V_3$. If $W = V_1 \cup V_3$, observe that the definition of the permutation s_χ yields that

$$q = \max_{\leq} W = \max_{\leq}\{s_\chi(i-1), s_\chi(i+j+1)\}.$$

Consider the case $q = s_\chi(i-1)$ so that $q \in V_1$, and let $m = s_\chi(i+j+1)$. Notice that if

$$X = \Psi_{\pi|_{V_2}}((Z_1, \ldots, Z_{n-1}, \xi)|_{V_2}),$$

then by Definition 4.2, we obtain that

$$\Psi_\pi(Z_1, \ldots, Z_{n-1}, \xi) = \Psi_{\pi|_W}((Z_1, \ldots, Z_{q-1}, Z_q X, \ldots, Z_{n-1}, \xi)|_W).$$

Thus, if

$$Y = \Psi_{\pi|_{V_1}}((Z_1, \ldots, Z_{q-1}, Z_q X, \ldots, Z_{n-1}, \xi)|_{V_1}),$$

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then again Definition 4.2 implies that

$$\begin{aligned}\Psi_\pi(Z_1,\ldots,Z_{n-1},\xi) &= \Psi_{\pi|_{V_3}}\big((Z_1,\ldots,Z_{m-1},Z_mY,\ldots,Z_{n-1},\xi)|_{V_3}\big)\\ &= \widetilde{E}\left(Z_{s_\chi(n)}Z_{s_\chi(n-1)}\ldots Z_{s_\chi(i+j+1)}Y\right).\end{aligned}$$

Therefore, since $Z_{s_\chi(n)}Z_{s_\chi(n-1)}\ldots Z_{s_\chi(i+j+1)} \in A_r$, Proposition 3.8 implies that

$$\Psi_\pi(Z_1,\ldots,Z_{n-1},\xi) = YE\left(Z_{s_\chi(n)}Z_{s_\chi(n-1)}\ldots Z_{s_\chi(i+j+1)}\right) = YE_{\pi|_{V_3}}((Z_1,\ldots,Z_{n-1},\xi)|_{V_3}).$$

Since $Z_{s_\chi(1)}, Z_{s_\chi(2)},\ldots,Z_{s_\chi(i-1)} \in A_r$, Proposition 3.8 implies that

$$Y = E(Z_{s_\chi(1)}Z_{s_\chi(2)}\ldots Z_{s_\chi(i-1)})X = E_{\pi|_{V_1}}((Z_1,\ldots,Z_{n-1},\xi)|_{V_1})X,$$

so the result follows. Note that the case $q = s_\chi(i+j+1)$ is handled similarly by interchanging the orders of the χ- intervals V_1 and V_3. This argument can be extended via induction to any bi-non-crossing partition π all of whose blocks are χ-interval.

By the same argument as [2, Lemma 5.2.1], one need only consider the case in property (iii) that for each χ-interval, the χ-maximal and χ-minimal elements belong to the same block. When using the recursive procedure in Definition 4.2 to reduce Ψ_π, one of the χ-intervals (which will either be entirely on the left or entirely on the right) will have the $L_2(A,\tau)$ term added to the last entry as above. This $L_2(A,\tau)$ entry can be pulled out on the appropriate side leaving only the bi-moment function expression for the χ-interval, which can be undone as usual. By repetition, eventually, all that remains is the expression for the χ-interval containing n as desired. ∎

Proof of Theorem 4.7 property (iv) The proof that property (iv) holds for the operator-valued bi-free moment function is one of the longest of [2] consisting of [2, Lemmas 5.3.1–5.3.4]. As such, we will only sketch the details here.

First, one proceeds to show that properties (i) and (ii) of Definition 4.1 hold for Ψ when 1_χ is replaced with an arbitrary bi-non-crossing partition. This effectively makes use of the same arguments as in Lemma 4.4; that is, one uses the recursive algorithm to reduce down and then note that the proofs of properties (i) and (ii) above still apply and lets one move elements around as needed. In particular, the same arguments used in [2, Lemmas 5.3.2 and 5.3.3] transfer with the use of Proposition 3.8.

Next, using property (iii), we need only prove property (iv) under the assumption that $s_\chi(1)$ and $s_\chi(n)$ are in the same block W_0 of W. One then follows many of the same ideas as [2, Lemmas 5.3.1 and 5.3.4] by applying the recursive definition from Definition 4.2, moving around the appropriate B-elements using the more general (i) and (ii), and combining the appropriate elements using (iii) as needed. ∎

4.2 Analytical operator-valued bi-free cumulant function

By convolving the analytical bi-moment function with the bi-non-crossing Möbius function, we obtain the following, which is essential to our study of conjugate variables in the subsequent section.

Definition 4.3 Let (A,E,ε,τ) be an analytical B-B-noncommutative probability space, and denote by Ψ the analytical bi-moment function. The *analytical bi-cumulant*

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function

$$\widetilde{\kappa} : \bigcup_{n\in\mathbb{N}} \bigcup_{\chi\in\{\ell,r\}^n} \mathrm{BNC}(\chi) \times A_{\chi(1)} \times \cdots \times A_{\chi(n-1)} \times L_2(A,\tau) \to L_2(B,\tau_B)$$

is defined by

$$\widetilde{\kappa}_\pi(Z_1,\ldots,Z_{n-1},\xi) = \sum_{\substack{\sigma\in\mathrm{BNC}(\chi)\\ \sigma\leq\pi}} \Psi_\sigma(Z_1,\ldots,Z_{n-1},\xi)\mu_{\mathrm{BNC}}(\sigma,\pi),$$

for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, $\xi \in L_2(A,\tau)$, and $Z_k \in A_{\chi(k)}$.

Remark 4.8 (1) In the case when $\pi = 1_\chi$, we will denote the map $\widetilde{\kappa}_{1_\chi}$ simply by $\widetilde{\kappa}_\chi$. By Möbius inversion, we obtain that

$$\Psi_\sigma(Z_1,\ldots,Z_n) = \sum_{\substack{\pi\in\mathrm{BNC}(\chi)\\ \pi\leq\sigma}} \widetilde{\kappa}_\pi(Z_1,\ldots,Z_{n-1},\xi)$$

for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\sigma \in \mathrm{BNC}(\chi)$, and $Z_k \in A_{\chi(k)}$.

(2) In the case that $B = \mathbb{C}$, the analytical bi-cumulant functions are precisely the $L_2(A,\tau)$-valued bi-free cumulants that were used in [4].

Unsurprisingly, the analytic extension of the operator-valued bi-free cumulant function lives up to its name.

Theorem 4.9 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space. Then, the analytical bi-cumulant function is the analytic extension of the operator-valued bi-free cumulant function.*

Proof Recall by [2, Theorem 6.2.1] that the convolution of a bi-multiplicative function with a scalar-valued multiplicative function on the lattice of non-crossing partitions (e.g., the bi-free Möbius function) produces a bi-multiplicative function. As the properties of an analytic extension of a bi-multiplicative function are analogous to those of a bi-multiplicative function, we obtain that the convolution of an analytic extension of a bi-multiplicative function with a scalar-valued multiplicative function on the lattice of non-crossing partitions (e.g., the bi-free Möbius function) produces the analytic extension of the corresponding bi-multiplicative function obtained via the same convolution. Hence, the result follows. ∎

Theorem 4.10 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space. For all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $\pi \in \mathrm{BNC}(\chi)$, and $Z_k \in A_{\chi(k)}$, we have that*

$$\widetilde{\kappa}_\pi(Z_1,\ldots,Z_{n-1},Z_n + N_\tau) = \kappa_\pi^B(Z_1,\ldots,Z_{n-1},Z_n) + N_{\tau_B}.$$

Proof By Theorem 4.6, we know that

$$\widetilde{\Psi}_\pi(Z_1,\ldots,Z_{n-1},Z_n + N_\tau) = E_\pi(Z_1,\ldots,Z_{n-1},Z_n) + N_{\tau_B}$$

for all $\pi \in \mathrm{BNC}(\chi)$. Therefore, as $\widetilde{\kappa}$ and κ^B are the convolution of $\widetilde{\Psi}$ and E against the bi-free Möbius function, respectively, the result follows. ∎

Of course, as [2, Theorem 8.1.1] demonstrated that bi-freeness with amalgamation over B is equivalent to the mixed operator-valued bi-free cumulants vanishing, Theorem 4.10 immediately implies the following.

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Corollary 4.11 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space containing a family of pairs of B-algebras $\{(C_k, D_k)\}_{k\in K}$. Consider the following two conditions:*

(1) *The family $\{(C_k, D_k)\}_{k\in K}$ is bi-free with amalgamation over B with respect to E.*
(2) *For all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, $Z_1, \ldots, Z_n \in A$, and nonconstant maps $\gamma : \{1, \ldots, n\} \to K$ such that*

$$Z_k \in \begin{cases} C_{\gamma(k)}, & \text{if } \chi(k) = \ell, \\ D_{\gamma(k)}, & \text{if } \chi(k) = r, \end{cases}$$

it follows that

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau) = 0.$$

Then, (1) implies (2). In the case that $\tau_B : B \to \mathbb{C}$ is faithful, (2) implies (1).

Proof Note that (1) implies (2) follows from [2, Theorem 8.1.1] and Theorem 4.10. In the case that τ_B is faithful, (2) immediately implies $\kappa_\pi^B(Z_1, \ldots, Z_{n-1}, Z_n) = 0$ where $\{Z_k\}_{k=1}^n$ are as in (2) via Theorem 4.10. Hence, [2, Theorem 8.1.1] completes the argument. ∎

4.3 Vanishing analytical cumulants

However, something stronger than Corollary 4.11 holds. Indeed, note that the analytic operator-valued bi-free cumulant function has the added benefit that the last entry can be an element of $L_2(A, \tau)$ and thus the L_2-image of a product of left and right operators. As such, it is possible to verify that additional analytic bi-cumulants vanish.

The desired result is analogous to the scalar-valued result demonstrated in [4, Proposition 2.30] and proved in a similar manner. Thus, we begin with a generalization of [2, Theorem 9.1.5] where we can expand out a cumulant involving products of operators. In [2, Theorem 9.1.5], only products of left and right operators were considered in the operator-valued setting, whereas [4, Lemma 2.29] expanded out scalar-valued cumulants involving a product of left and right operator in the last entry.

To begin, fix $m < n \in \mathbb{N}$, $\chi \in \{\ell, r\}^m$, integers

$$k(0) = 0 < k(1) < \cdots < k(m) = n,$$

and any function $\widehat{\chi} \in \{\ell, r\}^n$ such that for all $q \in \{1, \ldots, n\}$ for which there exists a (necessarily unique) $p_q \in \{1, \ldots, m-1\}$ with $k(p_q - 1) < q \leq k(p_q)$, we have

$$\widehat{\chi}(q) = \chi(p_q).$$

Thus, $\widehat{\chi}$ is constant from $k(p-1)+1$ to $k(p)$, whereas $\widehat{\chi}$ does not need to be constant from $k(m-1)+1$ to $k(m)$.

We may embed $\mathrm{BNC}(\chi)$ into $\mathrm{BNC}(\widehat{\chi})$ via $\pi \mapsto \widehat{\pi}$ where the blocks of $\widehat{\pi}$ are formed by taking each block V of π and forming a block

$$\widehat{V} = \bigcup_{p\in V} \{k(p-1)+1, \ldots, k(p)\}$$

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of $\widehat{\pi}$. It is not difficult to see that $\widehat{\pi} \in \mathrm{BNC}(\widehat{\chi})$ as $\widehat{\chi}$ is constant on $\{k(p-1)+1,\ldots,k(p)\}$ for all $p \in V\backslash\{m\}$ and although the block containing $\{k(n-1)+1,\ldots,k(n)\}$ has both left and right entries, it occurs at the bottom of the bi-non-crossing diagram and thus poses no problem. Alternatively, this map can be viewed as an analog of the map on non-crossing partitions from [9, Notation 11.9] after applying s_χ^{-1}.

It is easy to see that $\widehat{1_\chi} = 1_{\widehat{\chi}}$,

$$\widehat{0_\chi} = \bigcup_{p=1}^{m} \{k(p-1)+1,\ldots,k(p)\},$$

and that the map $\pi \mapsto \widehat{\pi}$ is injective and order-preserving. Furthermore, the image of $\mathrm{BNC}(\chi)$ under this map is

$$\widehat{\mathrm{BNC}}(\chi) = \left[\widehat{0_\chi}, \widehat{1_\chi}\right] = \left[\widehat{0_\chi}, 1_{\widehat{\chi}}\right] \subseteq \mathrm{BNC}(\widehat{\chi}).$$

Remark 4.12 Recall that since μ_{BNC} is the bi-non-crossing Möbius function, we have for each $\sigma, \pi \in \mathrm{BNC}(\chi)$ with $\sigma \leq \pi$ that

$$\sum_{\substack{\upsilon \in \mathrm{BNC}(\chi) \\ \sigma \leq \upsilon \leq \pi}} \mu_{\mathrm{BNC}}(\upsilon, \pi) = \begin{cases} 1, & \text{if } \sigma = \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Since the lattice structure is preserved under the map defined above, we see that $\mu_{\mathrm{BNC}}(\sigma, \pi) = \mu_{\mathrm{BNC}}(\widehat{\sigma}, \widehat{\pi})$.

It is also easy to see that the partial Möbius inversion from [9, Proposition 10.11] holds in the bi-free setting; that is, if $f, g : \mathrm{BNC}(\chi) \to \mathbb{C}$ are such that

$$f(\pi) = \sum_{\substack{\sigma \in \mathrm{BNC}(\chi) \\ \sigma \leq \pi}} g(\sigma)$$

for all $\pi \in \mathrm{BNC}(\chi)$, then for all $\pi, \sigma \in \mathrm{BNC}(\chi)$ with $\sigma \leq \pi$, we have the relation

$$\sum_{\substack{\upsilon \in \mathrm{BNC}(\chi) \\ \sigma \leq \upsilon \leq \pi}} f(\upsilon)\mu_{\mathrm{BNC}}(\upsilon, \pi) = \sum_{\substack{\omega \in \mathrm{BNC}(\chi) \\ \omega \vee \sigma = \pi}} g(\omega),$$

where $\pi \vee \sigma$ denotes the smallest element of $\mathrm{BNC}(\chi)$ greater than π and σ.

Thus, by following the proofs of either [9, Theorem 11.12], [2, Theorem 9.1.5], or [4, Lemma 2.29], we arrive at the following.

Proposition 4.13 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space. Under the above notation, if $\pi \in \mathrm{BNC}(\chi)$ and $Z_k \in A_{\widehat{\chi}(k)}$, then*

$$\begin{aligned} &\widetilde{\kappa}_\pi(Z_1 \cdots Z_{k(1)}, \ldots, Z_{k(m-2)+1} \cdots Z_{k(m-1)}, Z_{k(m-1)+1} \cdots Z_{k(m)} + N_\tau) \\ &\quad = \sum_{\substack{\sigma \in \mathrm{BNC}(\widehat{\chi}) \\ \sigma \vee \widehat{0_\chi} = \widehat{\pi}}} \widetilde{\kappa}_\sigma(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau). \end{aligned}$$

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In particular, when $\sigma = 1_\chi$, we have

$$\widetilde{\kappa}_\chi(Z_1 \cdots Z_{k(1)}, \ldots, Z_{k(m-2)+1} \cdots Z_{k(m-1)}, Z_{k(m-1)+1} \cdots Z_{k(m)} + N_\tau) = \sum_{\substack{\sigma \in \mathrm{BNC}(\widehat{\chi}) \\ \sigma \vee \widehat{0_\chi} = 1_{\widehat{\chi}}}} \widetilde{\kappa}_\sigma(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau),$$

Proof First, it is not difficult to verify using the recursive definition of the analytic operator-valued bi-moment function that

$$\Psi_v(Z_1 \cdots Z_{k(1)}, \ldots, Z_{k(m-2)+1} \cdots Z_{k(m-1)}, Z_{k(m-1)+1} \cdots Z_{k(m)} + N_\tau) = \Psi_{\widehat{v}}(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau)$$

for all $v \in \mathrm{BNC}(\chi)$. Therefore, we have

$$\begin{aligned}
&\widetilde{\kappa}_\pi(Z_1 \cdots Z_{k(1)}, \ldots, Z_{k(m-2)+1} \cdots Z_{k(m-1)}, Z_{k(m-1)+1} \cdots Z_{k(m)} + N_\tau) \\
&= \sum_{\substack{v \in \mathrm{BNC}(\chi) \\ v \le \pi}} \Psi_v(Z_1 \cdots Z_{k(1)}, \ldots, Z_{k(m-2)+1} \cdots Z_{k(m-1)}, Z_{k(m-1)+1} \cdots Z_{k(m)} + N_\tau) \mu_{\mathrm{BNC}}(v, \pi) \\
&= \sum_{\substack{v \in \mathrm{BNC}(\chi) \\ v \le \pi}} \Psi_{\widehat{v}}(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau) \mu_{\mathrm{BNC}}(\widehat{v}, \widehat{\pi}) \\
&= \sum_{\substack{\sigma \in \mathrm{BNC}(\widehat{\chi}) \\ \widehat{0_\chi} \le \sigma \le \widehat{\pi}}} \Psi_\sigma(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau) \mu_{\mathrm{BNC}}(\sigma, \widehat{\pi}) \\
&= \sum_{\substack{\sigma \in \mathrm{BNC}(\widehat{\chi}) \\ \sigma \vee \widehat{0_\chi} = \widehat{\pi}}} \widetilde{\kappa}_\sigma(Z_1, \ldots, Z_{n-1}, Z_n + N_\tau),
\end{aligned}$$

where the last line follows from Remark 4.12. ∎

Theorem 4.14 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space containing a family of pairs of B-algebras $\{(C_k, D_k)\}_{k \in K}$ that are bi-free with amalgamation over B with respect to E. For each $k \in K$, let $L_2(A_k, \tau)$ be the closed subspace of $L_2(A, \tau)$ generated by*

$$\mathrm{alg}(C_k, D_k) + N_\tau.$$

Then, for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, nonconstant maps $\gamma : \{1, \ldots, n\} \to K$, $\xi \in L_2(A_{\gamma(n)}, \tau)$, and $Z_1, \ldots, Z_{n-1} \in A$ such that

$$Z_k \in \begin{cases} C_{\gamma(k)}, & \text{if } \chi(k) = \ell, \\ D_{\gamma(k)}, & \text{if } \chi(k) = r, \end{cases}$$

it follows that

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, \xi) = 0.$$

Proof Fix an $m \in \mathbb{N}$, $\chi \in \{\ell, r\}^m$, nonconstant map $\gamma : \{1, \ldots, m\} \to K$, and $Z_1, \ldots, Z_{m-1} \in A$. For any $n \ge m$, if $Z_m, \ldots, Z_n \in C_{\gamma(m)} \cup D_{\gamma(m)}$ and $\widehat{\chi}$ is defined by

$$\widehat{\chi}(k) = \begin{cases} \chi(k), & \text{if } k \le m, \\ \ell, & \text{if } k > m \text{ and } Z_k \in C_{\gamma(m)}, \\ r, & \text{if } k > m \text{ and } Z_k \in D_{\gamma(m)}, \end{cases}$$

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then by Proposition 4.13 and Theorem 4.6 imply that

$$\begin{aligned}\widetilde{\kappa}_{\chi}(Z_1,\ldots,Z_{m-1},Z_m\cdots Z_n+N_\tau) &= \sum_{\substack{\sigma\in \mathrm{BNC}(\widehat{\chi})\\ \sigma\vee\widehat{0_\chi}=1_{\widehat{\chi}}}} \widetilde{\kappa}_{\sigma}(Z_1,\ldots,Z_{n-1},Z_n+N_\tau)\\ &= \sum_{\substack{\sigma\in \mathrm{BNC}(\widehat{\chi})\\ \sigma\vee\widehat{0_\chi}=1_{\widehat{\chi}}}} \kappa_{\sigma}^{B}(Z_1,\ldots,Z_{n-1},Z_n)+N_{\tau_B}.\end{aligned}$$

As the conditions $\sigma \in \mathrm{BNC}(\widehat{\chi})$ and $\sigma \vee \widehat{0_\chi} = 1_{\widehat{\chi}}$ automatically imply that each block of σ containing one of $\{1,2,\ldots,m-1\}$ must also contain an element of $\{m,\ldots,n\}$, the bi-multiplicative properties of the operator-valued bi-free cumulant function imply that each cumulant $\kappa_\sigma^B(Z_1,\ldots,Z_{n-1},Z_n)$ appearing in the sum above can be reduced down to an expression involving a mixed κ^B term which must be 0 by [2, Theorem 8.1.1]. Hence,

$$\widetilde{\kappa}_{\chi}(Z_1,\ldots,Z_{m-1},Z_m\cdots Z_n+N_\tau)=0.$$

Since $\widetilde{E}$ is a continuous function and left multiplication of A on $L_2(A,\tau)$ yields bounded operators, due to the recursive nature of Ψ, we see that Ψ is continuous in the $L_2(A,\tau)$ entry. Therefore, by Möbius inversion, $\widetilde{\kappa}$ is continuous in the $L_2(A,\tau)$ entry. Hence, the result follows. ∎

Corollary 4.15 *Let (A,E,ε,τ) be an analytical B-B-noncommutative probability space, and let $n\geq 2$, $\chi\in\{\ell,r\}^n$, $\xi\in L_2(A,\tau)$, $b\in B$, and $Z_k\in A_{\chi(k)}$. Suppose that either there exists $p\in\{1,\ldots,n-1\}$ such that*

$$Z_p=\begin{cases}L_b, & \text{if } \chi(p)=\ell\\ R_b, & \text{if } \chi(p)=r\end{cases}$$

or that $\xi\in L_2(B,\tau_B)$. Then,

$$\widetilde{\kappa}_{\chi}(Z_1,\ldots,Z_{n-1},\xi)=0.$$

Proof If $Z_p=L_b$ or $Z_p=R_b$ for some p, then we may proceed as in the proof of Theorem 4.14 by assuming that ξ is an element of A, expanding out the analytic operator-valued bi-free cumulant function with the aid of Proposition 4.13, and using the fact that non-singleton operator-valued bi-free cumulants involving L_b or R_b terms are zero by [2, Proposition 6.4.1] and then taking a limit at the end.

In the case where $\xi\in L_2(B,\tau_B)$, ξ is a limit of terms of the form L_b+N_τ. As

$$\widetilde{\kappa}_{\chi}(Z_1,\ldots,Z_{n-1},L_b+N_\tau)=\kappa_\chi(Z_1,\ldots,Z_{n-1},L_b)+N_{\tau_B}=0+N_{\tau_B},$$

by Theorem 4.10 and [2, Proposition 6.4.1], the result follows by taking a limit. ∎

5 Bi-free conjugate variables with respect to completely positive maps

In this section, we develop the appropriate notion of conjugate variables in order to define bi-free Fisher information and entropy with respect to completely positive maps. This can be viewed as both an extension of the bi-free conjugate variables

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developed in [4] and of the free conjugate variables with respect to a completely positive map developed in [10]. We will focus on both the moment and cumulant characterizations of these conjugate variables, whereas [10] focused on the moment and derivation characterizations of free conjugate variables. Although [4] analyzed the moment, cumulant, and bi-free difference quotient characterizations of bi-free conjugate variables, we will forgo trying to generalize the bi-free difference quotient characterization in this setting as it was the cumulant characterization that was found most effective and as the bi-module structures of [10] necessary for the derivation characterization using adjoints are less clear in this setting.

We refer the reader to [8, Definition 2.7] as an equivalent description to [10] of the free conjugate variables with respect to a completely positive map that we mimic below.

Definition 5.1 Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, let $Y \in A_r$, and let $\eta : B \to B$ be a completely positive map.

An element $\xi \in L_2(A, \tau)$ is said to satisfy the *left bi-free conjugate variable relations for X with respect to η and τ in the presence of* (C_ℓ, C_r) if for all $n \in \mathbb{N} \cup \{0\}$, $Z_1, \ldots, Z_n \in \{X\} \cup C_\ell \cup C_r$ we have

$$\tau(Z_1 \cdots Z_n \xi) = \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \tau\left(\left(\prod_{p \in V_k^c \setminus \{k, n+1\}} Z_p\right) L_{\eta\left(E\left(\prod_{p \in V_k} Z_p\right)\right)}\right),$$

where $V_k = \{k < m < n+1 \mid Z_m \in \{X\} \cup C_\ell\}$ and where all products are taken in numeric order (with the empty product being 1). If, in addition,

$$\xi \in \overline{\operatorname{alg}(X, C_\ell, C_r)}^{\|\cdot\|_\tau},$$

we call ξ the *left bi-free conjugate variable for X with respect to η and τ in the presence of* (C_ℓ, C_r) and denote ξ by $J_\ell(X : (C_\ell, C_r), \eta)$.

Similarly, an element $\nu \in L_2(A, \tau)$ is said to satisfy the *right bi-free conjugate variable relations for Y with respect to η and τ in the presence of* (C_ℓ, C_r) if for all $n \in \mathbb{N} \cup \{0\}$, $Z_1, \ldots, Z_n \in \{Y\} \cup C_\ell \cup C_r$ we have

$$\tau(Z_1 \cdots Z_n \nu) = \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \tau\left(\left(\prod_{p \in V_k^c \setminus \{k, n+1\}} Z_p\right) R_{\eta\left(E\left(\prod_{p \in V_k} Z_p\right)\right)}\right),$$

where $V_k = \{k < m < n+1 \mid Z_m \in \{Y\} \cup C_r\}$. If, in addition,

$$\nu \in \overline{\operatorname{alg}(Y, C_\ell, C_r)}^{\|\cdot\|_\tau},$$

we call ν the *right bi-free conjugate variable for Y with respect to η and τ in the presence of* (C_ℓ, C_r) and denote ν by $J_r(Y : (C_\ell, C_r), \eta)$.

Example 5.1 For an example of Definition 5.1, consider $X \in A_\ell$, $Y \in A_r$, $Z_2, Z_3 \in C_\ell$, and $Z_1, Z_4 \in C_r$. If $\xi = J_\ell(X : (C_\ell, \operatorname{alg}(C_r, Y)), \eta)$, then

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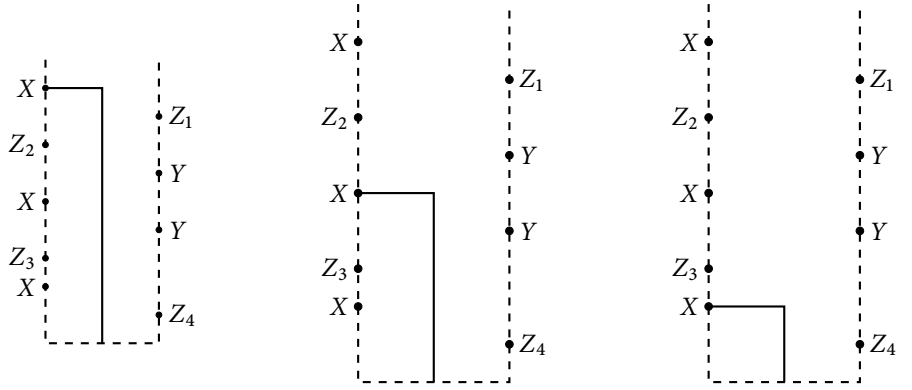
$$\begin{aligned}\tau(XZ_1Z_2YXYZ_3XZ_4\xi) = \tau\left(Z_1YYZ_4L_{\eta(E(Z_2XZ_3X))}\right)\\ + \tau\left(XZ_1Z_2YYZ_4L_{\eta(E(Z_3X))}\right)\\ + \tau\left(XZ_1Z_2YXYZ_3Z_4L_{\eta(E(1))}\right).\end{aligned}$$

This can be observed diagrammatically by drawing $X, Z_1, Z_2, Y, X, Y, Z_3, X, Z_4$ as one would in a bi-non-crossing diagram (i.e., drawing two vertical lines and placing the variables on these lines starting at the top and going down with left variables on the left line and right variables on the right line), drawing all pictures connecting the center of the bottom of the diagram to any X, taking the product of the elements starting from the top and going down in each of the two isolated components of the diagram, taking the expectation of the bounded region and applying η to the result to obtain a $b \in B$, appending L_b to the end of the product of operators from the unbounded region, and applying τ to the result.



This is analogous to applying the left bi-free difference quotient $\partial_{\ell,X}$ defined in [4] on a suitable algebraic free product to $XZ_1Z_2YXYZ_3XZ_4$ to obtain

$$Z_1YYZ_4 \otimes Z_2XZ_3X + XZ_1Z_2YYZ_4 \otimes Z_3X + XZ_1Z_2YXYZ_3Z_4 \otimes 1,$$

applying $\mathrm{Id} \otimes (\eta \circ E)$, collapsing the tensor, and applying τ to the result.

Similarly, if $\nu = J_r(Y : (\mathrm{alg}(C_\ell, X), C_r), \eta)$, then

$$\tau(XZ_1Z_2YXYZ_3XZ_4\nu) = \tau\left(XZ_1Z_2XZ_3XR_{\eta(E(YZ_4))}\right) + \tau\left(XZ_1Z_2YXZ_3XR_{\eta(E(Z_4))}\right).$$

This can be observed diagrammatically in a similar fashion by drawing all pictures connecting the center of the bottom of the diagram to any Y on the right.

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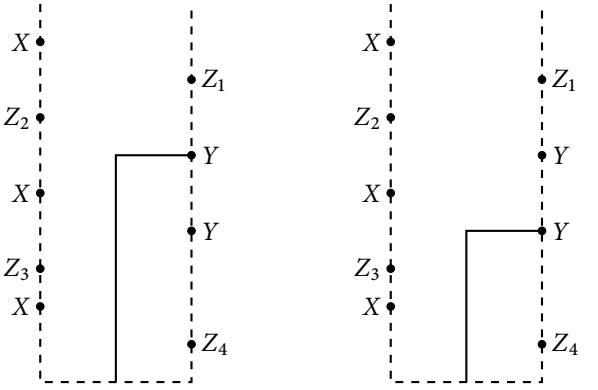

This is analogous to applying the right bi-free difference quotient $\partial_{r,Y}$ defined in [4] on a suitable algebraic free product to $XZ_1Z_2YXYZ_3XZ_4$ to obtain

$$XZ_1Z_2XZ_3X \otimes YZ_4 + XZ_1Z_2YXZ_3X \otimes Z_4,$$

applying $\mathrm{Id} \otimes (\eta \circ E)$, collapsing the tensor, and applying τ to the result.

Remark 5.2 (1) As $\tau(aL_b) = \tau(aR_b)$ for all $a \in A$ and $b \in B$, one may use either $L_{\eta\circ E}$ or $R_{\eta\circ E}$ in either part of Definition 5.1. In fact, one may simply use $\eta \circ E$ if one views the resulting element of B as an element of $L_2(B, \tau_B) \subseteq L_2(A, \tau)$, since $\tau(aL_b) = \tau(a(b + N_\tau))$ for all $a \in A$ and $b \in B$ by construction.

(2) The element $J_\ell(X : (C_\ell, C_r), \eta)$ is unique in the sense that if $\xi_0 \in \overline{\mathrm{alg}(X, C_\ell, C_r)}^{\|\cdot\|_\tau}$ satisfies the left bi-free conjugate variable relations for X with respect to (C_ℓ, C_r), then $\xi_0 = J_\ell(X : (C_\ell, C_r), \eta)$ as the left bi-free conjugate variable relations causes the inner products in $L_2(A, \tau)$ of both ξ_0 and $J_\ell(X : (C_\ell, C_r), \eta)$ against any element of $\mathrm{alg}(X, C_\ell, C_r)$ to be equal.

(3) In the case where $B = \mathbb{C}$, E reduces down to a unital, linear map $\varphi : A \to \mathbb{C}$ and, as τ is compatible with E, one obtains that $\tau = \varphi$. As φ is linear, Definition 5.1 immediately reduces down to the left and right conjugate variables with respect to φ in the presence of (C_ℓ, C_r) as in [4], provided η is unital.

(4) In the setting of Example 2.3, we note that $J_\ell(X : (B_\ell, B_r), \eta)$ exists if and only if the free conjugate variable of X with respect to (B, η) from [10] exists. This immediately follows as B_r commutes with X and B_ℓ and τ is tracial, so the expressions for either conjugate variable can be modified into the expressions of the other conjugate variable.

As with the bi-free conjugate variables in [4], any moment expression should be equivalent to certain cumulant expressions via Möbius inversion. Thus, we obtain the following equivalent characterization of conjugate variables. Note in that which follows, it does not matter whether the last entry in the analytical operator-valued bi-free cumulant function is treated as a left or as a right operator by Definition 4.1.

Theorem 5.3 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, let $Y \in A_r$, and let $\eta : B \to B$ be a completely positive map. For $\xi \in L_2(A, \tau)$, the following are equivalent:*

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(1) *ξ satisfies the left bi-free conjugate variable relations for X (resp. ξ satisfies the right bi-free conjugate variable relations for Y) with respect to η and τ in the presence of (C_ℓ, C_r).*
(2) *The following four cumulant conditions hold:*
   (a) $\widetilde{\kappa}_{1_{(\ell)}}(\xi) = 0 + N_{\tau_B}$.
   (b) $\widetilde{\kappa}_{1_{(\ell,\ell)}}(XL_b, \xi) = \eta(b) + N_{\tau_B}$ *(resp.* $\widetilde{\kappa}_{1_{(r,r)}}(YR_b, \xi) = \eta(b) + N_{\tau_B}$*) for all* $b \in B$.
   (c) $\widetilde{\kappa}_{1_{(\ell,\ell)}}(c_1, \xi) = \widetilde{\kappa}_{1_{(r,\ell)}}(c_2, \xi) = 0 + N_{\tau_B}$ *for all* $c_1 \in C_\ell$ *and* $c_2 \in C_r$.
   (d) *For all* $n \geq 3$, $\chi \in \{\ell, r\}^n$, *and all* $Z_1, Z_2, \ldots, Z_{n-1} \in A$ *such that*

$$Z_k \in \begin{cases} \{X\} \cup C_\ell, & \text{if } \chi(k) = \ell \\ C_r, & \text{if } \chi(k) = r \end{cases} \quad \left(\text{resp.} \quad Z_k \in \begin{cases} C_\ell, & \text{if } \chi(k) = \ell \\ \{Y\} \cup C_r, & \text{if } \chi(k) = r \end{cases} \right),$$

we have that

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{n-1}, \xi) = 0 + N_{\tau_B}.$$

Proof We will prove that the result for the left bi-free conjugate variable as the proof for the right bi-free conjugate variable is analogous.

Suppose that ξ satisfies (ii). To see that ξ satisfies the left bi-free conjugate variable relations, let $n \in \mathbb{N} \cup \{0\}$ and let $Z_1, \ldots, Z_n \in \{X\} \cup C_\ell \cup C_r$. Fix $\chi \in \{\ell, r\}^{n+1}$ such that

$$\chi(k) = \begin{cases} \ell, & \text{if } Z_k \in \{X\} \cup C_\ell \\ r, & \text{if } Z_k \in C_r \end{cases}$$

(note that the value of $\chi(n+1)$ does not matter in that which follows). By the relation between the analytic extensions of the bi-moment and bi-cumulant functions, we obtain that

$$\widetilde{E}(Z_1 \cdots Z_n \xi) = \sum_{\pi \in \mathrm{BNC}(\chi)} \widetilde{\kappa}_\pi(Z_1, \ldots, Z_n, \xi).$$

Due to the cumulant conditions in (ii), the only way $\widetilde{\kappa}_\pi(Z_1, \ldots, Z_n, \xi)$ is nonzero is if the block of π containing $n+1$ contains a single other index k with $Z_k = X$. Moreover, there is a bijection between such partitions and partitions of the form

$$\pi = \{k, n+1\} \cup \pi_1 \cup \pi_2,$$

where π_1 is a bi-non-crossing partition on $V_k = \{k < m < n+1 \mid Z_m \in \{X\} \cup C_\ell\}$ with respect to $\chi|_{V_k}$ and where π_2 is a bi-non-crossing partition on $W_k = V_k \backslash \{k, n+1\}$ with respect to $\chi|_{W_k}$. Using this decomposition, the properties of bi-analytic extensions of bi-multiplicative functions and the moment-cumulant formulas yield that

$$\begin{aligned} &\widetilde{E}(Z_1 \cdots Z_n \xi) \\ &= \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \sum_{\pi_2 \in \mathrm{BNC}(\chi|_{W_k})} \sum_{\pi_1 \in \mathrm{BNC}(\chi|_{V_k})} \widetilde{\kappa}_{\{k, n+1\} \cup \pi_1 \cup \pi_2}(Z_1, \ldots, Z_n, \xi) \\ &= \sum_{\substack{1 \leq k \leq n \\ Z_k = X}} \sum_{\pi_2 \in \mathrm{BNC}(\chi|_{W_k})} \widetilde{\kappa}_{\{k, n+1\} \cup \pi_1} \left(\left(Z_1, \ldots, Z_{k-1}, Z_k L_{E\left(\prod_{p \in V_k} Z_p\right)} Z_{k+1}, \ldots, Z_n, \xi \right) \Big| \, W_k \cup \{k, n+1\} \right) \end{aligned}$$

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$$= \sum_{\substack{1\leq k\leq n\\ Z_k=X}} \widetilde{\kappa}_{\pi_1}\left(\left(Z_1,\ldots,Z_{\max_{\leq}(W_k)-1},Z_{\max_{\leq}(W_k)}\eta\left(E\left(\prod_{p\in V_k}Z_p\right)\right)\right)\middle|W_k\right)$$

$$= \sum_{\substack{1\leq k\leq n\\ Z_k=X}} \tilde{E}\left(\left(\prod_{p\in V_k^c\setminus\{k,n+1\}}Z_p\right)\eta\left(E\left(\prod_{p\in V_k}Z_p\right)\right)\right).$$

Hence, by applying τ_B to both sides of this equation, the left bi-free conjugate variable relations from Definition 5.1 are obtained via part (i) of Proposition 3.8.

For the converse direction, suppose that ξ satisfies the left bi-free conjugate variable relations for X. Thus, for all $b\in B$, $\tau(L_b\xi)=0$ by the conjugate variable relations. Hence, $\widetilde{\kappa}_{1_{(\ell)}}(\xi)=\widetilde{E}(\xi)=0$ by part (vi) of Proposition 3.8 and therefore (a) holds.

To see that (b) holds, note for all $b_0,b\in B$ that

$$\begin{aligned} b_0\widetilde{\kappa}_{1_{(\ell,\ell)}}(XL_b,\xi)=\widetilde{\kappa}_{1_{(\ell,\ell)}}(L_{b_0}XL_b,\xi)&=\Psi_{1_{(\ell,\ell)}}(L_{b_0}XL_b,\xi)-\Psi_{0_{(\ell,\ell)}}(L_{b_0}XL_b,\xi)\\ &=\widetilde{E}(L_{b_0}XL_b\xi)-E(L_{b_0}XL_b)\widetilde{E}(\xi)=\widetilde{E}(L_{b_0}XL_b\xi).\end{aligned}$$

Therefore, by applying τ_B to both sides, we obtain that

$$\tau_B\left(b_0\widetilde{\kappa}_{1_{(\ell,\ell)}}(XL_b,\xi)\right)=\tau_B\left(\widetilde{E}(L_{b_0}XL_b\xi)\right)=\tau(L_{b_0}XL_b\xi).$$

By the left bi-free conjugate variable relations, we obtain that

$$\tau_B\left(b_0\widetilde{\kappa}_{1_{(\ell,\ell)}}(XL_b,\xi)\right)=\tau\left(L_{b_0}L_{\eta(E(L_b))}\right)=\tau(L_{b_0}L_{\eta(b)})=\tau(L_{b_0\eta(b)})=\tau_B(b_0\eta(b)).$$

As this holds for all $b_0\in B$, we obtain that $\widetilde{\kappa}_{1_{(\ell,\ell)}}(XL_b,\xi)=\eta(b)+N_{\tau_B}$ as desired.

To see that (c) holds, note for all $b\in B$ and $c_1\in C_\ell$ that

$$\tau_B\left(b\widetilde{\kappa}_{1_{(\ell,\ell)}}(c_1,\xi)\right)=\tau(L_bc_1\xi)=0$$

by similar computations as above. Since this holds for all $b\in B$, we see that $\widetilde{\kappa}_{1_{(\ell,\ell)}}(c_1,\xi)=0+N_{\tau_B}$. Similarly, for all $c_2\in C_r$, we see that

$$\begin{aligned}\widetilde{\kappa}_{1_{(r,\ell)}}(c_2,\xi)b=\widetilde{\kappa}_{1_{(r,\ell)}}(R_bc_2,\xi)&=\Psi_{1_{(r,\ell)}}(R_bc_2,\xi)-\Psi_{0_{(r,\ell)}}(R_bc_2,\xi)\\ &=\widetilde{E}(R_bc_2\xi)-\widetilde{E}(\xi)E(R_bc_2)=\widetilde{E}(R_bc_2\xi).\end{aligned}$$

Therefore, by applying τ_B to both sides, we obtain that

$$\tau_B\left(\widetilde{\kappa}_{1_{(r,\ell)}}(c_2,\xi)b\right)=\tau_B\left(\widetilde{E}(R_bc_2\xi)\right)=\tau(R_bc_2\xi).$$

By the left bi-free conjugate variable relations, we obtain that

$$\tau_B\left(\widetilde{\kappa}_{1_{(r,\ell)}}(c_2,\xi)b\right)=0.$$

Therefore, as τ_B is tracial and the above holds for all $b\in B$, we obtain that $\widetilde{\kappa}_{1_{(r,\ell)}}(c_2,\xi)=0+N_{\tau_B}$ as desired.

For (d), we proceed by induction on n. To do so, we will prove the base case $n=3$ and the inductive step simultaneously. Fix $n\geq 3$ and suppose when $n>3$ that (d) holds for all $m<n$. Let $\chi\in\{\ell,r\}^n$, and let $Z_1,Z_2,\ldots,Z_{n-1}\in A$ be as in the assumptions of (d). We will assume that $\chi(1)=r$ as the case $\chi(1)=\ell$ will be handled similarly. Thus, for all $b\in B$, we know that

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$$\begin{aligned}\widetilde{\kappa}_\chi(Z_1,\ldots,Z_{n-1},\xi)b &= \widetilde{\kappa}_\chi(R_bZ_1,Z_2,\ldots,Z_{n-1},\xi)\\ &= \widetilde{E}(R_bZ_1Z_2\cdots Z_{n-1}\xi) - \sum_{\substack{\pi\in \mathrm{BNC}(\chi)\\ \pi\neq 1_\chi}} \widetilde{\kappa}_\pi(R_bZ_1,Z_2,\ldots,Z_{n-1},\xi)\\ &= \widetilde{E}(R_bZ_1Z_2\cdots Z_{n-1}\xi) - \sum_{\substack{\pi\in \mathrm{BNC}(\chi)\\ \pi\neq 1_\chi}} \widetilde{\kappa}_\pi(Z_1,Z_2,\ldots,Z_{n-1},\xi)b.\end{aligned}$$

Using the fact that (a)–(c) hold and that (d) holds for all $m < n$, we obtain using the same arguments used in the other direction of the proof that

$$\sum_{\substack{\pi\in \mathrm{BNC}(\chi)\\ \pi\neq 1_\chi}} \widetilde{\kappa}_\pi(Z_1,\ldots,Z_{n-1},\xi)b = \sum_{\substack{1\leq k<n\\ Z_k=X}} \tilde{E}\left(R_b\left(\prod_{p\in V_k^c\setminus\{k,n\}} Z_p\right)\eta\left(E\left(\prod_{p\in V_k} Z_p\right)\right)\right),$$

where $V_k = \{k < m < n \mid Z_m \in \{X\} \cup C_\ell\}$. Therefore, by applying τ_B to both sides of our initial equation, we obtain that

$$\begin{aligned}&\tau_B\left(\widetilde{\kappa}_\chi(Z_1,\ldots,Z_{n-1},\xi)b\right)\\ &= \tau_B\left(\widetilde{E}(R_bZ_1Z_2\cdots Z_{n-1}\xi) - \sum_{\substack{1\leq k<n\\ Z_k=X}} \tilde{E}\left(R_b\left(\prod_{p\in V_k^c\setminus\{k,n\}} Z_p\right)\eta\left(E\left(\prod_{p\in V_k} Z_p\right)\right)\right)\right)\\ &= \tau(R_bZ_1Z_2\cdots Z_{n-1}\xi) - \sum_{\substack{1\leq k<n\\ Z_k=X}} \tau\left(R_b\left(\prod_{p\in V_k^c\setminus\{k,n\}} Z_p\right)\eta\left(E\left(\prod_{p\in V_k} Z_p\right)\right)\right) = 0,\end{aligned}$$

by the left bi-free conjugate variable relations. Therefore, as the above holds for all $b \in B$ and τ_B is tracial, the result follows. ∎

The cumulant approach to conjugate variables has merits as it is very simple to check that most cumulants vanish and the values of others. For instance, an observant reader might have noticed that the operators X and Y in Definition 5.1 were not required to be self-adjoint. This is for later use in the paper and can be converted to studying self-adjoint operators as follows.

Lemma 5.4 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. The left bi-free conjugate variables*

$$J_\ell(X : (\mathrm{alg}(C_\ell, X^*), C_r), \eta) \qquad \text{and} \qquad J_\ell(X^* : (\mathrm{alg}(C_\ell, X), C_r), \eta)$$

exist if and only if

$$J_\ell(\Re(X) : (\mathrm{alg}(C_\ell, \Im(X)), C_r), \eta) \qquad \text{and} \qquad J_\ell(\Im(X) : (\mathrm{alg}(C_\ell, \Re(X)), C_r), \eta)$$

exist where $\Re(X) = \frac{1}{2}(X + X^*)$ and $\Im(X) = \frac{1}{2i}(X - X^*)$. Furthermore,

$$J_\ell(\Re(X) : (\mathrm{alg}(C_\ell, \Im(X)), C_r), \eta) = J_\ell(X : (\mathrm{alg}(C_\ell, X^*), C_r), \eta) + J_\ell(X^* : (\mathrm{alg}(C_\ell, X), C_r), \eta),$$
$$J_\ell(\Im(X) : (\mathrm{alg}(C_\ell, \Re(X)), C_r), \eta) = iJ_\ell(X : (\mathrm{alg}(C_\ell, X^*), C_r), \eta) - iJ_\ell(X^* : (\mathrm{alg}(C_\ell, X), C_r), \eta).$$

A similar result holds for right bi-free conjugate variables.

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Proof Suppose that

$$\xi_1 = J_\ell(X : (\mathrm{alg}(C_\ell, X^*), C_r), \eta) \qquad \text{and} \qquad \xi_2 = J_\ell(X^* : (\mathrm{alg}(C_\ell, X), C_r), \eta)$$

exist. Hence, ξ_1 and ξ_2 satisfy the appropriate analytic cumulant equations from Theorem 5.3. Let

$$h_1 = \xi_1 + \xi_2 \qquad \text{and} \qquad h_2 = i\xi_1 - i\xi_2.$$

As

$$\xi_1 \in \overline{\mathrm{alg}(X, \mathrm{alg}(C_\ell, X^*), C_r)}^{\|\cdot\|_\tau} \qquad \text{and} \qquad \xi_2 \in \overline{\mathrm{alg}(X^*, \mathrm{alg}(C_\ell, X), C_r)}^{\|\cdot\|_\tau},$$

we easily see that

$$h_1 \in \overline{\mathrm{alg}(\Re(X), \mathrm{alg}(C_\ell, \Im(X)), C_r)}^{\|\cdot\|_\tau} \qquad \text{and} \qquad \xi_2 \in \overline{\mathrm{alg}(\Im(X), \mathrm{alg}(C_\ell, \Re(X)), C_r)}^{\|\cdot\|_\tau}.$$

Thus, by Theorem 5.3, it suffices to show that h_1 and h_2 satisfy the appropriate conjugate variable formulae. Indeed, property (a) of Theorem 5.3 holds as

$$\widetilde{\kappa}_{1_{(\ell)}}(h_1) = \widetilde{\kappa}_{1_{(\ell)}}(h_2) = 0 + N_{\tau_B}.$$

Next, notice for all $b \in B$ that

$$\begin{aligned}
&\widetilde{\kappa}_{1_{(\ell,\ell)}}(\Re(X)L_b, h_1) \\
&= \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}XL_b, \xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}X^*L_b, \xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}XL_b, \xi_2\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}X^*L_b, \xi_2\right) \\
&= \frac{1}{2}\eta(b) + 0 + 0 + \frac{1}{2}\eta(b) = \eta(b)
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\kappa}_{1_{(\ell,\ell)}}(\Im(X)L_b, h_2) \\
&= \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2i}XL_b, i\xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(-\frac{1}{2i}X^*L_b, i\xi_1\right) \\
&\quad + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2i}XL_b, -i\xi_2\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(-\frac{1}{2i}X^*L_b, -i\xi_2\right) \\
&= \frac{1}{2i}i\eta(b) - 0 + 0 - \frac{1}{2i}(-i)\eta(b) = \eta(b).
\end{aligned}$$

Hence, property (b) of Theorem 5.3 holds.

To see that properties (c) and (d) of Theorem 5.3 hold, note for all $b \in B$ that

$$\begin{aligned}
&\widetilde{\kappa}_{1_{(\ell,\ell)}}(\Re(X)L_b, h_2) \\
&= \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}XL_b, i\xi_1\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}X^*L_b, i\xi_1\right) \\
&\quad + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}XL_b, -i\xi_2\right) + \widetilde{\kappa}_{1_{(\ell,\ell)}}\left(\frac{1}{2}X^*L_b, -i\xi_2\right) \\
&= \frac{1}{2}i\eta(b) + 0 + 0 + \frac{1}{2}(-i)\eta(b) = 0,
\end{aligned}$$

and similarly $\widetilde{\kappa}_{1_{(\ell,\ell)}}(\Im(X)L_b, h_1) = 0$. Therefore, Proposition 4.13 along with the linearity of the cumulants in each entry yields properties (c) and (d).

The converse direction is proved analogously. ∎

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Of course, many other results follow immediately from the cumulant definition of the conjugate variables.

Lemma 5.5 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. If*

$$\xi = J_\ell(X : (C_\ell, C_r), \eta)$$

exists, then, for all $\lambda \in \mathbb{C}\backslash\{0\}$, the conjugate variable $J_\ell(\lambda X : (C_\ell, C_r), \eta)$ exists and is equal to $\frac{1}{\lambda}\xi$.

A similar result holds for right bi-free conjugate variables.

Proposition 5.6 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta_1, \eta_2 : B \to B$ be completely positive maps. If*

$$\xi_1 = J_\ell(X : (C_\ell, C_r), \eta_1) \qquad \text{and} \qquad \xi_2 = J_\ell(X : (C_\ell, C_r), \eta_2)$$

exist, then $\xi = J_\ell(X : (C_\ell, C_r), \eta_1 + \eta_2)$ exists and $\xi = \xi_1 + \xi_2$.

A similar result holds for right bi-free conjugate variables.

Proposition 5.7 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. For fixed $b_1, b_2 \in B$, define $\eta_{\ell,b_1,b_2} : B \to B$ by*

$$\eta_{\ell,b_1,b_2}(b) = \eta(bb_2)b_1$$

for all $b \in B$. If $\xi = J_\ell(X : (C_\ell, C_r), \eta)$ exists and η_{ℓ,b_1,b_2} is completely positive, then $J_\ell(X : (C_\ell, C_r), \eta_{\ell,b_1,b_2})$ exists and

$$J_\ell(X : (C_\ell, C_r), \eta_{\ell,b_1,b_2}) = R_{b_1}L_{b_2}\xi.$$

Similarly, if $Y \in A_r$ and $\eta_{r,b_1,b_2} : B \to B$ is defined by

$$\eta_{r,b_1,b_2} = b_2\eta(b_1 b)$$

for all $b \in B$ is completely positive, and $J_r(Y : (C_\ell, C_r), \eta)$ exists, then $J_r(Y : (C_\ell, C_r), \eta_{r,b_1,b_2})$ exists and

$$J_r(Y : (C_\ell, C_r), \eta_{r,b_1,b_2}) = R_{b_1}L_{b_2}J_r(Y : (C_\ell, C_r), \eta).$$

Proof By Theorem 5.3, it suffices to show that $R_{b_1}L_{b_2}\xi$ satisfies the appropriate analytical operator-valued bi-free cumulant formula. Indeed, clearly,

$$\widetilde{\kappa}_{1_{(\ell)}}(R_{b_1}L_{b_2}\xi) = b_2\widetilde{\kappa}_{1_{(\ell)}}(\xi)b_1 = 0,$$

and for all $b \in B$,

$$\widetilde{\kappa}_{1_{(\ell,\ell)}}(XL_b, R_{b_1}L_{b_2}\xi) = \widetilde{\kappa}_{1_{(\ell,\ell)}}(XL_bL_{b_2}, \xi)b_1 = \widetilde{\kappa}_{1_{(\ell,\ell)}}(XL_{bb_2}, \xi)b_1 = \eta(bb_2)b_1 = \eta_{b_1,b_2}(b).$$

To show that the other analytical operator-valued bi-free cumulants from Theorem 5.3 vanish, one simply needs to use the analytical extension properties of bi-multiplicative functions together with Proposition 4.13. The result for right bi-free conjugate variables is analogous. ∎

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Similarly, many results pertaining to conjugate variables from [4, 10, 14] immediately generalize to the conjugate variables in Definition 5.1. However, one result from [10] requires additional setup. In the context of Example 2.3, one can always consider a further von Neumann subalgebra D of B and ask how the conjugate variables react. To analyze the comparable situation in our setting, we need the following example.

Example 5.8 Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, and let D be a unital $*$-subalgebra of B (with $1_D = 1_B$). If $F : B \to D$ is a conditional expectation in the sense that $F(d) = d$ for all $d \in D$ and $F(d_1 b d_2) = d_1 F(b) d_2$ for all $d_1, d_2 \in D$ and $b \in B$, then $(A, F \circ E, \varepsilon|_{D \otimes D^{\mathrm{op}}})$ is a D-D-noncommutative probability space by [12, Section 3].

Note that $\tau_D = \tau_B|_D : D \to \mathbb{C}$ is a tracial state being the restriction of a tracial state. Moreover, if τ_B is compatible with F in the sense that $\tau_B(F(b)) = \tau_B(b)$ for all $b \in B$, we easily see that τ is compatible with $F \circ E$, as for all $a \in A$ we have that

$$\tau(a) = \tau\left(L_{E(a)}\right) = \tau_B(E(a)) = \tau_B(F(E(a))) = \tau\left(L_{F(E(a))}\right)$$

and similarly $\tau(a) = \tau\left(R_{F(E(a))}\right)$. Hence, $(A, F \circ E, \varepsilon|_{D \otimes D^{\mathrm{op}}}, \tau)$ is an analytical D-D-noncommutative probability space.

Proposition 5.9 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, and let $X \in A_\ell$. In addition, let D and F be as in Example 5.8, let (C_ℓ, C_r) be a pair of B-algebras (and thus automatically a pair of D-algebras), and let $\eta : D \to D$ be a completely positive map. Moreover, suppose that F is completely positive (and hence $\eta \circ F : B \to D \subseteq B$ is also completely positive).*

Then, the conjugate variable $J_\ell(X : (C_\ell, C_r), \eta \circ F)$ exists in the analytical B-B-noncommutative probability space $(A, E, \varepsilon, \tau)$ if and only if the conjugate variable $J_\ell(X : (C_\ell, C_r), \eta)$ exists in the analytical D-D-noncommutative probability space $(A, F \circ E, \varepsilon|_{D \otimes D^{\mathrm{op}}}, \tau)$, in which case they are the same element of $L_2(A, \tau)$.

A similar result holds for right bi-free conjugate variables.

Proof As $(\eta \circ F) \circ E = \eta \circ (F \circ E)$, the bi-free conjugate variable relations from Definition 5.1 are precisely the same and thus there is nothing to prove. ∎

With Proposition 5.9 out of the way, we turn our attention to proving that the expected generalizations of conjugate variable properties from [4, 10, 14] hold.

Lemma 5.10 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. Suppose further that $D_\ell \subseteq C_\ell$ and $D_r \subseteq C_r$ are such that (D_ℓ, D_r) is a pair of B-algebras in A. If*

$$\xi = J_\ell(X : (C_\ell, C_r), \eta)$$

exists, then

$$\xi' = J_\ell(X : (D_\ell, D_r), \eta)$$

exists. In particular, if P is the orthogonal projection of $L_2(A, \tau)$ onto

$$\overline{\mathrm{alg}(X, D_\ell, D_r)}^{\|\cdot\|_\tau},$$

then $\xi' = P(\xi)$.

A similar result holds for right bi-free conjugate variables.

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Proof Notice that if ξ satisfies the left bi-free conjugate variable relations for X with respect to E and η in the presence of (C_ℓ, C_r), ξ satisfies the left bi-free conjugate variable relations for X with respect to E and η in the presence of (D_ℓ, D_r). Therefore, since $\tau(ZP(\xi)) = \tau(Z\xi)$ for all $Z \in \mathrm{alg}(X, D_\ell, D_r)$, it follows that $P(\xi) = J_\ell(X : (D_\ell, D_r), \eta)$ as desired. ∎

The following generalizes [14, Proposition 3.6], [10, Proposition 3.8], and [4, Proposition 4.3].

Proposition 5.11 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let (C_ℓ, C_r) be a pair of B-algebras in A, let $X \in A_\ell$, and let $\eta : B \to B$ be a completely positive map. If (D_ℓ, D_r) is another pair of B-algebras such that*

$$(\mathrm{alg}(X, C_\ell), C_r) \qquad \textit{and} \qquad (D_\ell, D_r)$$

are bi-free with amalgamation over B with respect to E, then

$$\xi = J_\ell(X : (C_\ell, C_r), \eta)$$

exists if and only if

$$\xi' = J_\ell(X : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), \eta)$$

exists, in which case they are equal.

A similar result holds for right bi-free conjugate variables.

Proof Note by Lemma 5.10 that if ξ' exists, then ξ exists.

Conversely suppose that ξ exists. Hence, ξ is a $\|\cdot\|_\tau$-limit of elements from $\mathrm{alg}(X, C_\ell, C_r)$. Since the analytical operator-valued bi-free cumulants are $\|\cdot\|_\tau$-continuous in the last entry, it follows that any analytical operator-valued bi-free cumulant involving ξ at the end and at least one element of D_ℓ or D_r must be zero by Theorem 4.14 as

$$(\mathrm{alg}(X, C_\ell), C_r) \qquad \text{and} \qquad (D_\ell, D_r)$$

are bi-free with amalgamation over B with respect to E. Therefore, as

$$\xi \in \overline{\mathrm{alg}(X, C_\ell, C_r)}^{\|\cdot\|_\tau} \subseteq \overline{\mathrm{alg}(X, \mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r))}^{\|\cdot\|_\tau},$$

it follows that ξ' exists and $\xi = \xi'$. ∎

The following generalizes [14, Proposition 3.7], [10, Proposition 3.11], and [4, Proposition 4.4]. In that which follows, we will use $\mathbf{Z}$ to denote a tuple of operators $(Z_1, \ldots, Z_k)$. Furthermore, given another tuple $\mathbf{Z}' = (Z_1', \ldots, Z_k')$, we will use $\mathbf{Z} + \mathbf{Z}'$ to denote the tuple $(Z_1 + Z_1', \ldots, Z_k + Z_k')$ and we will use $\widehat{\mathbf{Z}_p}$ to denote the tuple $(Z_1, \ldots, Z_{p-1}, Z_{p+1}, \ldots, Z_k)$ obtained by removing Z_p from the list.

Proposition 5.12 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, and let $\eta : B \to B$ be a completely positive map. Suppose that $\mathbf{X}$ and $\mathbf{X}'$ are n-tuples of operators from A_ℓ, $\mathbf{Y}$ and $\mathbf{Y}'$ are m-tuples of operators from A_r, and (C_ℓ, C_r) and (D_ℓ, D_r) are pairs of B-algebras such that*

$$(\mathrm{alg}(\mathbf{X}, C_\ell), \mathrm{alg}(\mathbf{Y}, C_r)) \qquad \textit{and} \qquad (\mathrm{alg}(\mathbf{X}', D_\ell), \mathrm{alg}(\mathbf{Y}', D_r))$$

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are bi-free with amalgamation over B with respect to E. If

$$\xi = J_\ell\left(X_1 : \left(\operatorname{alg}\left(\widehat{\mathbf{X}}_1, C_\ell\right), \operatorname{alg}(\mathbf{Y}, C_r)\right), \eta, \tau\right)$$

exists, then

$$\xi' = J_\ell\left(X_1 + X_1' : \left(\operatorname{alg}\left(\widehat{(\mathbf{X}+\mathbf{X}')}_1, C_\ell, D_\ell\right), \operatorname{alg}(\mathbf{Y}+\mathbf{Y}', C_r, D_r)\right)\right)$$

exists. Moreover, if P is the orthogonal projection of $L_2(\mathcal{A}, \varphi)$ onto

$$\overline{\operatorname{alg}(\mathbf{X}+\mathbf{X}', \mathbf{Y}+\mathbf{Y}', C_\ell, C_r, D_\ell, D_r)}^{\|\cdot\|_\tau},$$

then

$$\xi' = P(\xi).$$

A similar result holds for right bi-free conjugate variables.

Proof Suppose that ξ exists. For notation purposes, let $\mathcal{A} = \operatorname{alg}(\mathbf{X}+\mathbf{X}', \mathbf{Y}+\mathbf{Y}', C_\ell, C_r, D_\ell, D_r)$.

Since $\tau(L_b Z P(\xi)) = \tau(L_b Z \xi)$ for all $Z \in \mathcal{A}$ and $b \in B$ (as $B_\ell \subseteq C_\ell$), we obtain by Proposition 3.8 that $\widetilde{E}(ZP(\xi)) = \widetilde{E}(Z\xi)$ for all $Z \in \mathcal{A}$. Thus, as $B_\ell, B_r \subseteq \mathcal{A}$, we obtain for all $\chi \in \{\ell, r\}^p$ with $\chi(p) = \ell$, for all $\pi \in \mathrm{BNC}(\chi)$, and for all $Z_k \in \mathcal{A}$ with

$$Z_k \in \begin{cases} \operatorname{alg}(\mathbf{X}+\mathbf{X}', C_\ell, D_\ell), & \text{if } \chi(k) = \ell \\ \operatorname{alg}(\mathbf{Y}+\mathbf{Y}', C_r, D_r), & \text{if } \chi(k) = r \end{cases}$$

that $\Psi_\pi(Z_1, \ldots, Z_{p-1}P(\xi)) = \Psi_\pi(Z_1, \ldots, Z_{p-1}, \xi)$ and thus

$$\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{p-1}, P(\xi)) = \widetilde{\kappa}_\chi(Z_1, \ldots, Z_{p-1}, \xi).$$

To show that $P(\xi)$ is the appropriate left bi-free conjugate variable, it suffices to consider expressions of the form $\widetilde{\kappa}_\chi(Z_1, \ldots, Z_{p-1}, P(\xi))$ and show that they obtain the correct values as dictated in Theorem 5.3. By the above, said cumulant is equal to an analytic operator-valued bi-free cumulant involving elements from C_ℓ, D_ℓ, C_r, D_r, $\mathbf{X}+\mathbf{X}'$, and $\mathbf{Y}+\mathbf{Y}'$ (in the appropriate positions) and a ξ at the end. By expanding using linearity, said cumulant can be modified to a sum of cumulants involving only elements from C_ℓ, D_ℓ, C_r, D_r, $\mathbf{X}$, $\mathbf{X}'$, $\mathbf{Y}$, and $\mathbf{Y}'$ with a ξ at the end. By a similar argument to that in Lemma 5.4, these cumulants then obtain the necessary values for $P(\xi)$ to be the appropriate left bi-free conjugate variable due to Theorem 5.3 applied to ξ and the fact that

$$(\operatorname{alg}(\mathbf{X}, C_\ell), \operatorname{alg}(\mathbf{Y}, C_r)) \qquad \text{and} \qquad (\operatorname{alg}(\mathbf{X}', D_\ell), \operatorname{alg}(\mathbf{Y}', D_r))$$

are bi-free with amalgamation over B with respect to E, so mixed cumulants vanish by Theorem 4.14. ∎

6 Bi-semicircular operators with completely positive covariance

One essential example of conjugate variables in [4, 10, 14] comes from central limit distributions. Thus, this section is devoted to defining the operator-valued bi-semicircular operators with covariance coming from a completely positive map,

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showing that one may add in certain bi-semicircular operators into analytical B-B-noncommutative probability spaces, and showing the bi-free conjugate variables behave in the appropriate manner.

To begin, let B be a unital *-algebra and let K be a finite index set. For each $k \in K$, let Z_k be a symbol. Recall that the *full Fock space* $\mathcal{F}(B, K)$ is the algebraic free product of B and $\{Z_k\}_{k\in K}$; that is,

$$\mathcal{F}(B, K) = B \oplus H_1 \oplus H_2 \oplus \cdots,$$

where

$$H_m = \{b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m \mid b_0, b_1, \ldots, b_m \in B, k_1, \ldots, k_m \in K\}.$$

Note that $\mathcal{F}(B, K)$ is a B-B-bimodule with the obvious left and right actions of B on B and H_m. Moreover, as $\mathcal{F}(B, K)$ is a direct sum of B and another B-B-bimodule, $\mathcal{F}(B, K)$ is a B-B-bimodule with the specified vector state $p : \mathcal{F}(B, K) \to B$ (as in the sense of [2, Definition 3.1.1]) defined by taking the B-term in the above direct product. Therefore, the set A of linear maps on $\mathcal{F}(B, K)$ is a B-B-noncommutative probability space with respect to the expectation $E : A \to B$ defined by $E(T) = p(T1_B)$ (see [2, Remark 3.2.2]).

Let $\{\eta_{i,j}\}_{i,j\in K}$ be linear maps on B. For each $k \in K$, the *left creation and annihilation operators* l_k and l_k^* are the linear maps defined such that

$$\begin{aligned} l_k b &= 1_B Z_k b, \\ l_k(b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m) &= 1_B Z_k b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m, \\ l_k^* b &= 0, \\ l_k^*(b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m) &= \eta_{k,k_1}(b_0) b_1 \cdots Z_{k_m} b_m, \end{aligned}$$

and the *right creation and annihilation operators* r_k and r_k^* are the linear maps defined such that

$$\begin{aligned} r_k b &= b Z_k 1_B, \\ r_k(b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m) &= b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m Z_k 1_B, \\ r_k^* b &= 0, \\ r_k^*(b_0 Z_{k_1} b_1 \cdots Z_{k_m} b_m) &= b_0 Z_{k_1} b_1 \cdots b_{m-1} \eta_{k_m,k}(b_m). \end{aligned}$$

It is elementary to see that $l_k, l_k^* \in A_\ell$ and $r_k, r_k^* \in A_r$. With these operators in hand, we make the following definition.

Definition 6.1 Using the above notation, write K as the disjoint union of two sets I and J. For each $i \in I$ and $j \in J$, let

$$S_i = l_i + l_i^* \qquad \text{and} \qquad D_j = r_j + r_j^*.$$

The pair $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$ is called the *operator-valued bi-semicircular operators with covariance* $\{\eta_{i,j}\}_{i,j\in K}$.

In the case that $\eta_{k_1,k_2} = 0$ for all $k_1, k_2 \in K$ with $k_1 \neq k_2$, we say that $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$ is a collection of $(\{\eta_{i,i}\}_{i\in I}, \{\eta_{j,j}\}_{j\in J})$ *bi-semicircular operators*.

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Remark 6.1 It is natural to ask what are the necessary conditions for operator-valued bi-semicircular operators with covariance $\{\eta_{i,j}\}_{i,j\in K}$ to sit inside an analytical B-B-noncommutative probability space. One may hope that a condition similar to [10, Theorem 4.3.1] would work; that is, the answer is yes if τ_B is tracial and $\eta : M_{|K|}(B) \to M_{|K|}(B)$ defined by

$$\eta([b_{i,j}]_{i,j\in K}) = [\eta_{i,j}(b_{i,j})]_{i,j\in K}$$

is completely positive. However, if $i \in I$, $j \in J$, and $b_1, b_2 \in B$, it is not difficult to verify that

$$\tau_B\left(E((S_iD_jL_{b_1}R_{b_2})^*(S_iD_jL_{b_1}R_{b_2})\right) = \tau_B\left(b_2^*\eta_{i,i}(1_B)b_1b_2\eta_{j,j}(1_B)b_1^* + b_2^*\eta_{i,j}(\eta_{i,j}(b_1b_2))b_1^*\right)$$

and it is not clear if this is positive (even if the outer $\eta_{i,j}$ was a $\eta_{j,i}$).

Only certain operator-valued bi-semicircular operators are required in this paper. Indeed, we will need only the case where $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$ are $(\{\eta_{i,i}\}_{i\in I}, \{\eta_{j,j}\}_{j\in J})$ bi-semicircular operators, as in this setting the pairs of algebras

$$\{(\mathrm{alg}(B_\ell, S_i), B_r)\}_{i\in I} \cup \{(B_\ell, \mathrm{alg}(B_r, D_j))\}_{j\in J}$$

are bi-free with amalgamation over B with respect to E, as the following result shows.

Theorem 6.2 *Using the above notation, if B is a $*$-algebra, τ_B is a tracial state on B, $\{\eta_i\}_{i\in I} \cup \{\eta_j\}_{j\in J}$ are completely positive maps from B to B, $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$ are $(\{\eta_i\}_{i\in I}, \{\eta_j\}_{j\in J})$ bi-semicircular operators, $A_\ell = \mathrm{alg}(B_\ell, \{S_i\}_{i\in I})$, $A_r = \mathrm{alg}(B_r, \{D_j\}_{j\in J})$, A is generated as a $*$-algebra by A_ℓ and A_r, and $\tau : A \to \mathbb{C}$ is defined by $\tau = \tau_B \circ E$, then $(A, E, \varepsilon, \tau)$ is an analytical B-B-noncommutative probability space. Moreover, all operator-valued bi-free cumulants involving $\{S_i\}_{i\in I}$ and $\{D_j\}_{j\in J}$ of order not two are zero and for all $i, i_1, i_2 \in I$ and $j, j_1, j_2 \in J$,*

$$\kappa_{1_{(\ell,\ell)}}(S_{i_1}L_b, S_{i_2}) = \delta_{i_1,i_2}\eta_{i_1}(b), \qquad \kappa_{1_{(r,r)}}(D_{j_1}R_b, D_{j_2}) = \delta_{j_1,j_2}\eta_{j_1}(b),$$
$$\kappa_{1_{(\ell,r)}}(S_iL_b, D_j) = 0, \qquad \kappa_{1_{(r,\ell)}}(D_jR_b, S_i) = 0.$$

Proof As shown above, (A, E, ε) is a B-B-noncommutative probability space. Next, note that E is completely positive when restricted A_ℓ and A_r by [13, Remark 4.3.2] as the expectations reduce to the free case. In fact, this same idea can be used to show that τ is positive. Indeed, first note that S_i and D_j commute. Thus, every element of A can be written as sum of elements of the form

$$Z = L_{b_0}S_{i_1}L_{b_1}\cdots S_{i_n}L_{b_n}R_{b_{n+1}}D_{i_{n+1}}R_{b_{n+2}}\cdots D_{j_{n+m}}R_{b_{n+m}},$$

where $b_0, \ldots b_{n+m} \in B$. Moreover, we can write

$$\mathcal{F}(B, K) \cong \mathcal{F}(B, I) \otimes_B \mathcal{F}(B, J)$$

in such a way that Z acts via

$$L_{b_0}S_{i_1}L_{b_1}\cdots S_{i_n}L_{b_n} \otimes L_{b_{n+m}}S_{j_{n+m}}L_{b_{n+m-1}}\cdots S_{j_{n+1}}L_{b_{n+1}},$$

so that if $E_I : \mathcal{L}(\mathcal{F}(B, I)) \to B$ and $E_J : \mathcal{L}(\mathcal{F}(B, J)) \to B$ are the corresponding expectations, then

$$E(Z) = E_I(L_{b_0}S_{i_1}L_{b_1}\cdots S_{i_n}L_{b_n})E_J(L_{b_{n+m}}S_{j_{n+m}}L_{b_{n+m-1}}\cdots S_{j_{n+1}}L_{b_{n+1}}).$$

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Therefore, if we have $Z = \sum_{k=1}^{d} X_k Y_k$ where X_k is a product of L_b's and S_i's and Y_k is a product of R_b's and D_j's, then

$$\tau(Z^*Z) = \sum_{k_1,k_2=1}^{d} \tau_B\left(E_I\left(X_{k_1}^* X_{k_2}\right) E_J\left(\widetilde{Y}_{k_2}\widetilde{Y}_{k_1}^*\right)\right),$$

where $\widetilde{Y}$ represents the monomial obtained by reversing the order and changing R's to L's and D's to S's. However, as E_I and E_J are completely positive on the algebras generated by L's and S's, we can find $b_{X,k_1,k_3}, b_{Y,k_1,k_4} \in B$ such that

$$E_I\left(X_{k_1}^* X_{k_2}\right) = \sum_{k_3=1}^{d_2} b_{X,k_1,k_3}^* b_{X,k_2,k_3} \quad \text{and} \quad E_J\left(\widetilde{Y}_{k_2}\widetilde{Y}_{k_1}^*\right) = \sum_{k_4=1}^{d_3} b_{Y,k_2,k_4} b_{Y,k_1,k_4}^*,$$

and thus

$$\begin{aligned}\tau(Z^*Z) &= \sum_{k_1,k_2=1}^{d}\sum_{k_3=1}^{d_2}\sum_{k_4=1}^{d_3} \tau_B(b_{X,k_1,k_3}^* b_{X,k_2,k_3} b_{Y,k_2,k_4} b_{Y,k_1,k_4}^*) \\ &= \sum_{k_3=1}^{d_2}\sum_{k_4=1}^{d_3} \tau_B\left(\left(\sum_{k_1=1}^{d} b_{X,k_1,k_3} b_{Y,k_1,k_4}\right)^* \left(\sum_{k_2=1}^{d} b_{X,k_2,k_3} b_{Y,k_2,k_4}\right)\right) \geq 0.\end{aligned}$$

Hence, τ is positive.

Next, one can verify in $L_2(A,\tau)$ that S_i and D_j are the sum of an isometry and its adjoint (see [13, Proposition 4.6.9]) and thus define bounded linear operators. Hence, $(A, E, \varepsilon, \tau)$ is an analytical B-B-noncommutative probability space.

To see the cumulant condition, one can proceed in two ways. One can immediately realize that

$$(\mathrm{alg}(B_\ell, \{S_i\}_{i\in I}), B_r) \qquad \text{and} \qquad (B_\ell, \mathrm{alg}(B_r, \{D_j\}_{j\in J})$$

are bi-free over B due to the above tensor-product relation. This implies that mixed cumulants are zero. The other cumulants then follow from the free case in [13]. Alternatively, one can analyze the actions of L_b, R_b, S_i, and D_j as one would on the operator-valued reduced free product space in an identical way to the LR-diagrams of [2, 3] to obtain a diagrammatic description of the elements of $\mathcal{F}(B,K)$ produced, note that the ones that contribute to a B-element are exactly the bi-non-crossing diagrams that correspond to pair bi-non-crossing partitions, and use induction to deduce the values of the operator-valued cumulants. ∎

We immediately obtain the following using Theorems 6.2 and 5.3.

Lemma 6.3 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, and let $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ be $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators in A. Then,*

$$J_\ell\left(S_1 : \left(\mathrm{alg}\left(B_\ell, \{S_i\}_{j=2}^n\right), \mathrm{alg}\left(B_r, \{D_j\}_{j=1}^m\right)\right), \eta_{\ell,1}\right) = S_1.$$

A similar result holds for the other left and the right conjugate variables.

In order to obtain more examples of bi-free conjugate variables, it would be typical to perturb by bi-semicircular operators and use Proposition 5.12. To do this,

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we must have the collection of bi-semicircular operators in the same analytical B-B-noncommutative probability space. Thus, it is natural to ask whether given two analytical B-B-noncommutative probability spaces there is a bi-free product which causes the pairs of left and right algebras to be bi-freely independent over B and preserve the analytical properties.

Unfortunately, we do not have an answer to this question. The proof of positivity in the operator-valued free case requires the characterization of the vanishing of alternating centered moments in [13, Proposition 3.3.3] to ensure positivity in the end. One may attempt to use the bi-free analogue of "alternating centered moments vanish" from [1]; however, the bi-free formula generalization of [13, Proposition 3.3.3] is far more complicated. In particular, the proof from [13] will not immediately generalize, as Example 3.3 shows that E will not be positive and the traciality of τ_B will need to come into play.

Luckily, if we deal only with bi-semicircular operators, which is all that is required for this paper, there is no issue. In fact, in the case one is working with von Neumann factors as in Example 2.3, the following is trivial as we can add the corresponding collection of bi-semicircular operators using factors by [13].

Theorem 6.4 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space with A_ℓ and A_r generated by isometries, let $\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, and let $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ be $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators. Then, there exists an analytical B-B-noncommutative probability space $(A', E', \varepsilon', \tau')$ with $A \subseteq A'$, $E'|_A = E$, $\tau'|_A = \tau$, $A_\ell \subseteq A'_\ell$, $A_r \subseteq A'_r$, $\{S_i\}_{i=1}^n \subseteq A'_\ell$, $\{D_j\}_{j=1}^m \subseteq A'_r$ and such that the pairs of algebras*

$$(A_\ell, A_r) \qquad \text{and} \qquad \left(\operatorname{alg}(B_\ell, \{S_i\}_{i=1}^n), \operatorname{alg}(B_r, \{D_j\}_{j=1}^m)\right)$$

are bi-free with amalgamation over B with respect to E'.

As, for any $(\{\eta_{i,i}\}_{i\in I}, \{\eta_{j,j}\}_{j\in J})$ bi-semicircular operators $(\{S_i\}_{i\in I}, \{D_j\}_{j\in J})$, we know that

$$\{(\operatorname{alg}(B_\ell, S_i), B_r)\}_{i\in I} \cup \{(B_\ell, \operatorname{alg}(B_r, D_j))\}_{j\in J}$$

are bi-free over B, to prove Theorem 6.4, it suffices to use the following lemma and an analogous result on the right iteratively, or simply adapt the proof to multiple operators simultaneously.

Lemma 6.5 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space with A_ℓ and A_r generated by isometries, let $\eta : B \to B$ be a completely positive map, and let S be an η-semicircular operator. Then, there exists an analytical B-B-noncommutative probability space $(A', E', \varepsilon', \tau')$ such that $A' = \operatorname{alg}(A_\ell, A_r, S)$, $E'|_A = E$, $\tau'|_A = \tau$, $A'_\ell = \operatorname{alg}(A_\ell, S)$, $A'_r = A_r$, and*

$$(A_\ell, A_r) \qquad \text{and} \qquad (\operatorname{alg}(B_\ell, S), B_r)$$

are bi-free with amalgamation over B with respect to E'.

Proof By taking the operator-valued bi-free product of B-B-noncommutative probability spaces, we obtain a B-B-noncommutative probability space (A', E', ε') such

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that $A' = \text{alg}(A_\ell, A_r, S)$, $A'_\ell = \text{alg}(A_\ell, S)$, $A'_r = A_r$, $E'|_A = E$, and

$$(A_\ell, A_r) \qquad \text{and} \qquad (\text{alg}(B_\ell, S), B_r)$$

are bi-free with amalgamation over B with respect to E'. Note that E' restricted to A_r is trivially completely positive and E' restricted to A'_ℓ is completely positive by the free result from [13]. Thus, to verify Definition 3.1, it suffices to verify that if $\tau' = \tau_B \circ E$, then τ' is positive and elements of A'_ℓ and A'_r define bounded operators on $L_2(A', \tau')$.

By analyzing the reduced free product construction, we can realize A_ℓ, A_r, and S as operators acting on

$$\mathcal{F} = B \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots,$$

where

$$\mathcal{H}_m = \{a_0 Z a_1 \cdots Z a_m \mid a_0, a_1, \ldots, a_{m-1} \in A_\ell, a_m \in A\}$$

and if $p : \mathcal{F} \to B$ is defined by taking the B-term in $\mathcal{F}$, then

$$E'(T) = p(T1_B).$$

Define a function $\langle \cdot, \cdot \rangle : \mathcal{F} \times \mathcal{F} \to A$ by setting $B, \mathcal{H}_1, \mathcal{H}_2, \ldots$ to be pairwise orthogonal, $\langle b_1, b_2 \rangle = L_{b_2^* b_1}$, and

$$\langle a'_0 Z a'_1 \cdots Z a'_m, a_0 Z a_1 \cdots Z a_m \rangle = a_m^* L_\eta(a_{m-1}^* \cdots L_\eta(a_1^* L_\eta(a_0^* a'_0) a'_1) \cdots a'_{m-1}) a'_m,$$

where $L_\eta(T) = L_{\eta(E(T))}$. As η is completely positive and E is completely positive when restricted to A_ℓ, we obtain that $\langle \cdot, \cdot \rangle$ is an A-valued inner product by the same arguments as [13, Proposition 4.6.6]. To elaborate slightly, given $\sum_{k=1}^n a_{k,0} Z a_{k,1} \cdots Z a_{k,m}$, the matrix $[\eta(a_{i,0}^* a_{j,0})]$ is positive and thus can be written as $[\sum_{k=1}^n b_{k,i}^* b_{k,j}]$ for some $b_{i,j} \in B$. One then substitutes $L_\eta(a_{i,0}^* a_{j,0}) = \sum_{k=1}^n L_{b_{k,i}}^* L_{b_{k,j}}$ and continues until one ends with a sum of products of elements of A with their adjoints.

As $\tau : A \to \mathbb{C}$ is positive and as for all $T \in A'$,

$$\tau'(T^* T) = \tau(\langle T1_B, T1_B \rangle),$$

we obtain that τ' is positive as desired. To see that elements of A'_ℓ and A'_r define bounded operators on $L_2(A', \tau')$, note if $T \in A_r$, then, using the above description,

$$T(a_0 Z a_1 \cdots Z a_m) = a_0 Z a_1 \cdots Z T a_m.$$

As any of the terms,

$$L_\eta(a_{m-1}^* \cdots L_\eta(a_1^* L_\eta(a_0^* a'_0) a'_1) \cdots a'_{m-1})$$

in the above A-valued inner product will be able to be written as sums involving terms of the form $L_{b_1}^* L_{b_2}$ and will then produce terms of the form

$$a_m^* T^* L_{b_1}^* L_{b_2} T a'_m = a_m^* L_{b_1}^* T^* T L_{b_2} a'_m$$

in the A-valued inner product when T acts on the left, the fact that A_r is generated by isometries yields that A_r acts as bounded operators on $L_2(A, \tau)$. The fact that A_ℓ is generated by isometries immediately yields that A_ℓ acts as bounded operators on

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$L_2(A,\tau)$, and it is not difficult to see that S acts as the sum of an isometry and its adjoint on $L_2(A,\tau)$ and thus is bounded. ∎

With Theorem 6.4 establishing we can always assume that our *B-B*-noncommutative probability spaces have $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators, we can proceed with the following.

Theorem 6.6 *Let (A,E,ε,τ) be an analytical B-B-noncommutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X}\in A_\ell^n$ and $\mathbf{Y}\in A_r^m$ be tuples of self-adjoint operators, let $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ be $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators in A, and let (C_ℓ, C_r) be pairs of B-algebras of A such that*

$$\{(\operatorname{alg}(C_\ell,\mathbf{X}),\operatorname{alg}(C_r,\mathbf{Y}))\}\cup\{(\operatorname{alg}(B_\ell,S_i),B_r)\}_{i=1}^n\cup\{(B_\ell,\operatorname{alg}(B_r,D_j))\}_{j=1}^m$$

are bi-free. If P is the orthogonal projection of $L_2(A,\varphi)$ onto

$$\overline{\operatorname{alg}\left(C_\ell,C_r,\mathbf{X}+\sqrt{\varepsilon}\mathbf{S},\mathbf{Y}+\sqrt{\varepsilon}\mathbf{D}\right)}^{\|\cdot\|_\tau},$$

then

$$\xi = J_\ell\left(X_1+\sqrt{\varepsilon}S_1 : \left(\operatorname{alg}\left(C_\ell,\widehat{(\mathbf{X}+\sqrt{\varepsilon}\mathbf{S})_1}\right),\operatorname{alg}\left(C_r,\mathbf{Y}+\sqrt{\varepsilon}\mathbf{D}\right)\right),\eta_{\ell,1}\right) = \frac{1}{\sqrt{\varepsilon}}P(S_1).$$

Thus,

$$\|\xi\|_\tau \le \frac{1}{\sqrt{\varepsilon}}\sqrt{\tau_B(\eta(1_B))}.$$

A similar computation holds for the other entries of the tuples and the right conjugate variables.

Proof By Lemmas 6.3 and 5.5, we have that

$$J_\ell\left(\sqrt{\varepsilon}S_1 : \left(\operatorname{alg}\left(B_\ell,\sqrt{\varepsilon}\hat{\mathbf{S}}_1\right),\operatorname{alg}\left(B_r,\sqrt{\varepsilon}\mathbf{D}\right)\right),\eta_{\ell,1}\right) = \frac{1}{\sqrt{\varepsilon}}S_1.$$

The conjugate variable result then follows from Propositions 5.11 and 5.12, whereas the τ-norm computation is trivial. ∎

7 Bi-free fisher information with respect to a completely positive map

With the above technology, the bi-free Fisher information with respect to completely positive maps can be constructed and has similar properties to the bi-free Fisher information from [4] and the free Fisher information with respect to a completely positive map from [10]. We highlight the main results and properties in this section.

Definition 7.1 Let (A,E,ε,τ) be an analytical *B-B*-noncommutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from *B* to *B*, let $\mathbf{X}\in A_\ell^n$ and $\mathbf{Y}\in A_r^m$ and let (C_ℓ,C_r) be a pair of *B*-algebras of *A*. The *relative bi-free Fisher information of* $(\mathbf{X},\mathbf{Y})$ *with respect to* $(\{\eta_{\ell,i}\}_{i=1}^n,\{\eta_{r,j}\}_{j=1}^m)$ *in the presence of* (C_ℓ,C_r)

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is

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \sum_{i=1}^n \|\xi_i\|_\tau^2 + \sum_{j=1}^m \|\nu_j\|_\tau^2,$$

where, for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$,

$$\xi_i = J_\ell\left(X_i : \left(\mathrm{alg}\left(C_\ell, \widehat{\mathbf{X}}_i\right), \mathrm{alg}\left(C_r, \mathbf{Y}\right)\right), \eta_{\ell,i}\right) \quad \text{and}$$
$$\nu_j = J_r\left(Y_j : \left(\mathrm{alg}\left(C_\ell, \mathbf{X}\right), \mathrm{alg}\left(C_r, \widehat{\mathbf{Y}}_j\right)\right), \eta_{r,j}\right)$$

provided these variables exist, and otherwise is defined as ∞.

In the case that $\eta_{\ell,i} = \eta_{r,j} = \eta$ for all i and j, we use $\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), \eta)$ to denote the above bi-free Fisher information. In the case that $C_\ell = B_\ell$ and $C_r = B_r$, we use $\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))$. In the case both occur, we use $\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : \eta)$.

Note that the bi-free Fisher with respect to completely positive maps exists in many settings due to Theorems 6.4 and 6.6. Furthermore, the properties of the bi-free Fisher with respect to completely positive maps are in analogy with those from [4, 10, 14] as the following shows.

Remark 7.1 (1) In the case that $B = \mathbb{C}$ and η is unital, Definition 7.1 immediately reduces down to the bi-free Fisher information in [4, Definition 5.1] by Remark 5.2.

(2) In the case we are in the context of Example 2.3 with $m = 0$, $C_\ell = B_\ell$, and $C_r = B_r$, Definition 7.1 immediately reduces down to the free Fisher information with respect to a complete positive map from [10, Definition 4.1].

(3) Note that

$$\begin{aligned}\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \\ = \sum_{i=1}^n \Phi^*\left(X_i \sqcup \varnothing : \left(\mathrm{alg}\left(C_\ell, \widehat{\mathbf{X}}_i\right), \mathrm{alg}\left(C_r, \mathbf{Y}\right)\right), \eta_{\ell,i}, \tau\right) \\ + \sum_{j=1}^m \Phi^*\left(\varnothing \sqcup Y_j : \left(\mathrm{alg}\left(C_\ell, \mathbf{X}\right), \mathrm{alg}\left(C_r, \widehat{\mathbf{Y}}_j\right)\right), \eta_{r,j}, \tau\right).\end{aligned}$$

(4) If $\mathbf{X} = (X_1, X_1^*, \ldots, X_n, X_n^*)$ and $\mathbf{Y} = (Y_1, Y_1^*, \ldots, Y_m, Y_m^*)$, then Lemma 5.4 implies that

$$\begin{aligned}\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \\ = \frac{1}{2}\Phi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)),\end{aligned}$$

where $\mathbf{X}' = (\Re(X_1), \Im(X_1), \ldots, \Re(X_n), \Im(X_n))$ and $\mathbf{Y}' = (\Re(Y_1), \Im(Y_1), \ldots, \Re(Y_m), \Im(Y_m))$.

(5) In the context of Proposition 5.9 (i.e., reducing the B-B-noncommutative probability space to a D-D-noncommutative probability space),

$$\begin{aligned}\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \\ = \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i} \circ F\}_{i=1}^n, \{\eta_{r,j} \circ F\}_{j=1}^m)).\end{aligned}$$

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(6) By Lemma 5.5, for all $\lambda \in \mathbb{C}\backslash\{0\}$,

$$\Phi^*(\lambda\mathbf{X} \sqcup \lambda\mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \frac{1}{|\lambda|^2}\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(7) In the context of Lemma 5.10 (i.e., (D_ℓ, D_r) a smaller pair of B-algebras than (C_ℓ, C_r)),

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (D_\ell, D_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \leq \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(8) In the context of Proposition 5.11 (i.e., adding in a bi-free pair of B-algebras),

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (\mathrm{alg}(D_\ell, C_\ell), \mathrm{alg}(D_r, C_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(9) If in addition to the assumptions of Definition 7.1 $\mathbf{X}' \in A_\ell^{n'}$, $\mathbf{Y}' \in A_r^{m'}$, $(\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})$ is a collection of completely positive maps on B, and (D_ℓ, D_r) is a pair of B-algebras, then, by (iii) and (vii),

$$\Phi^*(\mathbf{X}, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta_{r,j}\}_{j=1}^m \cup \{\eta'_{r,j}\}_{j=1}^{m'})) \geq \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) + \Phi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (D_\ell, D_r), (\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})).$$

(10) In the context of (ix) with the additional assumption that

$$(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y})) \qquad \text{and} \qquad (\mathrm{alg}(D_\ell, \mathbf{X}'), \mathrm{alg}(D_r, \mathbf{Y}'))$$

are bi-free with amalgamation over B with respect to E, Proposition 5.11 implies that

$$\Phi^*(\mathbf{X}, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta_{r,j}\}_{j=1}^m \cup \{\eta'_{r,j}\}_{j=1}^{m'})) = \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) + \Phi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (D_\ell, D_r), (\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})).$$

Unsurprisingly, more complicated properties of free Fisher information extend.

Proposition 7.2 (Bi-free Stam inequality) *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X}, \mathbf{X}' \in A_\ell^n$ and $\mathbf{Y}, \mathbf{Y}' \in A_r^m$, and let (C_ℓ, C_r) and (D_ℓ, D_r) be pairs of B-algebras of A such that*

$$(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y})) \qquad \textit{and} \qquad (\mathrm{alg}(D_\ell, \mathbf{X}'), \mathrm{alg}(D_r, \mathbf{Y}'))$$

are bi-free with amalgamation over B with respect to E. Then,

$$\begin{aligned}&\left(\Phi^*(\mathbf{X} + \mathbf{X}' \sqcup \mathbf{Y} + \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))\right)^{-1} \\ &\geq \left(\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))\right)^{-1} \\ &\quad + \left(\Phi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))\right)^{-1}.\end{aligned}$$

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Proof Let

$$P_0 : L_2(\mathcal{A}, \varphi) \to L_2(B, \tau_B),$$
$$P_1 : L_2(\mathcal{A}, \varphi) \to \overline{\mathrm{alg}(C_\ell, C_r, \mathbf{X}, \mathbf{Y})}^{\|\cdot\|_\tau},$$
$$P_2 : L_2(\mathcal{A}, \varphi) \to \overline{\mathrm{alg}(D_\ell, D_r, \mathbf{X}', \mathbf{Y}')}^{\|\cdot\|_\tau}$$

be the orthogonal projections onto their co-domains. Note that if

$$Z \in \mathrm{alg}(C_\ell, C_r, \mathbf{X}, \mathbf{Y}) \quad \text{and} \quad \mathrm{alg}(D_\ell, D_r, \mathbf{X}', \mathbf{Y}'),$$

then bi-freeness implies that

$$\widetilde{E}(ZZ') = \widetilde{E}\left(Z\widetilde{E}(Z')\right).$$

Indeed, this is easily seen as if Z and Z' are monomials, then any cumulant of the monomial ZZ' corresponding to a bi-non-crossing partition is nonzero if and only if it decomposes into a bi-non-crossing partition on Z union a bi-non-crossing partition on Z'. Thus, $P_1P_2 = P_2P_1 = P_0$.

The remainder of the proof can then be read from [10, Proposition 4.5], [4, Proposition 5.8], or even [14, Proposition 6.5]. ∎

Proposition 7.3 (Bi-free Cramer–Rao inequality) *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ consist of self-adjoint operators, and let (C_ℓ, C_r) be a pair of B-algebras of A. Then,*

$$\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))\tau\left(\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2\right)$$
$$\geq \left(\sum_{i=1}^n \tau_B(\eta_{\ell,i}(1)) + \sum_{j=1}^m \tau_B(\eta_{r,j}(1))\right)^2.$$

Moreover, equality holds if $(\mathbf{X}, \mathbf{Y})$ are $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$-bi-semicircular elements and

$$\{(C_\ell, C_r)\} \cup \{(\mathrm{alg}(B_\ell, X_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \mathrm{alg}(B_r, Y_j))\}_{j=1}^m$$

are bi-free with amalgamation over B with respect to E. The converse holds when $C_\ell = B_\ell$ and $C_r = B_r$.

Proof The result follows from the obvious modifications to [4, Proposition 5.10]. Also, see [10, Proposition 4.6] and [14, Proposition 6.9]. ∎

Similarly, limits behave as one expects based on [4, 10, 14].

Proposition 7.4 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ consist of self-adjoint operators, and let (C_ℓ, C_r) be a pair of B-algebras of A. Suppose further for each $k \in \mathbb{N}$ that $\mathbf{X}^{(k)} \in A_\ell^n$ and $\mathbf{Y}^{(k)} \in A_r^m$ are tuples of self-adjoint*

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elements in A such that

$$\limsup_{k\to\infty} \left\| X_i^{(k)} \right\| < \infty,$$
$$\limsup_{k\to\infty} \left\| Y_j^{(k)} \right\| < \infty,$$
$$s\text{-}\lim_{k\to\infty} X_i^{(k)} = X_i, \text{ and}$$
$$s\text{-}\lim_{k\to\infty} Y_j^{(k)} = Y_j$$

for all $1 \leq i \leq n$ *and* $1 \leq j \leq m$ *(where the strong limit is computed as bounded linear maps acting on* $L_2(A,\tau)$*). Then,*

$$\liminf_{k\to\infty} \Phi^*\left(\mathbf{X}^{(k)} \sqcup \mathbf{Y}^{(k)} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right)$$
$$\geq \Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

The proof of Proposition 7.4 becomes identical to [4, Proposition 5.12] once the following lemma is established. Also, see [10, Proposition 4.7] and [14, Proposition 6.10].

Lemma 7.5 *Under the assumptions of Proposition 7.4 along with the additional assumptions that*

$$\xi_k = J_\ell\left(X_1^{(k)} : \left(\operatorname{alg}\left(C_\ell, \widehat{\mathbf{X}}_1^{(k)}\right), \operatorname{alg}(C_r, \mathbf{Y}^{(k)})\right), \eta_{\ell,1}\right)$$

exist and are bounded in L_2*-norm by some constant* $K > 0$*, it follows that*

$$\xi = J_\ell\left(X_1 : \left(\operatorname{alg}\left(C_\ell, \widehat{\mathbf{X}}_1\right), \operatorname{alg}(C_r, \mathbf{Y})\right), \eta_{\ell,1}\right)$$

exists and is equal to

$$w\text{-}\lim_{k\to\infty} P(\xi_k),$$

where P *is the orthogonal projection of* $L_2(A,\tau)$ *onto* $\overline{\operatorname{alg}(C_\ell, C_r, \mathbf{X}, \mathbf{Y})}^{\|\cdot\|_\tau}$.

If, in addition,

$$\limsup_{k\to\infty} \|\xi_k\|_2 \leq \|\xi\|_2,$$

then

$$\lim_{k\to\infty} \|\xi_k - \xi\|_2 = 0.$$

The same holds with X_1 *replaced with* X_i*, and a similar result holds for the right.*

Proof The proof of this result follows from the same sequence of steps as [4, Lemma 5.13] using the analytical operator-valued bi-free cumulants. ∎

Corollary 7.6 *Under the assumptions of Proposition 7.4, if in addition*

$$(B_\ell(\mathbf{X}), B_r(\mathbf{Y})) \qquad \text{and} \qquad \left(\operatorname{alg}\left(C_\ell, \mathbf{X}^{(k)}\right), \operatorname{alg}\left(C_r, \mathbf{Y}^{(k)}\right)\right)$$

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are bi-free for all k and

$$\lim_{k\to\infty}\left\|X_i^{(k)}\right\| = \lim_{k\to\infty}\left\|Y_j^{(k)}\right\| = 0$$

for all $1 \le i \le n$ and $1 \le j \le m$, then

$$\begin{aligned}\lim_{k\to\infty}\Phi^*\left(\mathbf{X}+\mathbf{X}^{(k)} \sqcup \mathbf{Y}+\mathbf{Y}^{(k)} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right)\\ = \Phi^*(\mathbf{X}\sqcup\mathbf{Y} : (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).\end{aligned}$$

Furthermore, if $C_\ell = B_\ell$, $C_r = B_r$, and

$$\Phi^*(\mathbf{X}\sqcup\mathbf{Y} : (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) < \infty,$$

then

$$J_\ell\left(X_i^{(k)} : \left(B_\ell\left(\widehat{(\mathbf{X}+\mathbf{X}^{(k)})}_i\right), B_r\left(\mathbf{Y}+\mathbf{Y}^{(k)}\right)\right), \eta_{\ell,i}\right)$$

tends to

$$J_\ell\left(X_i : \left(B_\ell\left(\hat{\mathbf{X}}_i\right), B_r\left(\mathbf{Y}\right)\right), \eta_{\ell,i}\right)$$

in τ-norm. A similar result holds for right bi-free conjugate variables.

Proof The proof is identical to [4, Corollary 5.14] and thus is omitted. ∎

Theorem 7.7 *Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ be tuples of self-adjoint operators, let $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ be a collection of $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators in A, and let (C_ℓ, C_r) be pairs of B-algebras of A such that*

$$\{(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y}))\} \cup \{(\mathrm{alg}(B_\ell, S_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \mathrm{alg}(B_r, D_j))\}_{j=1}^m$$

are bi-free. Then, the map

$$h : [0,\infty) \ni t \mapsto \Phi^*\left(\mathbf{X}+\sqrt{t}\mathbf{S}\sqcup\mathbf{Y}+\sqrt{t}\mathbf{D} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right)$$

is decreasing, right continuous, and

$$\frac{K_2^2}{K_1 + K_2 t} \le h(t) \le \frac{1}{t}K_3,$$

where

$$\begin{aligned}K_1 &= \tau\left(\sum_{i=1}^n X_i^2 + \sum_{j=1}^m Y_j^2\right),\\ K_2 &= \sum_{i=1}^n \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^m \tau_B(\eta_{r,j}(1_B)), \text{ and}\\ K_3 &= \sum_{i=1}^n \tau_B(\eta_{\ell,i}(1_B))^2 + \sum_{j=1}^m \tau_B(\eta_{r,j}(1_B))^2.\end{aligned}$$

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Moreover, if $(\mathbf{X}, \mathbf{Y})$ *is the* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$*-bi-semicircular distribution and*

$$\{(C_\ell, C_r)\} \cup \{(\mathrm{alg}(B_\ell, X_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \mathrm{alg}(B_r, Y_j))\}_{j=1}^m$$

are bi-free with amalgamation over B with respect to E, then $h(t) = \frac{K_2^2}{K_1^2 + K_2 t}$ *for all t.*

Finally, if $C_\ell = B_\ell$, $C_r = B_r$, *and* $h(t) = \frac{K_2^2}{K_1 + K_2 t}$ *for all t, then* $(\mathbf{X}, \mathbf{Y})$ *is the* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$*-bi-semicircular distribution.*

Proof The proof becomes identical to [4, Theorem 5.15] using the above and the fact that

$$\tau\left(\left(X_i + \sqrt{t}S_i\right)^2\right) = \tau\left(X_i^2\right) + t\tau\left(S_i^2\right) = \tau\left(X_i^2\right) + t\tau_B\left(\eta_{\ell,i}(1_B)\right),$$

with an analogous computation on the right for use in the lower bound computation, in conjunction with the bi-free Cramer–Rao inequality (Proposition 7.3). ∎

8 Bi-free entropy with respect to completely positive maps

With the construction and properties of the bi-free Fisher information with respect to completely positive maps complete, the construction and properties of bi-free entropy with respect to completely positive maps follows easily by extending results from [4, 10] with similar proofs.

Definition 8.1 Let $(A, E, \varepsilon, \tau)$ be an analytical B-B-noncommutative probability space, let $\{\eta_{\ell,i}\}_{i=1}^n$ and $\{\eta_{r,j}\}_{j=1}^m$ be completely positive maps from B to B, let (C_ℓ, C_r) be pairs of B-algebras of A, and let $\mathbf{X} \in A_\ell^n$ and $\mathbf{Y} \in A_r^m$ be tuples of self-adjoint operators. The *relative bi-free entropy of* $(\mathbf{X}, \mathbf{Y})$ *with respect to* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ *in the presence of* (B_ℓ, B_r) is defined to be

$$\begin{aligned}\chi^*&(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \\ &= \frac{K}{2}\ln(2\pi e) + \frac{1}{2}\int_0^\infty \left(\frac{K}{1+t} - \Phi^*\left(\mathbf{X} + \sqrt{t}\mathbf{S} \sqcup \mathbf{Y}\right.\right. \\ &\quad \left.\left. + \sqrt{t}\mathbf{D} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right)\right) dt,\end{aligned}$$

where

$$K = \sum_{i=1}^n \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^m \tau_B(\eta_{r,j}(1_B))$$

and $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ is a collection of $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ bi-semicircular operators such that

$$\{(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y})\rangle)\} \cup \{(B_\ell(S_i), B_r)\}_{i=1}^n \cup \{(B_\ell, B_r(D_j))\}_{j=1}^m$$

are bi-free (note that such semicircular operators can be included in A by Theorem 6.4).

In the case that $C_\ell = B_\ell$ and $C_r = B_r$, we use $\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))$ to denote the bi-free entropy. If in addition $\eta_{\ell,i} = \eta_{r,j} = \eta$ for all i and j, we use $\chi^*(\mathbf{X} \sqcup \mathbf{Y} : \eta)$ to denote the bi-free entropy.

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We note that there is a slight change in the normalization used in Definition 8.1 over that used in [10, Definition 8.1]. Generally, this makes no real difference other than making some of the bounds in this section nice, such as the following one.

Proposition 8.1 *In the context of Definition 8.1, if*

$$K_1 = \tau\left(\sum_{i=1}^{n} X_i^2 + \sum_{j=1}^{m} Y_j^2\right),$$

then

$$\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \leq \frac{K}{2} \ln\left(\frac{2\pi e}{K} K_1\right).$$

Moreover, equality holds when $(\mathbf{X}, \mathbf{Y})$ *are* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$*-bi-semicircular operators such that* $\{(C_\ell, C_r)\} \cup \{(\mathrm{alg}(B_\ell, X_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \mathrm{alg}(B_r, Y_j))\}_{j=1}^m$ *are bi-free and if* $C_\ell = B_\ell$ *and* $C_r = B_r$*, this is the only setting where equality holds.*

Proof The proof is identical to [4, Proposition 6.5] in conjunction with Theorem 7.7. ∎

Remark 8.2 (1) In the case that $B = \mathbb{C}$ and η is unital, Definition 8.1 produces the non-microstate bi-free entropy from [4, Definition 6.1].

(2) In the setting of Example 2.3, when $C_r = B_r$ and $\eta_{\ell,i} = \eta_{r,j} = \eta$ for all i and j, Definition 8.1 produces the free entropy with respect to a completely positive map from [10, Definition 8.1] modulo an additive constant (which is 0 in the case η is unital).

Of course, due to the fact that the bi-free Fisher information from Section 7 behaves analogously to the Fisher information considered in [4, 10], results for the behavior of entropy automatically generalize.

Proposition 8.3 *Using Remark 7.1, the following hold:*

(v) *In the context of Proposition 5.9 (i.e., reducing the B-B-noncommutative probability space to a D-D-noncommutative probability space),*

$$\begin{aligned}&\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \\ &\quad = \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i} \circ F\}_{i=1}^n, \{\eta_{r,j} \circ F\}_{j=1}^m)).\end{aligned}$$

(vi) *For all* $\lambda \in \mathbb{R}\backslash\{0\}$,

$$\begin{aligned}&\chi^*(\lambda\mathbf{X} \sqcup \lambda\mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \\ &\quad = \left(\sum_{i=1}^{n} \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^{m} \tau_B(\eta_{r,j}(1_B))\right) \ln|\lambda| \\ &\qquad + \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).\end{aligned}$$

(vii) *In the context of Lemma 5.10 (i.e.,* (D_ℓ, D_r) *is a smaller pair of B-algebras than* (C_ℓ, C_r)*),*

$$\begin{aligned}&\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (D_\ell, D_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) \\ &\quad \geq \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).\end{aligned}$$

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(viii) *In the context of Proposition 5.11 (i.e., adding in a bi-free pair of B-algebras),*

$$\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (\mathrm{alg}(D_\ell, C_\ell), \mathrm{alg}(D_r, C_r)), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) = \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

(ix) *If in addition to the assumptions of Definition 8.1 we have* $\mathbf{X}' \in A_\ell^{n'}$, $\mathbf{Y}' \in A_r^{m'}$, $(\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})$ *is a collection of completely positive maps on B, and* (D_ℓ, D_r) *is a pair of B-algebras, then*

$$\chi^*(\mathbf{X}, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta_{r,j}\}_{j=1}^m \cup \{\eta'_{r,j}\}_{j=1}^{m'})) \leq \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) + \chi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (D_\ell, D_r), (\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})).$$

(x) *In the context of (ix) with the additional assumption that*

$$(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y})) \qquad \textit{and} \qquad (\mathrm{alg}(D_\ell, \mathbf{X}'), \mathrm{alg}(D_r, \mathbf{Y}'))$$

are bi-free with amalgamation over B with respect to E, Proposition 5.11 implies that

$$\chi^*(\mathbf{X}, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}' : (\mathrm{alg}(C_\ell, D_\ell), \mathrm{alg}(C_r, D_r)), (\{\eta_{\ell,i}\}_{i=1}^n \cup \{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta_{r,j}\}_{j=1}^m \cup \{\eta'_{r,j}\}_{j=1}^{m'})) = \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)) + \chi^*(\mathbf{X}' \sqcup \mathbf{Y}' : (D_\ell, D_r), (\{\eta'_{\ell,i}\}_{i=1}^{n'}, \{\eta'_{r,j}\}_{j=1}^{m'})).$$

Using Proposition 7.4, Theorem 7.7, and the same arguments as [4, Proposition 6.7], the following holds.

Proposition 8.4 *Under the assumptions of Definition 8.1, if for each* $k \in \mathbb{N}$ *there exist self-adjoint tuples* $\mathbf{X}^{(k)} \in A_\ell^n$ *and* $\mathbf{Y}^{(k)} \in A_r^m$ *such that*

$$\limsup_{k\to\infty} \left\| X_i^{(k)} \right\| < \infty,$$
$$\limsup_{k\to\infty} \left\| Y_j^{(k)} \right\| < \infty,$$
$$s\text{-}\lim_{k\to\infty} X_i^{(k)} = X_i, \text{ and}$$
$$s\text{-}\lim_{k\to\infty} Y_j^{(k)} = Y_j$$

for all $1 \leq i \leq n$ *and* $1 \leq j \leq m$ *(with the strong limit computed as bounded linear maps acting on* $L_2(A, \tau)$*), then*

$$\limsup_{k\to\infty} \chi^*\left(\mathbf{X}^{(k)} \sqcup \mathbf{Y}^{(k)} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right) \leq \chi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)).$$

Using the previous proposition together with Theorem 7.7 and the same arguments as [4, Proposition 6.8], the following holds.

Proposition 8.5 *Under the assumptions of Definition 8.1, suppose that* $(\{S_i\}_{i=1}^n, \{D_j\}_{j=1}^m)$ *is a collection of* $(\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)$ *bi-semicircular operators such that*

$$(\mathrm{alg}(C_\ell, \mathbf{X}), \mathrm{alg}(C_r, \mathbf{Y}))\} \cup \{(\mathrm{alg}(B_\ell, S_i), B_r)\}_{i=1}^n \cup \{(B_\ell, \mathrm{alg}(B_r, D_j))\}_{j=1}^m$$

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are bi-free. For $t \in [0, \infty)$, *let*

$$g(t) = \chi^*\left(\mathbf{X} + \sqrt{t}\mathbf{S} \sqcup \mathbf{Y} + \sqrt{t}\mathbf{D} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right).$$

Then, $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ *is a concave, continuous, increasing function such that* $g(t) \geq \frac{K}{2}\ln(2\pi e t)$, *where*

$$K = \sum_{i=1}^n \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^m \tau_B(\eta_{r,j}(1_B))$$

and, when $g(t) \neq -\infty$,

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon}(g(t+\varepsilon) - g(t)) = \frac{1}{2}\Phi^*\left(\mathbf{X} + \sqrt{t}\mathbf{S} \sqcup \mathbf{Y} + \sqrt{t}\mathbf{D} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right).$$

Finally, using the Bi-free Stam inequality (Proposition 7.2) together with the same proof as [4, Proposition 6.11] yields the following.

Proposition 8.6 *Under the assumptions of Definition 8.1, if*

$$\Phi^*\left(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m)\right) < \infty,$$

then

$$\chi^*(\mathbf{X} \sqcup \mathbf{Y} : (B_\ell, B_r)) \geq \frac{K}{2}\ln\left(\frac{2\pi K e}{\Phi^*(\mathbf{X} \sqcup \mathbf{Y} : (C_\ell, C_r), (\{\eta_{\ell,i}\}_{i=1}^n, \{\eta_{r,j}\}_{j=1}^m))}\right) > -\infty,$$

where

$$K = \sum_{i=1}^n \tau_B(\eta_{\ell,i}(1_B)) + \sum_{j=1}^m \tau_B(\eta_{r,j}(1_B)).$$

9 Minimizing bi-free fisher information

In this section, we will prove Theorem 9.5, thereby describing the minimal value of the bi-free Fisher information of non-self-adjoint pairs of operators under certain distribution conditions. Throughout the section, we will be working under the situation from Example 3.3 where $\mathcal{A}$ is a unital C*-algebra, $\varphi : \mathcal{A} \to \mathbb{C}$ is a state, $B = M_d(\mathbb{C})$ (the $d \times d$ matrices with complex entries), and $\tau_B = \mathrm{tr}_d$ (the normalized trace on $M_d(\mathbb{C})$). Thus, $A_d = \mathcal{A} \otimes M_d(\mathbb{C}) \otimes M_d(\mathbb{C})^{\mathrm{op}}$, $E_d : A_d \to M_d(\mathbb{C})$ and $\tau_d : A_d \to \mathbb{C}$ are defined such that

$$E_d(Z \otimes b_1 \otimes b_2) = \varphi(Z)b_1b_2 \qquad \text{and} \qquad \tau_d(Z \otimes b_1 \otimes b_2) = \varphi(Z)\,\mathrm{tr}_d(b_1b_2),$$

for all $Z \in \mathcal{A}$ and $b_1, b_2 \in M_d(\mathbb{C})$. We recall the following result that aids in computing moments in (A_d, E_d, ε) where $\{E_{i,j}\}_{i,j=1}^d \subseteq M_d(\mathbb{C})$ are the canonical matrix units and I_d is the identity of $M_d(\mathbb{C})$.

Lemma 9.1 [11, Lemma 3.7] *Let* $(\mathcal{A}, \varphi)$ *be a C*-noncommutative probability space, let* $\chi \in \{\ell, r\}^n$, *and let*

$$Z_k = \begin{cases} \sum_{i,j=1}^d z_{k;i,j} \otimes E_{i,j} \otimes I_d, & \text{if } \chi(k) = \ell, \\ \sum_{i,j=1}^d z_{k;i,j} \otimes I_d \otimes E_{i,j}, & \text{if } \chi(k) = r. \end{cases}$$

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Then,

$$E_d(Z_1 \cdots Z_n) = \sum_{\substack{i_1,\ldots,i_n=1 \\ j_1,\ldots,j_n=1}}^{d} \varphi(z_{1;i_1,j_1} \cdots z_{n;i_n,j_n}) E_\chi((i_1,\ldots,i_n),(j_1,\ldots,j_n)),$$

where

$$E_\chi((i_1,\ldots,i_n),(j_1,\ldots,j_n)) := E_{i_{s_\chi(1)},j_{s_\chi(1)}} \cdots E_{i_{s_\chi(n)},j_{s_\chi(n)}} \in M_d(\mathbb{C}).$$

To discuss conjugate variables, we need to consider $L_2(A_d, \tau_d)$. It is not difficult to verify that $L_2(A_d, \tau_d)$ can be identified with the $d \times d$ matrices with entries in $L_2(\mathcal{A}, \varphi)$ where

$$(Z \otimes I_d \otimes I_d)[\xi_{i,j}] = [Z\xi_{i,j}],$$

$1_\mathcal{A} \otimes b \otimes I_d$ acts via left multiplication on $[\xi_{i,j}]$, and $1_\mathcal{A} \otimes I_d \otimes b$ acts via right multiplication on $[\xi_{i,j}]$ for all $[\xi_{i,j}] \in M_d(L_2(\mathcal{A}, \varphi))$ and $b \in M_d(\mathbb{C})$.

Next, we consider the subalgebra $D_d \subseteq M_d(\mathbb{C})$ of the diagonal matrices. Clearly, if $F_d : M_d(\mathbb{C}) \to D_d$ is the canonical conditional expectation onto the diagonal, then $(A_d, F_d \circ E_d, \varepsilon|_{D_d \otimes D_d^{\mathrm{op}}}, \tau_d)$ is also an analytical D_d-D_d-noncommutative probability space (see Example 5.8).

To begin stating our main result, we first require two definitions.

Definition 9.1 [12, Definition 4.4] Let

$$Z_\ell = \sum_{i,j=1}^{d} z_{\ell,i,j} \otimes E_{i,j} \otimes I_d \quad \text{and} \quad Z_r = \sum_{i,j=1}^{d} z_{\ell,i,j} \otimes I_d \otimes E_{i,j}.$$

The pair (Z_ℓ, Z_r) is said to be *bi-R-cyclic* if for all $n \geq 1$, $\chi \in \{\ell, r\}^n$ and $1 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq d$,

$$\kappa_\chi^{\mathbb{C}}(z_{\chi(1),i_1,j_1}, z_{\chi(2),i_2,j_2}, \ldots, z_{\chi(n),i_n,j_n}) = 0$$

whenever at least one of $j_{s_\chi(1)} = i_{s_\chi(2)}$, $j_{s_\chi(2)} = i_{s_\chi(3)}$, …, $j_{s_\chi(n-1)} = i_{s_\chi(n)}$, $j_{s_\chi(n)} = i_{s_\chi(1)}$ fails.

Definition 9.2 [12, Example 4.7] Let $x, y \in \mathcal{A}$. The pair (x, y) is said to be *bi-R-diagonal* if all odd length $\mathbb{C}$-valued bi-free cumulants involving $(\{x, x^*\}, \{y, y^*\})$ vanish and

$$\kappa_\chi^{\mathbb{C}}(z_1, \ldots, z_{2n}) = 0$$

(where $\chi \in \{\ell, r\}^{2n}$ is such that $\chi(k) = \ell$ if $z_k \in \{x, x^*\}$ and $\chi(k) = r$ if $z_k \in \{y, y^*\}$) unless $(z_{s_\chi(1)}, \ldots, z_{s_\chi(n)})$ is of one of the following forms:

- $(Z, Z^*, Z, Z^*, \ldots, Z, Z^*)$ with $Z \in \{x, x^*, y, y^*\}$,
- $(x, x^*, x, x^*, \ldots, x, x^*, y, y^*, y, y^*, \ldots, y, y^*)$,
- $(x, x^*, x, x^*, \ldots, x, x^*, x, y^*, y, y^*, y, y^*, \ldots, y, y^*)$,
- $(x^*, x, x^*, x, \ldots, x^*, x, y^*, y, y^*, y, \ldots, y^*, y)$, or
- $(x^*, x, x^*, x, \ldots, x^*, x, x^*, y, y^*, y, y^*, y, \ldots, y^*, y)$.

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To reduce the above to our desired context, let $x, y \in \mathcal{A}$. For $d = 2$, let

$$X = x \otimes E_{1,2} \otimes I_2 + x^* \otimes E_{2,1} \otimes I_2 \qquad \text{and} \qquad Y = y \otimes I_2 \otimes E_{1,2} + y^* \otimes I_2 \otimes E_{2,1},$$

which are then self-adjoint elements of A_2. The pair (X, Y) is intimately related to whether or not (x, y) is bi-R-diagonal. Indeed, combining [7, Proposition 2.21] and [12, Theorem 4.9], we obtain the following proposition.

Proposition 9.2 *As described above, the following conditions are equivalent:*

(1) *The pair (x, y) is bi-R-diagonal.*
(2) *The pair (X, Y) is bi-R-cyclic.*
(3) *The pair of algebras $(\mathrm{alg}(D_2, X), \mathrm{alg}(D_2, Y))$ is bi-free from $(M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r)$ with amalgamation over D_2 with respect to $F_2 \circ E_2$.*

In addition, the joint moments of (X, Y) with respect to τ_d are not too difficult to describe. Indeed, for any $n \in \mathbb{N}$, $Z_1, Z_2, \ldots, Z_n \in \{X, Y\}$, $\chi \in \{\ell, r\}^n$, and $z_1, \ldots, z_n \in \{x, y\}$ such that

$$\chi(k) = \begin{cases} \ell, & \text{if } Z_k = X \\ r, & \text{if } Z_k = Y \end{cases} \qquad \text{and} \qquad z_k = \begin{cases} x, & \text{if } Z_k = X, \\ y, & \text{if } Z_k = Y, \end{cases}$$

then

$$E_2(Z_1 \cdots Z_n) = \sum_{k=1}^{n} \sum_{p_k \in \{1, *\}} \varphi(z_1^{p_1} \cdots z_n^{p_n})(E_{1,2})^{p_{s_\chi(1)}}(E_{1,2})^{p_{s_\chi(2)}} \cdots (E_{1,2})^{p_{s_\chi(n)}}.$$

Thus, if n is odd, we see that $\tau_2(Z_1 \cdots Z_n) = 0$ and if n is even, we see that

$$\tau_2(Z_1 \cdots Z_n) = \frac{1}{2}\left(\varphi(z_1^{p_1} \cdots z_n^{p_n}) + \varphi(z_1^{q_1} \cdots z_n^{q_n})\right),$$

where

$$p_{s_\chi(k)} = \begin{cases} 1, & \text{if } k \text{ is odd} \\ *, & \text{if } k \text{ is even} \end{cases} \qquad \text{and} \qquad q_{s_\chi(k)} = \begin{cases} *, & \text{if } k \text{ is odd} \\ 1, & \text{if } k \text{ is even} \end{cases}$$

(that is, the 1's and $*$'s alternate in the χ-ordering). Hence, the joint moments of (X, Y) with respect to τ_d depend only on specific moments of (x, y). We let $\Delta_{X,Y}$ denote the set of all pairs (x_0, y_0) in a C*-noncommutative probability space $(\mathcal{A}_0, \varphi_0)$ such that if we apply the above procedure to (x_0, y_0) resulting in (X_0, Y_0), then (X_0, Y_0) has the same joint distribution as (X, Y) (so $\Delta_{X,Y} = \Delta_{X_0,Y_0}$).

One specific case worth mentioning is when x and y are normal operators with $[\mathrm{alg}(x, x^*), \mathrm{alg}(y, y^*)] = 0$, thus defining a probability measure μ on $\mathbb{C}^2$. Then, X and Y will be commuting self-adjoint operators and therefore their joint distribution gives rise to a compactly supported probability measure μ_0 on $\mathbb{R}^2$ with moments

$$\tau_2(X^n Y^m) = \begin{cases} 0, & \text{if } n + m \text{ is odd}, \\ \varphi((x^*x)^i (y^*y)^j), & \text{if } n = 2i \text{ and } m = 2j, \\ \frac{1}{2}\varphi((x^*x)^i (xy^* + x^*y)(y^*y)^j), & \text{if } n = 2i + 1 \text{ and } m = 2j + 1. \end{cases}$$

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One additional property is required in this section. In particular, as we are attempting to generalize [8, Theorem 1.1], which makes heavy use of traciality, we need a condition that lets us bypass the issue that τ_2 is not tracial on A_2.

Definition 9.3 Let $(\mathcal{A}, \varphi)$ be a C*-noncommutative probability space, and let $x, y \in \mathcal{A}$. We say that (x, y) is *alternating adjoint flipping with respect to* φ if for any $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^{2n}$, and $z_1, \ldots, z_{2n} \in \{x, y\}$ such that

$$z_k = \begin{cases} x, & \text{if } \chi(k) = \ell, \\ y, & \text{if } \chi(k) = r, \end{cases}$$

we have that

$$\varphi(z_1^{p_1} \cdots z_{2n}^{p_{2n}}) = \varphi(z_1^{q_1} \cdots z_{2n}^{q_{2n}}),$$

where

$$p_{s_\chi(k)} = \begin{cases} 1, & \text{if } k \text{ is odd} \\ *, & \text{if } k \text{ is even} \end{cases} \qquad \text{and} \qquad q_{s_\chi(k)} = \begin{cases} *, & \text{if } k \text{ is odd}, \\ 1, & \text{if } k \text{ is even}. \end{cases}$$

Remark 9.3 If (x, y) is alternating adjoint flipping, then the description of the joint moments of (X, Y) above reduces to a nicer expression. Furthermore, we see that

$$\varphi((x^*x)^m) = \varphi((xx^*)^m) \qquad \text{and} \qquad \varphi((y^*y)^m) = \varphi((yy^*)^m)$$

for all $m \in \mathbb{N}$, so that x^*x and xx^* have the same distribution and y^*y and yy^* have the same distribution, which would be automatic if φ was tracial when restricted to $\text{alg}(x, x^*)$ and when restricted to $\text{alg}(y, y^*)$ (a common assumption in bi-free probability).

Recall that a pair (u_ℓ, u_r) is said to be a *bi-Haar unitary* if u_ℓ and u_r are commuting normal operators such that

$$\varphi(u_\ell^m u_r^k) = \begin{cases} 1, & \text{if } m + k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Of course, bi-Haar unitary pairs are trivially seen to be alternating adjoint flipping, since any joint moment with an equal number of adjoint and non-adjoint terms is 1 and any joint moment with a differing number of adjoint and non-adjoint terms is 0. Here is another example which is of use in this paper.

Example 9.4 Let $\mathcal{H}$ be any Hilbert space of dimension at least 4, let $\mathcal{F}(\mathcal{H})$ denote the Fock space generated by $\mathcal{H}$, let φ_0 be the vacuum vector state on $\mathcal{B}(\mathcal{F}(\mathcal{H}))$, and let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal set. For $i = 1, 2$, let $s_i = l(e_i) + l^*(e_i)$ (i.e., left creation plus annihilation by e_i), and for $j = 1, 2$, let $d_j = r(e_{j+2}) + r^*(e_{j+2})$ (i.e., right creation and annihilation by e_{j+2}). Thus, $(\{s_1, s_2\}, \{d_1, d_2\})$ is a bi-free central limit distribution with variance 1 and covariance 0.

Let

$$c_\ell = \frac{1}{\sqrt{2}}(s_1 + is_2) \qquad \text{and} \qquad c_r = \frac{1}{\sqrt{2}}(d_1 + id_2).$$

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We call the pair (c_ℓ, c_r) a *bi-free circular pair* (with mean 0, variance 1, and covariance 0).

We claim that (c_ℓ, c_r) is an example of a bi-R-diagonal pair that is alternating adjoint flipping with respect to φ_0. To see that (c_ℓ, c_r) is bi-R-diagonal, we note that any bi-free cumulant for $(\{s_1, s_2\}, \{d_1, d_2\})$ of order 1, of order greater than 3, or involving two different elements is 0. As

$$\kappa_{1_{(\ell,\ell)}}(c_\ell, c_\ell) = \frac{1}{2}\kappa_{1_{(\ell,\ell)}}(s_1, s_1) + (i)^2\frac{1}{2}\kappa_{1_{(\ell,\ell)}}(s_2, s_2) = 0,$$
$$\kappa_{1_{(\ell,\ell)}}(c_\ell^*, c_\ell^*) = \frac{1}{2}\kappa_{1_{(\ell,\ell)}}(s_1, s_1) + (-i)^2\frac{1}{2}\kappa_{1_{(\ell,\ell)}}(s_2, s_2) = 0,$$

and similar computations hold on the right, we have that (c_ℓ, c_r) is bi-R-diagonal.

To see that (c_ℓ, c_r) is alternating adjoint flipping with respect to φ_0, first note that φ_0 is tracial when restricted to $\mathrm{alg}(s_1, s_2)$ as s_1 and s_2 are freely independent with respect to φ_0. Hence, for all $n \in \mathbb{N}$,

$$\varphi_0((c_\ell^* c_\ell)^n) = \varphi_0((c_\ell c_\ell^*)^n).$$

Moreover, as any monomial of odd length involving freely independent semicircular variables is 0, we obtain that, for all $n \in \mathbb{N}$,

$$\varphi_0(c_\ell(c_\ell^* c_\ell)^n) = 0 = \varphi_0(c_\ell^*(c_\ell c_\ell^*)^n).$$

Similarly, for all $n \in \mathbb{N}$, we have that

$$\varphi_0((c_r^* c_r)^n) = \varphi_0((c_r c_r^*)^n) \qquad \text{and} \qquad \varphi_0(c_r(c_r^* c_r)^n) = 0 = \varphi_0(c_r^*(c_r c_r^*)^n).$$

To see the remaining moment conditions, first note that $\{c_\ell, c_\ell^*\}$ commutes with $\{c_r, c_r^*\}$. Thus, as the χ-ordering is not changed by commutation of left and right operators, it suffices to show that

$$\begin{aligned}
\varphi_0((c_\ell^* c_\ell)^n(c_r c_r^*)^m) &= \varphi_0((c_\ell c_\ell^*)^n(c_r^* c_r)^m),\\
\varphi_0(c_\ell(c_\ell^* c_\ell)^n(c_r c_r^*)^m) &= \varphi_0(c_\ell^*(c_\ell c_\ell^*)^n(c_r^* c_r)^m),\\
\varphi_0((c_\ell^* c_\ell)^n c_r^*(c_r c_r^*)^m) &= \varphi_0(c_\ell^*(c_\ell c_\ell^*)^n c_r(c_r^* c_r)^m), \text{ and}\\
\varphi_0(c_\ell(c_\ell^* c_\ell)^n c_r^*(c_r c_r^*)^m) &= \varphi_0(c_\ell^*(c_\ell c_\ell^*)^n c_r(c_r^* c_r)^m),
\end{aligned}$$

for all $n, m \in \mathbb{N} \cup \{0\}$. However, as $\{c_\ell, c_\ell^*\}$ is classically independent from $\{c_r, c_r^*\}$ since the joint bi-free cumulants vanish, each of the eight above moment expressions simplifies to the φ_0-moment of the $\{c_\ell, c_\ell^*\}$ term times the φ_0-moment of the $\{c_r, c_r^*\}$. Thus, the desired moments are equal by the above knowledge of the φ_0-moments of the $\{c_\ell, c_\ell^*\}$ and the φ_0-moment of the $\{c_r, c_r^*\}$.

With the above definitions, notation, and constructions out of the way, our main result is at hand.

Theorem 9.5 *Let $(\mathcal{A}, \varphi)$ be a C^*-noncommutative probability space, and let $x, y \in \mathcal{A}$ be such that x^*x and xx^* have the same distribution with respect to φ and y^*y and yy^* have the same distribution with respect to φ. With X and Y as described above,*

$$\min\{\Phi^*(\{x_0, x_0^*\} \sqcup \{y_0, y_0^*\} : (\mathbb{C}, \mathbb{C}), \varphi) \mid (x_0, y_0) \in \Delta_{X,Y}\} \geq 2\Phi^*(X \sqcup Y)$$

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and equality holds and is achieved for any pair (x_0, y_0) that is alternating adjoint flipping and bi-R-diagonal.

Remark 9.6 Note that Theorem 9.5 is a generalization of [8, Theorem 1.1] to the bi-free setting. Prior to the acknowledgements of [8], it is mentioned that the minimum in the free result can only be reached by an R-diagonal element via the result from [15] that $\Phi^*(x_1, \ldots, x_n : B) = \Phi^*(x_1, \ldots, x_n) < \infty$ implies $\{x_1, \ldots, x_n\}$ is free from B. As there is no such known analogous result in the bi-free case, we leave Theorem 9.5 as stated.

To begin the proof of Theorem 9.5, we note the following connecting the bi-free Fisher information of $(\{x, x^*\}, \{y, y^*\})$ and (X, Y) (and thereby demonstrating the necessity of considering bi-free Fisher information with respect to completely positive maps in this construction). We note that the following is a generalization of [8, Proposition 3.6] with a similar but more complicated proof due to the χ-ordering and additional variables present.

Proposition 9.7 *Under the assumptions of Theorem 9.5, if $\eta : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is defined by*

$$\eta\left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\right) = \begin{bmatrix} a_{2,2} & 0 \\ 0 & a_{1,1} \end{bmatrix},$$

then

$$\Phi^*(\{x, x^*\} \sqcup \{y, y^*\} : (\mathbb{C}, \mathbb{C}), \varphi) = 2\Phi^*(X \sqcup Y : (M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r), \eta).$$

Proof First, suppose that the bi-free Fisher information $\Phi^*(\{x, x^*\} \sqcup \{y, y^*\})$ is finite. Thus, there exist

$$\xi_1, \xi_2 \in \overline{\operatorname{alg}(x, x^*, y, y^*)}^{\|\cdot\|_\varphi}$$

such that ξ_1 is the left bi-free conjugate variable for x with respect to φ in the presence of $(x, \{y, y^*\})$ and ξ_2 is the left bi-free conjugate variable for x^* with respect to φ in the presence of $(x^*, \{y, y^*\})$. Let

$$\Xi = \begin{bmatrix} 0 & \xi_2 \\ \xi_1 & 0 \end{bmatrix} \in M_2(L_2(A_2, \tau_2)) \cong L_2(M_2(A_2), \tau_2 \otimes \operatorname{tr}_2).$$

We claim that $\Xi = J_\ell\left(X : \left(M_2(\mathbb{C})_\ell, \operatorname{alg}\left(M_2(\mathbb{C})_r, Y\right)\right), \eta\right)$. Since a similar result holds on the right and since

$$\|\Xi\|_{\tau_2}^2 = \frac{1}{2}\left(\|\xi_1\|_\varphi^2 + \|\xi_2\|_\varphi^2\right),$$

the result will follow in this case.

First, we claim that $\Xi \in \overline{\operatorname{alg}\left(X, Y, M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r\right)}^{\tau_2}$. Indeed, it is not difficult to verify that

$$z \otimes E_{i_1, j_1} \otimes E_{i_2, j_2} \in \operatorname{alg}\left(X, Y, M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r\right)$$

for all $z \in \{x, x^*, y, y^*\}$ and $i_1, i_2, j_1, j_2 \in \{1, 2\}$. Thus, since $\xi_1, \xi_2 \in \overline{\operatorname{alg}(x, x^*, y, y^*)}^{\|\cdot\|_\varphi}$, the claim follows.

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To complete the claim that Ξ is the appropriate left bi-free conjugate variable, we must show that Ξ satisfies the left bi-free conjugate variable relations; that is, for all $n \in \mathbb{N}$, $b_0, b_1, \ldots, b_n \in M_2(\mathbb{C})$, $\chi \in \{\ell, r\}^n$ with $\chi(n) = \ell$, and $Z_1, \ldots, Z_{n-1} \in A_2$ and $C_1, \ldots C_{n-1} \in 1_{\mathcal{A}} \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{\mathrm{op}}$ where

$$Z_k = \begin{cases} X, & \text{if } \chi(k) = \ell \\ Y, & \text{if } \chi(k) = r \end{cases} \qquad \text{and} \qquad C_k = \begin{cases} L_{b_k}, & \text{if } \chi(k) = \ell, \\ R_{b_k}, & \text{if } \chi(k) = r, \end{cases}$$

we have that

(9.1)

$$\tau_2\left(L_{b_0} R_{b_n} Z_1 C_1 \cdots Z_{n-1} C_{n-1} \Xi\right) = \sum_{\substack{1 \le k < n \\ \chi(k) = \ell}} \tau_2\left(L_{b_0} R_{b_n}\left(\prod_{p \in V_k^c \setminus \{k,n\}} Z_p C_p\right) L_{\eta\left(E_2\left(C_k \prod_{p \in V_k} Z_p C_p\right)\right)}\right),$$

where $V_k = \{k < m < n \mid \chi(m) = \ell\}$. By linearity, it suffices to consider $b_k = E_{i_k, j_k}$ for all k where $i_k, j_k \in \{1, 2\}$. In that which follows, the proof is near identical to that of [8, Proposition 3.6] taking into account the χ-order. For notational purposes, for $k \in \{1, 2\}$, let $\overline{k} = 3 - k$.

Let $q = s_\chi^{-1}(n)$ (i.e., Ξ appears qth in the χ-ordering). We begin by computing the left-hand side of (9.1). Using Lemma 9.1 (and recalling $\tau_2 = \mathrm{tr}_2 \circ E_2$), proceeding via χ-order using commutation, we obtain that:

- the only way the product produces a nonzero trace is if $i_0 = j_n$,
- the term $L_{b_0} X L_{b_{s_\chi(1)}}$ can be made to appear in the product and is nonzero only if $j_0 = \overline{i_{s_\chi(1)}}$,
- the term $L_{b_{s_\chi(k-1)}} X L_{b_{s_\chi(k)}}$ can be made to occur for all $2 \le k < q$ and is nonzero only if $j_{s_\chi(k-1)} = \overline{i_{s_\chi(k)}}$,
- the term $L_{b_{s_\chi(q-1)}} R_{b_{s_\chi(q+1)}} \Xi$ can be made to occur and is nonzero only if $j_{s_\chi(q-1)} = \overline{i_{s_\chi(q+1)}}$,
- the term $R_{b_{s_\chi(k+1)}} Y R_{b_{s_\chi(k)}}$ can be made to occur for all $q < k < n$ and is nonzero only if $j_{s_\chi(k)} = \overline{i_{s_\chi(k+1)}}$ (recall the opposite multiplication), and
- the term $R_{b_n} Y R_{b_{s_\chi(n)}}$ can be made to occur and is nonzero only if $j_{s_\chi(n)} = \overline{i_n}$.

Note the discrepancy in notation around the Ξ term due to the labeling of the left and right B-operators (i.e., $b_{s_\chi(q)} = b_n$ is in the wrong spot). Thus, with

$$(X)_{1,2} = x, \quad (X)_{2,1} = x^*, \quad (Y)_{1,2} = y, \quad (Y)_{2,1} = y^*, \quad (\Xi)_{1,2} = \xi_2, \quad \text{and} \quad (\Xi)_{2,1} = \xi_1,$$

and

$$\overline{Z_k} = \begin{cases} (X)_{j_0, \overline{j_0}}, & \text{if } s_\chi(k) = 1, \\ (X)_{j_{s_\chi(k-1)}, \overline{j_{s_\chi(k-1)}}}, & \text{if } 1 < s_\chi(k) < q, \\ (\Xi)_{j_{s_\chi(q-1)}, \overline{j_{s_\chi(q-1)}}}, & \text{if } s_\chi(k) = q, \\ (Y)_{j_{s_\chi(k)}, \overline{j_{s_\chi(k)}}}, & \text{if } q < s_\chi(k) < n, \\ (Y)_{j_{s_\chi(n)}, \overline{j_{s_\chi(n)}}}, & \text{if } s_\chi(k) = n, \end{cases}$$

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we see that the left-hand side of (9.1) is

$$\frac{1}{2}\delta_{j_n,i_0}\delta_{j_0,\overline{i_{s_\chi(1)}}}\delta_{j_{s_\chi(1)},\overline{i_{s_\chi(2)}}}\cdots\delta_{j_{s_\chi(q-2)},\overline{i_{s_\chi(q-1)}}}\delta_{j_{s_\chi(1-2)},\overline{i_{s_\chi(q+1)}}}\delta_{j_{s_\chi(q+1)},\overline{i_{s_\chi(q+2)}}}\cdots\delta_{j_{s_\chi(n-1)},\overline{i_{s_\chi(n)}}}\delta_{j_{s_\chi(n)},\overline{i_n}}$$
$$\times\varphi\left(\overline{Z_1}\cdots\overline{Z_n}\right), \tag{9.2}$$

where $\delta_{j,i}$ is the Kronecker delta. Moreover, using the conjugate variable relations for ξ_1 and ξ_2, we see that

$$\varphi\left(\overline{Z_1}\cdots\overline{Z_n}\right)=\sum_{\substack{1\le k<n\\ \chi(k)=\ell}}\delta_{j_{s_\chi(k-1)},\overline{j_{s_\chi(q-1)}}}\varphi\left(\prod_{p\in V_k^c\setminus\{k,n\}}\overline{Z_p}\right)\varphi\left(\prod_{p\in V_k}\overline{Z_p}\right), \tag{9.3}$$

where the $\delta_{j_{s_\chi(k-1)},\overline{j_{s_\chi(q-1)}}}$ should be $\delta_{j_{s_\chi(k-1)},\overline{j_{s_\chi(q-1)}}}$ when $k=s_\chi^{-1}(1)$.

To complete the proof that equation (9.1) holds, we compute the right-hand side of equation (9.1) and show the kth term in the sum equals the kth term obtain in equation (9.2) using equation (9.3). Indeed, for a fixed $1\le k<n$ for which $\chi(k)=\ell$, we can compute

$$M_k=E_2\left(C_k\prod_{p\in V_k}Z_pC_p\right),$$

in a similar fashion to the above. Thus, to obtain a nonzero value, the relations $j_{s_\chi(p-1)}=\overline{i_{s_\chi(p)}}$ for all $p\in V_k$ must hold. Moreover, one immediately obtains when $M_k\neq 0$ that

$$M_k=\varphi\left(\prod_{p\in V_k}\overline{Z_p}\right)T_k,$$

for some $T_k\in M_2(\mathbb{C})$.

Next, notice that $\eta(M_k)$ is equivalent to multiplying M_k on the left by $U=E_{1,2}+E_{2,1}$ (for right conjugate variables, one would multiply on the right) and thus we consider UM_k in place of $\eta(M_k)$. At this point, notice by Lemma 9.1 that UT_k can be written as a product of b_p's with $b_{s_\chi(q-1)}$ being the right-most term. By commutation, $R_{b_{s_\chi(q+1)}}$ will act on the right of UT_k thereby multiplying by $b_{s_\chi(q+1)}$ on the right and forcing $j_{s_\chi(q-1)}=\overline{i_{s_\chi(q+1)}}$ for a nonzero value to be obtained. One then proceeds as above to show that a nonzero value is obtained only if the above relations are satisfied and that the term that is produced agrees with the kth term of (9.3). Hence, the proof is complete in the case that $\Phi^*(\{x,x^*\}\sqcup\{y,y^*\})<\infty$.

To prove the result in the case that $\Phi^*(\{x,x^*\}\sqcup\{y,y^*\})=\infty$, it suffices to show that if $\Phi^*(X\sqcup Y:\eta)<\infty$, then $\Phi^*(\{x,x^*\}\sqcup\{y,y^*\}:(\mathbb{C},\mathbb{C}),\varphi)<\infty$. Thus, suppose that $\Phi^*(X\sqcup Y:\eta)<\infty$. Hence, $\Xi=J_\ell\left(X:(M_2(\mathbb{C})_\ell,\operatorname{alg}\left(M_2(\mathbb{C})_r,Y\right)),\eta\right)$ exists and can be written as

$$\Xi=\begin{bmatrix}\xi_{1,1}&\xi_{1,2}\\ \xi_{2,1}&\xi_{2,2}\end{bmatrix}.$$

We claim that

$$\xi_{2,1}=J_\ell(x:(x^*,\{y,y^*\}),\varphi)\qquad\text{and}\qquad\xi_{1,2}=J_\ell(x^*:(x,\{y,y^*\}),\varphi).$$

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As a similar result will hold on the right, we will obtain that $\Phi^*(\{x, x^*\} \sqcup \{y, y^*\} : (\mathbb{C}, \mathbb{C}), \varphi) < \infty$ as desired.

As $\Xi \in \overline{\mathrm{alg}(X, Y, M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r)}^{\tau_2}$, it is not difficult to see that $\xi_{i,j} \in \overline{\mathrm{alg}(x, x^*, y, y^*)}^{\|\cdot\|_\varphi}$. To see that $\xi_{2,1}$ and $\xi_{1,2}$ satisfy the appropriate left bi-free conjugate variable relations, one need only use equation (9.1), choose $b_k = E_{i_k, j_k}$ satisfying the above required relations for a nonzero value, and expand both sides of equation (9.1) in an identical way to that above. The resulting equations are exactly the left bi-free conjugate variable relations required. ∎

Using the results from this paper, there is some immediate knowledge about the bi-free Fisher information with respect to η from Proposition 9.7. We note that the following is a generalization of [8, Proposition 3.7] with a similar but more complicated proof due to the χ-ordering and additional variables present.

Proposition 9.8 *Under the assumptions and notation of Proposition 9.7,*

$$\Phi^*(X \sqcup Y : \eta) \geq \Phi^*(X \sqcup Y : \eta|_{D_2})$$

and the equality holds when $(\mathrm{alg}((D_2)_\ell, X), \mathrm{alg}((D_2)_r, Y))$ *is bi-free from* $(M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r)$ *with amalgamation over* D_2 *with respect to* F_2*. Moreover,*

$$\Phi^*(X \sqcup Y : \eta|_{D_2}) \geq \Phi^*(X \sqcup Y)$$

and the equality holds if (x, y) *is alternating adjoint flipping.*

Proof Since $\eta = \eta \circ F$, we have that

$$\Phi^*(X \sqcup Y : \eta) = \Phi^*(X \sqcup Y : ((M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r), \eta \circ F)$$

by Remark 7.1(5). Moreover,

$$\Phi^*(X \sqcup Y : ((M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r), \eta \circ F) \geq \Phi^*(X \sqcup Y : \eta|_{D_2})$$

by Remark 7.1(7). Furthermore, equality holds if $(\mathrm{alg}((D_2)_\ell, X), \mathrm{alg}((D_2)_r, Y))$ is bi-free from $(M_2(\mathbb{C})_\ell, M_2(\mathbb{C})_r)$ over D_2 with respect to F_2 by Remark 7.1(8).

To see that $\Phi^*(X \sqcup Y : \eta|_{D_2}) \geq \Phi^*(X \sqcup Y)$, we assume that

$$\Xi = J_\ell\left(X : ((D_2)_\ell, \mathrm{alg}((D_2)_r, Y)), \eta|_{D_2}\right) \in \overline{\mathrm{alg}(X, Y, (D_2)_\ell, (D_2)_r)}^{\|\cdot\|_{\tau_2}}$$

exists and show that Ξ satisfies the left bi-free conjugate variable relations for X in the presence of Y. Thus, if P is the orthogonal projection of $L_2(A_2, \tau_2)$ onto $\overline{\mathrm{alg}(X, Y)}^{\|\cdot\|_{\tau_2}}$, then $P(\Xi)$ will also satisfy the left bi-free conjugate variable relations for X in the presence of Y. As a similar result will hold on the right, the inequality $\Phi^*(X \sqcup Y : \eta|_{D_2}) \geq \Phi^*(X \sqcup Y)$ will be demonstrated.

By the defining property of Ξ, we know for all $n \in \mathbb{N}$, $b_0, b_1, \ldots, b_n \in D_2$, $\chi \in \{\ell, r\}^n$ with $\chi(n) = \ell$, and $Z_1, \ldots, Z_{n-1} \in A_2$ and $C_1, \ldots C_{n-1} \in 1_{\mathcal{A}} \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})^{\mathrm{op}}$, where

$$Z_k = \begin{cases} X, & \text{if } \chi(k) = \ell \\ Y, & \text{if } \chi(k) = r \end{cases} \qquad \text{and} \qquad C_k = \begin{cases} L_{b_k}, & \text{if } \chi(k) = \ell, \\ R_{b_k}, & \text{if } \chi(k) = r, \end{cases}$$

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that

(9.4)
$$\tau_2\left(L_{b_0}R_{b_n}Z_1C_1\cdots Z_{n-1}C_{n-1}\Xi\right) = \sum_{\substack{1\leq k<n\\ \chi(k)=\ell}} \tau_2\left(L_{b_0}R_{b_n}\left(\prod_{p\in V_k^c\setminus\{k,n\}} Z_pC_p\right)L_{\eta\left((F\circ E_2)\left(C_k\prod_{p\in V_k}Z_pC_p\right)\right)}\right),$$

where $V_k = \{k < m < n \mid \chi(m) = \ell\}$. We will use equation (9.4) where $b_k = I_2$ for all k. To begin, notice that

$$\eta\left((F\circ E_2)\left(C_k\prod_{p\in V_k}Z_pC_p\right)\right) = \tau_2\left(X^{|V_k|}\right)I_2,$$

as odd moments of X are zero and as x^*x and xx^* have the same distribution with respect to φ. Therefore,

$$\varphi((x^*x)^m) = \varphi((xx^*)^m) = \tau_2(X^{2m}),$$

for all $m \in \mathbb{N}$. Hence, equation (9.4) reduces to

$$\tau_2\left(Z_1\cdots Z_{n-1}\Xi\right) = \sum_{\substack{1\leq k<n\\ \chi(k)=\ell}} \tau_2\left((Z_1,\ldots,Z_{n-1})|_{V_k^c\setminus\{k,n\}}\right)\tau_2\left(X^{|V_k|}\right),$$

which is exactly the desired formula.

To prove $\Phi^*(X \sqcup Y : \eta|_{D_2}) \leq \Phi^*(X \sqcup Y)$ when (x, y) is alternating adjoint flipping thereby completing the proof, we proceed in a similar (but more complicated) fashion. Suppose that

$$\Xi = J_\ell\left(X : (\mathbb{C}, \operatorname{alg}(Y))\right) \in \overline{\operatorname{alg}(X,Y)}^{\|\cdot\|_{\tau_2}} \subseteq \overline{\operatorname{alg}(X,Y,(D_2)_\ell,(D_2)_r)}^{\|\cdot\|_{\tau_2}}$$

exists. We will demonstrate that Ξ satisfies the left bi-free conjugate variable relations for X with respect to η in the presence of $((D_2)_\ell, \operatorname{alg}((D_2)_r, Y))$. As an analogous result will hold on the right, this will complete the proof.

Write

$$\Xi = \begin{bmatrix}\xi_{1,1} & \xi_{1,2}\\ \xi_{2,1} & \xi_{2,2}\end{bmatrix} \in M_2(L_2(\mathcal{A},\varphi)) = L_2(A_2,\tau_2).$$

First, we will demonstrate that $\xi_{1,1} = \xi_{2,2} = 0$. To begin, let

$$\mathcal{H}_e = \overline{\operatorname{span}(Z_1\cdots Z_{2n} \mid n\in\mathbb{N}, Z_k\in\{X,Y\})}^{\|\cdot\|_{\tau_2}} \text{ and}$$
$$\mathcal{H}_o = \overline{\operatorname{span}(Z_1\cdots Z_{2n-1} \mid n\in\mathbb{N}, Z_k\in\{X,Y\})}^{\|\cdot\|_{\tau_2}}.$$

By the defining property of Ξ, we know for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$ with $\chi(n) = \ell$, and $Z_1, \ldots, Z_{n-1} \in A_2$, where

$$Z_k = \begin{cases} X, & \text{if } \chi(k) = \ell,\\ Y, & \text{if } \chi(k) = r,\end{cases}$$

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that

$$\tau_2\left(Z_1\cdots Z_{n-1}\Xi\right)=\sum_{\substack{1\le k<n\\ \chi(k)=\ell}}\tau_2\left((Z_1,\ldots,Z_{n-1})|_{V_k^c\setminus\{k,n\}}\right)\tau_2\left(X^{|V_k|}\right), \tag{9.5}$$

where $V_k=\{k<m<n\mid \chi(m)=\ell\}$. Note that as τ_2 evaluates any odd product involving X and Y to 0 by Lemma 9.1, if $n-1$ is even, then $\tau_2\left(Z_1\cdots Z_{n-1}\Xi\right)=0$. Therefore, since $\Xi\in\mathcal{H}_e+\mathcal{H}_o$, we obtain that $\Xi\in\mathcal{H}_o$.

Note for $n\in\mathbb{N}$, $\chi\in\{\ell,r\}^{2n-1}$, $Z_1,\ldots,Z_{2n-1}\in\{X,Y\}$, and $z_1,\ldots,z_{2n-1}\in\{x,y\}$ where

$$Z_k=\begin{cases}X & \text{if } \chi(k)=\ell\\ Y & \text{if } \chi(k)=r\end{cases}\qquad\text{and}\qquad z_k=\begin{cases}x & \text{if } \chi(k)=\ell\\ y & \text{if } \chi(k)=r\end{cases}$$

that in $M_2(L_2(\mathcal{A},\varphi))$ we have

$$Z_1\cdots Z_{2n-1}=\begin{bmatrix}0 & z_1^{p_1}z_2^{p_2}\cdots z_{2n-1}^{p_{2n-1}}\\ z_1^{q_1}z_2^{q_2}\cdots z_{2n-1}^{q_{2n-1}} & 0\end{bmatrix},$$

where

$$p_{s_\chi(k)}=\begin{cases}1, & \text{if } k \text{ is odd}\\ *, & \text{if } k \text{ is even}\end{cases}\qquad\text{and}\qquad q_{s_\chi(k)}=\begin{cases}*, & \text{if } k \text{ is odd},\\ 1, & \text{if } k \text{ is even}.\end{cases}$$

Therefore, as $\mathcal{H}_o$ is the $\|\cdot\|_{\tau_2}$-limit of matrices of the above form and as $\Xi\in\mathcal{H}_o$, we obtain that $\xi_{1,1}=\xi_{2,2}=0$ as desired.

Let

$$H_1=\operatorname{span}\left(z_1\cdots z_{2n-1}\;\middle|\;\begin{array}{c}n\in\mathbb{N},z_k\in\{x,x^*,y,y^*\}\\ \text{the powers of the } z_k\text{'s alternate between 1 and } * \text{ in the } \chi\text{-ordering}\\ \text{and the first and last elements in the } \chi\text{-ordering have power 1}\end{array}\right)\quad\text{and}$$

$$H_*=\operatorname{span}\left(z_1\cdots z_{2n-1}\;\middle|\;\begin{array}{c}n\in\mathbb{N},z_k\in\{x,x^*,y,y^*\}\\ \text{the powers of the } z_k\text{'s alternate between 1 and } * \text{ in the } \chi\text{-ordering}\\ \text{and the first and last elements in the } \chi\text{-ordering have power } *\end{array}\right).$$

Using the above and the notation $\xi=\xi_{1,2}$ and $\xi^*=\xi_{2,1}$ (note that we do not claim that there is an involution operation on $L_2(\mathcal{A},\varphi)$ as we do not know φ is tracial), we see that $\xi\in\overline{H_1}^{\|\cdot\|_\varphi}$, $\xi^*\in\overline{H_*}^{\|\cdot\|_\varphi}$, and if we have a $\|\cdot\|_\varphi$-limiting sequence using $\{x,x^*,y,y^*\}$ producing ξ, we can obtain a $\|\cdot\|_\varphi$-limiting sequence using $\{x,x^*,y,y^*\}$ producing ξ^* by exchanging $x\leftrightarrow x^*$ and $y\leftrightarrow y^*$. This, in conjunction with the alternating adjoint flipping condition, lets us show if $n\in\mathbb{N}$, $\chi\in\{\ell,r\}^{2n}$, $Z_1,\ldots,Z_{2n-1}\in\{X,Y\}$, and $z_1,\ldots,z_{2n-1}\in\{x,y\}$, where

$$Z_k=\begin{cases}X, & \text{if } \chi(k)=\ell\\ Y, & \text{if } \chi(k)=r\end{cases}\qquad\text{and}\qquad z_k=\begin{cases}x, & \text{if } \chi(k)=\ell,\\ y, & \text{if } \chi(k)=r,\end{cases}$$

that

$$\varphi(z_1^{p_1}z_2^{p_2}\cdots z_{2n-1}^{p_{2n-1}}\xi^{p_{2n}})=\varphi(z_1^{q_1}z_2^{q_2}\cdots z_{2n-1}^{q_{2n-1}}\xi^{q_{2n}}), \tag{9.6}$$

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where

$$p_{s_\chi(k)} = \begin{cases} 1, & \text{if } k \text{ is odd} \\ *, & \text{if } k \text{ is even} \end{cases} \quad \text{and} \quad q_{s_\chi(k)} = \begin{cases} *, & \text{if } k \text{ is odd,} \\ 1, & \text{if } k \text{ is even.} \end{cases}$$

Indeed, consider $z_1^{p_1} \cdots z_{2n-1}^{p_{2n-1}} \xi^{p_{2n}}$ with $p_{2n} = 1$ (the case $p_{2n} = *$ is analogous). As the terms preceding ξ in the χ-ordering both must have $*$'s on them and as ξ is a $\|\cdot\|_\varphi$-limit of elements of H_1, we see that $z_1^{p_1} \cdots z_{2n-1}^{p_{2n-1}} \xi^{p_{2n}}$ is a $\|\cdot\|_\varphi$-limit of a linear combination of monomials in $\{x, x^*, y, y^*\}$ that alternate between $*$ and non-$*$-terms in the χ-ordering. As $x \leftrightarrow x^*$ and $y \leftrightarrow y^*$ produce the same φ-moment by the alternating adjoint flipping condition (as $z_1^{p_1} \cdots z_{2n-1}^{p_{2n-1}}$ and every element of H_1 is of odd length) and produce a sequence that converges to $z_1^{q_1} z_2^{q_2} \cdots z_{2n-1}^{q_{2n-1}} \xi^{q_{2n}}$ with respect to $\|\cdot\|_\varphi$, the claim is complete.

Returning to showing Ξ satisfies the left bi-free conjugate variable relations for X with respect to η in the presence of $((D_2)_\ell, \operatorname{alg}((D_2)_r, Y))$, it suffices to demonstrate that equation (9.4) holds for this Ξ. Furthermore, it suffices to verify that equation (9.4) holds when $b_k = E_{i_k, i_k}$ for all k. By the same computations as done in the proof of Proposition 9.7 with $j_k = i_k$ for all k, we see with $q = s_\chi^{-1}(n)$ that

$$\tau_2\left(L_{b_0} R_{b_n} Z_1 C_1 \cdots Z_{n-1} C_{n-1} \Xi\right)$$
$$= \begin{cases} \frac{1}{2}\varphi(z_1^{p_1} z_2^{p_2} \cdots z_{n-1}^{p_{n-1}} \xi^{p_n}), & \text{if } n \text{ is even and } \left(i_0, i_{s_\chi(1)}, \ldots, i_{s_\chi(q-1)}, i_{s_\chi(q+1)}, \ldots, i_{s_\chi(n)}\right) = (1, 2, \ldots, 1, 2) \\ \frac{1}{2}\varphi(z_1^{q_1} z_2^{q_2} \cdots z_{n-1}^{q_{n-1}} \xi^{q_n}), & \text{if } n \text{ is even and } \left(i_0, i_{s_\chi(1)}, \ldots, i_{s_\chi(q-1)}, i_{s_\chi(q+1)}, \ldots, i_{s_\chi(n)}\right) = (2, 1, \ldots, 2, 1) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sum_{\substack{1 \le k < n \\ \chi(k) = \ell}} \tau_2\left(L_{b_0} R_{b_n}\left(\prod_{p \in V_k^c \setminus \{k,n\}} Z_p C_p\right) L_{\eta\left((F \circ E_2)\left(C_k \prod_{p \in V_k} Z_p C_p\right)\right)}\right)$$
$$= \begin{cases} \frac{1}{2} \sum\limits_{\substack{1 \le k < n \\ \chi(k) = \ell \\ |V_k| \text{ even}}} \varphi\left((z_1^{p_1}, \ldots, z_{n-1}^{p_{n-1}})|_{V_k^c \setminus \{k,n\}}\right) \varphi\left((z_1^{p_1}, \ldots, z_{n-1}^{p_{n-1}})|_{V_k}\right), \\ \qquad \text{if } n \text{ is even and } \left(i_0, i_{s_\chi(1)}, \ldots, i_{s_\chi(q-1)}, i_{s_\chi(q+1)}, \ldots, i_{s_\chi(n)}\right) = (1, 2, \ldots, 1, 2), \\ \frac{1}{2} \sum\limits_{\substack{1 \le k < n \\ \chi(k) = \ell \\ |V_k| \text{ even}}} \varphi\left((z_1^{q_1}, \ldots, z_{n-1}^{q_{n-1}})|_{V_k^c \setminus \{k,n\}}\right) \varphi\left((z_1^{p_{qn-1}}, \ldots, z_{n-1}^{p_{qn-1}})|_{V_k}\right), \\ \qquad \text{if } n \text{ is even and } \left(i_0, i_{s_\chi(1)}, \ldots, i_{s_\chi(q-1)}, i_{s_\chi(q+1)}, \ldots, i_{s_\chi(n)}\right) = (2, 1, \ldots, 2, 1), \\ 0, \qquad \text{otherwise,} \end{cases}$$

where $V_k = \{k < m < n \mid \chi(m) = \ell\}$ (note only the terms where $|V_k|$ is even survive from the $\eta \circ F \circ E_2$ expression due to the form of X) and z_k, p_k, and q_k are defined as usual in this proof. Hence, it suffices to show when n is even that

(9.7)
$$\varphi(z_1^{p_1} z_2^{p_2} \cdots z_{n-1}^{p_{n-1}} \xi^{p_n}) = \sum_{\substack{1 \le k < n \\ \chi(k) = \ell \\ |V_k| \text{ even}}} \varphi\left((z_1^{p_1}, \ldots, z_{n-1}^{p_{n-1}})|_{V_k^c \setminus \{k,n\}}\right) \varphi\left((z_1^{p_1}, \ldots, z_{n-1}^{p_{n-1}})|_{V_k}\right),$$

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(9.8)
$$\varphi(z_1^{q_1} z_2^{q_2} \cdots z_{n-1}^{q_{n-1}} \xi^{q_n}) = \sum_{\substack{1 \le k < n \\ \chi(k) = \ell \\ |V_k| \text{ even}}} \varphi\left((z_1^{q_1}, \ldots, z_{n-1}^{q_{n-1}})|_{V_k^c \setminus \{k,n\}}\right) \varphi\left((z_1^{q_1}, \ldots, z_{n-1}^{p_{q_{n-1}}})|_{V_k}\right).$$

Note that equations (9.7) and (9.8) are the same equation by the alternating adjoint flipping condition and equation (9.6). Moreover, due to the defining property of Ξ, we know with n even that

$$\begin{aligned}\tau_2(Z_1 \cdots Z_{n-1}\Xi) &= \sum_{\substack{1 \le k < n \\ \chi(k) = \ell}} \tau_2\left(\left(\prod_{p \in V_k^c \setminus \{k,n\}} Z_p\right) L_{\eta\left((F \circ E_2)\left(\prod_{p \in V_k} Z_p\right)\right)}\right) \\ &= \sum_{\substack{1 \le k < n \\ \chi(k) = \ell}} \tau_2\left(\left(\prod_{p \in V_k^c \setminus \{k,n\}} Z_p\right)\right) \tau_2\left(X^{|V_k|}\right).\end{aligned}$$

Due to the form of X and the alternating adjoint flipping condition, we immediately see that

$$\tau_2\left(X^{|V_k|}\right) = \begin{cases} 0, & \text{if } |V_k| \text{ is odd,} \\ \varphi\left((z_1^{p_1}, \ldots, z_{n-1}^{p_{n-1}})|_{V_k}\right), & \text{if } |V_k| \text{ is even,} \end{cases}$$

and, for n even and k such that $|V_k|$ is even, we have

$$\begin{aligned}\tau_2\left(\left(\prod_{p \in V_k^c \setminus \{k,n\}} Z_p\right)\right) &= \frac{1}{2}\left(\varphi\left((z_1^{p_1}, \ldots, z_{n-1}^{p_{n-1}})|_{V_k^c \setminus \{k,n\}}\right) + \varphi\left((z_1^{q_1}, \ldots, z_{n-1}^{q_{n-1}})|_{V_k^c \setminus \{k,n\}}\right)\right) \\ &= \varphi\left((z_1^{p_1}, \ldots, z_{n-1}^{p_{n-1}})|_{V_k^c \setminus \{k,n\}}\right),\end{aligned}$$

thereby completing the proof. ∎

Proof of Theorem 9.5 The proof follows immediately by combining Propositions 9.2, 9.7, and 9.8. ∎

10 Maximizing bi-free entropy

In this section, we will prove Theorem 10.2 by obtaining an upper bound for the bi-free entropy of a pair of operators and their adjoints based on the entropy of a pair of matrices and demonstrate when equality is obtained. In particular, this generalizes an essential result from [8, Section 5].

To begin, we must establish a formula for the bi-free entropy of non-self-adjoint operators.

Definition 10.1 Let $(\mathcal{A}, \varphi)$ be a C*-noncommutative probability space, and let $\{X_i, X_i^*\}_{i=1}^n \cup \{X_i'\}_{i=1}^{n'} \cup \{Y_j, Y_j^*\}_{j=1}^m \cup \{Y_j'\}_{j=1}^{m'} \subseteq \mathcal{A}$ where X_i' and Y_j' are self-adjoint for all i and j. The *bi-free entropy of* $(\{\mathbf{X}, \mathbf{X}^*, \mathbf{X}'\}, \{\mathbf{Y}, \mathbf{Y}^*, \mathbf{Y}'\})$ is defined to be

$$\chi^*(\mathbf{X}, \mathbf{X}^*, \mathbf{X}' \sqcup \mathbf{Y}, \mathbf{Y}^*, \mathbf{Y}') = \frac{2n + 2m + n' + m'}{2} \ln(2\pi e) + \frac{1}{2} \int_0^\infty \left(\frac{2n + 2m + n' + m'}{1 + t} - g(t)\right) dt,$$

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where

$$g(t) = \Phi^*\left(\mathbf{X} + \sqrt{t}\mathbf{C}_\ell, \mathbf{X}^* + \sqrt{t}\mathbf{C}_\ell^*, \mathbf{X}' + \sqrt{t}\mathbf{S} \sqcup \mathbf{Y} + \sqrt{t}\mathbf{C}_r, \mathbf{Y}^* + \sqrt{t}\mathbf{C}_r^*, \mathbf{Y}' + \sqrt{t}\mathbf{D}\right),$$

where $\mathbf{S}$ and $\mathbf{D}$ consist of semicircular variables of mean 0, variance 1, and covariance 0 and $\mathbf{C}_\ell$ and $\mathbf{C}_r$ consist of circular variables of mean 0, variance 1, and covariance 0 such that

$$\begin{aligned}(\{\mathbf{X}, \mathbf{X}^*, \mathbf{X}'\}, \{\mathbf{Y}, \mathbf{Y}^*, \mathbf{Y}'\}) \cup \{(S_i, 1)\}_{i=1}^{n'} \cup \{(1, D_j)\}_{j=1}^{m'} \\ \cup \{(\{C_{\ell,i}, C_{\ell,i}^*\}, 1)\}_{i=1}^{n} \cup \{(1, \{C_{r,j}, C_{r,j}^*\})\}_{j=1}^{m}\end{aligned}$$

are bi-free.

Remark 10.1 Given any C*-noncommutative probability space $(\mathcal{A}, \varphi)$, it is always possible to find a larger C*-noncommutative probability space that contains the necessary bi-free elements from Definition 10.1. Indeed, one need only consider the scalar reduced free product of the appropriate spaces and use Definition 2.4 to obtain bi-freeness. The fact that the state is positive follows as it will be a vector state.

In the simplest case, one may ask why we do not simply define

$$\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) = \chi^*(\{\Re(x), \Im(x)\} \sqcup \{\Re(y), \Im(y)\})$$

to trivially reduce to the self-adjoint case in a similar fashion to Remark 7.1(4) and why the integrand in Definition 10.1 is well defined. Both of these questions are answered via Remark 7.1(4) as

$$\begin{aligned}&\Phi^*\left(\left\{x + \sqrt{t}c_\ell, x^* + \sqrt{t}c_\ell^*\right\} \sqcup \left\{y + \sqrt{t}c_r, y^* + \sqrt{t}c_r^*\right\}\right) \\ &= \frac{1}{2}\Phi^*\left(\left\{\Re(x) + \sqrt{t}\Re(c_\ell), \Im(x) + \sqrt{t}\Im(c_\ell)\right\} \sqcup \left\{\Re(y) + \sqrt{t}\Re(c_r), \Im(y) + \sqrt{t}\Im(c_r)\right\}\right) \\ &= \frac{1}{2}\Phi^*\left(\left\{\Re(x) + \frac{\sqrt{t}}{\sqrt{2}}s_1, \Im(x) + \frac{\sqrt{t}}{\sqrt{2}}s_2\right\} \sqcup \left\{\Re(y) + \frac{\sqrt{t}}{\sqrt{2}}d_1, \Im(y) + \frac{\sqrt{t}}{\sqrt{2}}d_2\right\}\right) \\ &= \Phi^*\left(\left\{\sqrt{2}\Re(x) + \sqrt{t}s_1, \sqrt{2}\Im(x) + \sqrt{t}s_2\right\} \sqcup \left\{\sqrt{2}\Re(y) + \sqrt{t}d_1, \sqrt{2}\Im(y) + \sqrt{t}d_2\right\}\right),\end{aligned}$$

where s_1, s_2, d_1, and d_2 are as in Example 9.4. Hence, the integrand in Definition 10.1 is well defined with

$$\begin{aligned}\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) &= \chi^*\left(\left\{\sqrt{2}\Re(x), \sqrt{2}\Im(x)\right\} \sqcup \left\{\sqrt{2}\Re(y), \sqrt{2}\Im(y)\right\}\right) \\ &= \chi^*(\{\Re(x), \Im(x)\} \sqcup \{\Re(y), \Im(y)\}) + 4\ln(\sqrt{2}).\end{aligned}$$

We normalize Definition 10.1 so that the following holds and generalizes [8, Theorem 1.4] in the case $d = 1$.

Theorem 10.2 *Let $(\mathcal{A}, \varphi)$ be a C*-noncommutative probability space, and let $x, y \in \mathcal{A}$ be such that x^*x and xx^* have the same distribution with respect to φ and y^*y and yy^* have the same distribution with respect to φ. With X and Y as in Section 9,*

$$\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) \leq 2\chi^*(X \sqcup Y)$$

and equality holds whenever the pair (x, y) is bi-R-diagonal and alternating adjoint flipping.

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To prove Theorem 10.2, we need two technical lemmata. For the first, note that the following does not immediately follow from Remark 2.4 as being bi-free over $M_2(\mathbb{C})$ with respect to E_2 does not imply being bi-free with respect to τ_2.

Lemma 10.3 *Let $(\mathcal{A}, \varphi)$ be a C*-noncommutative probability space, let $x, y \in \mathcal{A}$ be such that x^*x and xx^* have the same distribution with respect to φ and y^*y and yy^* have the same distribution with respect to φ, and let (c_ℓ, c_r) be a bi-free circular pair in $\mathcal{A}$ with mean 0, variance 1, and covariance 0 such that*

$$(\{x, x^*\}, \{y, y^*\}) \cup \{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$$

are bi-free with respect to φ. Using the notation of Section 10, if

$$S_\ell = c_\ell \otimes E_{1,2} \otimes I_2 + c_\ell^* \otimes E_{2,1} \otimes I_2 \in A_2 \quad \text{and} \quad S_r = c_r \otimes I_2 \otimes E_{1,2} + c_r^* \otimes I_2 \otimes E_{2,1} \in A_2,$$

then S_ℓ and S_r have semicircular distributions with respect to τ_2 of mean 0 and variance 1 and

$$\{(X, Y)\} \cup \{(S_\ell, 1_{A_2})\} \cup \{(1_{A_2}, S_r)\}$$

are bi-free with respect to τ_2.

Proof As $\{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$ are bi-free with respect to φ by Example 9.4, Remark 2.4 implies that $(S_\ell, 1)$ and $(1, S_r)$ are bi-free with respect to E_2. Moreover, as c_ℓ and c_r commute, s_ℓ and s_r commute. Hence, we see for all $n, m \in \mathbb{N}$ that

$$\begin{aligned}\tau_2(s_\ell^n s_r^m) &= \mathrm{tr}_2(E_2(s_\ell^n s_r^m)) = \mathrm{tr}_2(E_2(s_\ell^n)E_2(s_r^m)) \\ &= \begin{cases} 0, & \text{if } n \text{ or } m \text{ is odd,} \\ \varphi\left((c_\ell^* c_\ell)^{\frac{n}{2}}\right)\varphi\left((c_r^* c_r)^{\frac{m}{2}}\right), & \text{if } n \text{ and } m \text{ are even,} \end{cases}\end{aligned}$$

by Example 9.4 and the alternating adjoint flipping condition. Therefore, as $c_\ell^* c_\ell$ and $c_r^* c_r$ are known to have the same distributions as the square of a semicircular element of mean 0 and variance 1 (see [17, Section 5.1]), we obtain that (s_ℓ, s_r) is the bi-free central limit distribution with mean 0, variance 1, and covariance 0 with respect to τ_2. Hence, $\{(S_\ell, 1_{A_2})\} \cup \{(1_{A_2}, S_r)\}$ are bi-free with respect to τ_2.

To complete the proof, it suffices to show that $\{(X, Y)\} \cup \{(S_\ell, S_r)\}$ are bi-free with respect to τ_2. Therefore, by [3], it suffices to show for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^n$, nonconstant $\gamma \in \{1, 2\}^n$, and $Z_k \in \{X, Y, S_\ell, S_r\}$, where

$$Z_k = \begin{cases} X, & \text{if } \chi(k) = \ell \text{ and } \gamma(k) = 1, \\ Y, & \text{if } \chi(k) = r \text{ and } \gamma(k) = 1, \\ S_\ell, & \text{if } \chi(k) = \ell \text{ and } \gamma(k) = 2, \\ S_r, & \text{if } \chi(k) = r \text{ and } \gamma(k) = 2, \end{cases}$$

that

$$\tau_2(Z_1 \cdots Z_n) = \sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^{\tau_2}(Z_1, \ldots, Z_n), \tag{10.1}$$

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where γ is representing the partition $\{\{k \mid \gamma(k) = 1\}, \{k \mid \gamma(k) = 2\}\}$. Note if $z_1, \ldots, z_n \in \{x, y, c_\ell, c_r\}$ are such that

$$z_k = \begin{cases} x, & \text{if } Z_k = X, \\ y, & \text{if } Z_k = Y, \\ c_\ell, & \text{if } Z_k = S_\ell, \\ c_r, & \text{if } Z_k = S_r, \end{cases}$$

then by Lemma 9.1 and the fact that $(\{x, x^*\}, \{y, y^*\}) \cup \{(c_\ell, 1)\} \cup \{(1, c_r)\}$ are bi-free with respect to φ, we have that

$$\begin{aligned} \tau_2(Z_1 \cdots Z_n) &= \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \frac{1}{2}\left(\varphi\left(z_1^{p_1} \cdots z_n^{p_n}\right) + \varphi\left(z_1^{q_1} \cdots z_n^{q_n}\right)\right), & \text{if } n \text{ is even}, \end{cases} \\ &= \begin{cases} 0, & \text{if } n \text{ is odd}, \\ \frac{1}{2}\sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^\varphi\left(z_1^{p_1}, \ldots, z_n^{p_n}\right) + \kappa_\pi^\varphi\left(z_1^{q_1}, \ldots, z_n^{q_n}\right), & \text{if } n \text{ is even}, \end{cases} \end{aligned}$$

where

$$p_{s_\chi(k)} = \begin{cases} 1, & \text{if } k \text{ is odd} \\ *, & \text{if } k \text{ is even} \end{cases} \quad \text{and} \quad q_{s_\chi(k)} = \begin{cases} *, & \text{if } k \text{ is odd}, \\ 1, & \text{if } k \text{ is even}. \end{cases}$$

To show that this agrees with the right-hand side of equation (10.1), we divide the discussion into several cases. To this end, let

$$I_{X,Y} = \{k \mid \gamma(k) = 1\} \quad \text{and} \quad I_S = \{k \mid \gamma(k) = 2\}.$$

First, suppose that n is odd. If $|I_S|$ is odd, then the right-hand side of equation (10.1) is zero as there must be a cumulant involving an odd number of S_ℓ and S_r and $\{(S_\ell, S_r)\}$ is a bi-free central limit distribution with 0 mean. Otherwise, $|I_{X,Y}|$ is odd. In this case, we may rearrange the sum on the right-hand side of equation (10.1) to add over all $\pi \in \mathrm{BNC}(\chi)$ with $\pi \leq \gamma$ that form the same partition when restricted to I_S. Since summing over such partitions yields a product of moment terms in the X's and Y's where the sum of the lengths of the moments is $|I_{X,Y}|$ and since all odd moment terms involving only X's and Y's is zero by Lemma 9.1, this portion of the sum yields zero. Hence, equation (10.1) holds when n is odd.

In the case n is even, note if $|I_S|$ is odd, then the right-hand side of equation (10.1) is still zero. However,

$$\frac{1}{2}\sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^\varphi\left(z_1^{p_1}, \ldots, z_n^{p_n}\right) + \kappa_\pi^\varphi\left(z_1^{q_1}, \ldots, z_n^{q_n}\right) = 0$$

as there must be a cumulant involving an odd number of $(\{c_\ell, c_\ell^*\}, \{c_r, c_r^*\})$ and (c_ℓ, c_r) is a bi-free circular pair. Thus, we may assume that n, $|I_S|$, and $|I_{X,Y}|$ are even.

Under these assumptions, we claim that

$$\frac{1}{2}\sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^\varphi\left(z_1^{p_1}, \ldots, z_n^{p_n}\right) + \kappa_\pi^\varphi\left(z_1^{q_1}, \ldots, z_n^{q_n}\right) = \sum_{\substack{\pi \in \mathrm{BNC}(\chi) \\ \pi \leq \gamma}} \kappa_\pi^{\tau_2}(Z_1, \ldots, Z_n).$$

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To see this, again we need only consider $\pi \in \mathrm{BNC}(\chi)$ that form pair partitions when restricted to I_S and no block of π contains both an element of $\{k \mid \chi(k) = \ell\}$ and of $\{k \mid \chi(k) = r\}$, since $\{(S_\ell, 1)\} \cup \{(1, S_r)\}$ are bi-free with respect to τ_2 and $\{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$ are bi-free with respect to φ. For such a partition π, if we let $\widehat{\pi}$ be the largest partition on $I_{X,Y}$ such that $\widehat{\pi} \cup \pi|_{I_S}$ is an element of $\mathrm{BNC}(\chi)$, then by adding over all $\sigma \in \mathrm{BNC}(\chi)$ with $\sigma \leq \gamma$ and $\sigma|_{I_S} = \pi|_{I_S}$, it suffices to show that

$$\frac{1}{2}\left(\varphi_{\widehat{\pi}}\left((z_1^{p_1}, \ldots, z_n^{p_n})|_{I_{X,Y}}\right)\kappa^{\varphi}_{\pi|_{I_S}}\left((z_1^{p_1}, \ldots, z_n^{p_n})|_{I_S}\right) + \varphi_{\widehat{\pi}}\left((z_1^{q_1}, \ldots, z_n^{q_n})|_{I_{X,Y}}\right)\kappa^{\varphi}_{\pi|_{I_S}}\left((z_1^{q_1}, \ldots, z_n^{q_n})|_{I_S}\right)\right) \tag{10.2}$$
$$= (\tau_2)_{\widehat{\pi}}\left((Z_1, \ldots, Z_n)|_{I_{X,Y}}\right).$$

Note that

$$\kappa^{\varphi}_{\pi|_{I_S}}\left(\left(z_1^{p_1}, \ldots, z_n^{p_n}\right)|_{I_S}\right) = 0 \quad \text{or} \quad \kappa^{\varphi}_{\pi|_{I_S}}\left(\left(z_1^{q_1}, \ldots, z_n^{q_n}\right)|_{I_S}\right) = 0$$

if and only if π has a block with two $*$-terms or two non-$*$-terms, as $\pi|_{I_S}$ is a pair partition and (c_ℓ, c_r) is a bi-circular pair. In this case, we would have that $\widehat{\pi}$ has a block of odd length, and thus the right-hand side of equation (10.2) is also zero, as any odd τ_2-moment involving X and Y is zero. Otherwise, both φ-cumulants are 1 and this forces every block of $\widehat{\pi}$ to be of even length and alternate between 1 and $*$ in the χ-ordering. Since

$$\varphi((x^*x)^m) = \varphi((xx^*)^m) = \tau_2(X^{2m}) \quad \text{and} \quad \varphi((y^*y)^m) = \varphi((yy^*)^m) = \tau_2(Y^{2m})$$

and since (by the assumption that π does not contain a block containing elements of $\{k \mid \chi(k) = \ell\}$ and of $\{k \mid \chi(k) = r\}$) there is a single block of $\widehat{\pi}$ containing elements of $\{k \mid \chi(k) = \ell\}$ and $\{k \mid \chi(k) = r\}$, adding the two φ-terms together produces exactly the τ_2 term in equation (10.2). ∎

Lemma 10.4 *Let $(\mathcal{A}, \varphi)$ be a C^*-noncommutative probability space, let $x, y \in \mathcal{A}$, and let (c_ℓ, c_r) be a bi-free circular pair in $\mathcal{A}$ with mean 0, variance 1, and covariance 0 such that*

$$(\{x, x^*\}, \{y, y^*\}) \cup \{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$$

are bi-free with respect to φ. Then:

1. *If (x, y) is bi-R-diagonal, then $\left(x + \sqrt{t}c_\ell, y + \sqrt{t}c_r\right)$ is bi-R-diagonal for all $t \in (0, \infty)$.*
2. *If (x, y) is alternating adjoint flipping, then $\left(x + \sqrt{t}c_\ell, y + \sqrt{t}c_r\right)$ is alternating adjoint flipping for all $t \in (0, \infty)$.*
3. *If x^*x and xx^* (rsp.e y^*y and yy^*) have the same distribution with respect to φ, then $(x + \sqrt{t}c_\ell)^*(x + \sqrt{t}c_\ell)$ and $(x + \sqrt{t}c_\ell)(x + \sqrt{t}c_\ell)^*$ (resp. $(y + \sqrt{t}c_r)^*(y + \sqrt{t}c_r)$ and $(y + \sqrt{t}c_r)(y + \sqrt{t}c_r)^*$) have the same distribution with respect to φ.*

Proof As (c_ℓ, c_r) is bi-R-diagonal by Example 9.4 and as sums and scalar multiples of bi-R-diagonal pairs are bi-R-diagonal by [7, Proposition 3.1], (i) follows.

To see that (ii) holds, first, we claim that for all $n \in \mathbb{N}$, $\chi \in \{\ell, r\}^{2n}$, and $z_1, \ldots, z_n \in \{x, y, c_\ell, c_r\}$ such that

$$z_k \in \begin{cases} \{x, c_\ell\}, & \text{if } \chi(k) = \ell, \\ \{y, c_r\}, & \text{if } \chi(k) = r, \end{cases}$$

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we have that

$$\varphi(z_1^{p_1}\cdots z_{2n}^{p_{2n}}) = \varphi(z_1^{q_1}\cdots z_{2n}^{q_{2n}}),$$

where

$$p_{s_\chi(k)} = \begin{cases} 1, & \text{if } k \text{ is odd} \\ *, & \text{if } k \text{ is even} \end{cases} \qquad \text{and} \qquad q_{s_\chi(k)} = \begin{cases} *, , & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

Recall that

$$\varphi(z_1^{p_1}\cdots z_{2n}^{p_{2n}}) = \sum_{\pi\in \mathrm{BNC}(\chi)} \kappa_\pi(z_1^{p_1},\ldots,z_{2n}^{p_{2n}})$$

and the bi-free cumulant is zero if any block of π contains both an element of $\{x, x^*, y, y^*\}$ and an element of $\{c_\ell, c_\ell^*, c_r, c_r^*\}$. As the only cumulants involving $c_\ell, c_\ell^*, c_r, c_r^*$ with non-zero values are

$$\kappa_{1_{(\ell,\ell)}}(c_\ell, c_\ell^*) = 1 = \kappa_{1_{(\ell,\ell)}}(c_\ell^*, c_\ell) \qquad \text{and} \qquad \kappa_{1_{(r,r)}}(c_r, c_r^*) = 1 = \kappa_{1_{(r,r)}}(c_r^*, c_r),$$

for any fixed $\pi \in \mathrm{BNC}(\chi)$ for which the blocks containing $\{c_\ell, c_\ell^*, c_r, c_r^*\}$ do not cause the bi-free cumulant to be zero, we may add over all elements of $\mathrm{BNC}(\chi)$ with the same blocks as π for those indices corresponding to elements of $\{c_\ell, c_\ell^*, c_r, c_r^*\}$ to obtain a product of moments involving $\{x, x^*, y, y^*\}$, each of which is of even length and alternates between 1 and $*$ in the χ-ordering. We may then use the alternating adjoint flipping condition on (x, y) to exchange the powers and reverse this cumulant reduction process to obtain $\varphi(z_1^{q_1}\cdots z_{2n}^{q_{2n}})$, thereby completing the claim. Thus, (ii) then follows by linearity.

To see that (iii) holds, we desire to show that

$$\varphi\left(\left((x+\sqrt{t}c_\ell)^*(x+\sqrt{t}c_\ell)\right)^n\right) = \varphi\left(\left((x+\sqrt{t}c_\ell)(x+\sqrt{t}c_\ell)^*\right)^n\right)$$

for all $n \in \mathbb{N}$. To see how the left-hand side can be changed into the right-hand side, arguments similar to the proof of Lemma 10.3 are used. First, we expand out the product and expand the moment using linearity. Then, for each moment term, we expand via the free cumulants and use the fact that mixed free cumulants vanish. Cumulants involving an odd number of c_ℓ and c_ℓ^* vanish, and thus we can consider only pair partitions when restricted to entries involving c_ℓ and c_ℓ^*. Any cumulant involving just c_ℓ or just c_ℓ^* vanishes and can be ignored. By adding over all partitions with the same blocks on c_ℓ and c_ℓ^* that do not vanish yields a product of moment terms of the form $\varphi((x^*x)^m)$ and $\varphi((xx^*)^m)$. For any such terms, viewing the $(2n)$th term as the first term does not change the value, as the distributions of x^*x and xx^* are the same, thereby effectively moving the x or c_ℓ term at the end to the beginning. One then reverses the above process and obtains the right-hand side as desired. ∎

Proof of Theorem 10.2. As per Remark 10.1, we may assume without loss of generality that there exists a bi-free circular pair (c_ℓ, c_r) (with mean 0, variance 1, and covariance 0) in $\mathcal{A}$ such that

$$\{(\{x, x^*\}, \{y, y^*\})\} \cup \{(\{c_\ell, c_\ell^*\}, 1)\} \cup \{(1, \{c_r, c_r^*\})\}$$

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are bi-free. Therefore, as $\{(X, Y)\} \cup \{(S_\ell, S_r)\}$ are bi-free with respect to τ_2 by Lemma 10.3, we obtain that

$$\chi^*(X \sqcup Y) = \ln(2\pi e) + \frac{1}{2}\int_0^\infty \left(\frac{2}{1+t} - \Phi^*\left(X + \sqrt{t}S_\ell \sqcup Y + \sqrt{t}S_r\right)\right) dt.$$

However, as

$$X + \sqrt{t}S_\ell = (x + \sqrt{t}c_\ell) \otimes E_{1,2} \otimes I_2 + (x + \sqrt{t}c_\ell)^* \otimes E_{2,1} \otimes I_2,$$
$$Y + \sqrt{t}S_r = (y + \sqrt{t}c_r) \otimes I_2 \otimes E_{1,2} + (y + \sqrt{t}c_r)^* \otimes I_2 \otimes E_{2,1}$$

and as Lemma 10.4(iii) shows that the assumptions of Theorem 9.5 as satisfied, we obtain that

(10.3)
$$\Phi^*\left(\{x + \sqrt{t}c_\ell, (x + \sqrt{t}c_\ell)^*\} \sqcup \{y + \sqrt{t}c_r, (y + \sqrt{t}c_r)^*\}\right) \geq 2\Phi^*\left(X + \sqrt{t}S_\ell \sqcup Y + \sqrt{t}S_r\right)$$

for all $t \in (0, \infty)$. Hence, the inequality

$$\chi^*(\{x, x^*\} \sqcup \{y, y^*\}) \leq 2\chi^*(X \sqcup Y)$$

follows by comparing the above bi-free entropy formula with that from Definition 10.1.

In the case that (x, y) is bi-R-diagonal and alternating adjoint flipping, Lemma 10.4 implies that $(x + \sqrt{t}c_\ell, y + \sqrt{t}c_r)$ is bi-R-diagonal and alternating adjoint flipping for all $t \in (0, \infty)$; thus; equality holds in equation (10.3) by Theorem 9.5. ∎

11 Other results

In this section, we will examine other results from [8] that generalize to the bi-free setting. As these results are less connected to bi-free entropy with respect to a completely positive map and have proofs that can be adapted from [8] using the same modifications from Sections 9 and 10 to deal with the χ-ordering, we simply state these results.

Theorem 11.1 (Generalization of [8, Theorem 1.2]) *Let $(\mathcal{A}, \varphi)$ be a C^*-noncommutative probability space, let $d \in \mathbb{N}$, and let $(A_d, E_d, \varepsilon, \tau_d)$ be as in Section 9. Then:*

(1) *For all $\{x_{i,j}\}_{i,j=1}^d, \{y_{i,j}\}_{i,j=1}^d \subseteq \mathcal{A}$, if*

$$X = \sum_{i,j=1}^d x_{i,j} \otimes E_{i,j} \otimes I_d \qquad \text{and} \qquad Y = \sum_{i,j=1}^d y_{i,j} \otimes I_d \otimes E_{i,j},$$

then

$$\Phi^*\left(\{x_{i,j}, x_{i,j}^*\}_{i,j=1}^d \sqcup \{y_{i,j}, y_{i,j}^*\}_{i,j=1}^d\right) \geq d^3\Phi^*(X, X^* \sqcup Y, Y^*).$$

Moreover, equality holds if $(\{X, X^*\}, \{Y, Y^*\})$ is bi-free from $(M_d(\mathbb{C})_\ell, M_d(\mathbb{C})_r)$ with respect to τ_d.

(2) *If in (i) X and Y are self-adjoint, then*

$$\Phi^*\left(\{x_{i,j}\}_{i,j=1}^d \sqcup \{y_{i,j}\}_{i,j=1}^d\right) \geq d^3\Phi^*(X \sqcup Y)$$

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with equality holding if $(\{X\},\{Y\})$ *is bi-free from* $(M_d(\mathbb{C})_\ell, M_d(\mathbb{C})_r)$ *with respect to* τ_d.

Theorem 11.2 (Generalization of [8, Theorem 1.5]) *Let* $(\mathcal{A},\varphi)$ *be a* C^**-noncommutative probability space, let* $d \in \mathbb{N}$*, and let* $(A_d, E_d, \varepsilon, \tau_d)$ *be as in Section 9. Then:*

(1) *For all* $\{x_{i,j}\}_{i,j=1}^d, \{y_{i,j}\}_{i,j=1}^d \subseteq \mathcal{A}$*, if*

$$X = \sum_{i,j=1}^d x_{i,j} \otimes E_{i,j} \otimes I_d \qquad \text{and} \qquad Y = \sum_{i,j=1}^d y_{i,j} \otimes I_d \otimes E_{i,j},$$

then

$$\chi^*\left(\{x_{i,j}, x_{i,j}^*\}_{i,j=1}^d \sqcup \{y_{i,j}, y_{i,j}^*\}_{i,j=1}^d\right) \leq d^2\left(\chi^*(X, X^* \sqcup Y, Y^*) - 2\ln(d)\right).$$

Moreover, equality holds if $(\{X, X^*\},\{Y, Y^*\})$ *is bi-free from* $(M_d(\mathbb{C})_\ell, M_d(\mathbb{C})_r)$ *with respect to* τ_d.

(2) *If in (i)* X *and* Y *are self-adjoint, then*

$$\chi^*\left(\{x_{i,j}\}_{i,j=1}^d \sqcup \{y_{i,j}\}_{i,j=1}^d\right) \leq d^2\left(\chi^*(X \sqcup Y) - \ln(d)\right)$$

with equality holding if $(\{X\},\{Y\})$ *is bi-free from* $(M_d(\mathbb{C})_\ell, M_d(\mathbb{C})_r)$ *with respect to* τ_d.

To prove Theorem 11.2, we note that it is essential to prove the following.

Lemma 11.3 (Generalization of [8, Proposition 5.3]) *Let* $(\mathcal{A},\varphi)$ *be a* C^**-noncommutative probability space, let* D_ℓ, D_r *be* $*$*-subalgebras of* A*, let* $d \in \mathbb{N}$*, and let* $(A_d, E_d, \varepsilon, \tau_d)$ *be as in Section 9.*

(1) *If* $\{c_{\ell,i,j}\}_{i,j=1}^d$ *and* $\{c_{r,i,j}\}_{i,j=1}^d$ *are circular elements of* $\mathcal{A}$ *with mean 0 and variance 1 such that*

$$\{(D_\ell, D_r)\} \cup \{(\{c_{\ell,i,j}, c_{\ell,i,j}^*\}, 1)\}_{i,j=1}^d \cup \{(1, \{c_{r,i,j}, c_{r,i,j}^*)\}_{i,j=1}^d$$

are bi-free, then

$$C_\ell = \sum_{i,j=1}^d c_{\ell,i,j} \otimes E_{i,j} \otimes I_d \quad \text{and} \quad C_r = \sum_{i,j=1}^d c_{r,i,j} \otimes I_d \otimes E_{i,j}$$

are circular elements of mean 0 and variance d such that $(\{C_\ell, C_\ell^*\}, 1)$, $(1, \{C_r, C_r^*\})$, *and* $(D_\ell \otimes M_d(\mathbb{C}) \otimes I_d, D_r \otimes I_d \otimes M_d(\mathbb{C}))$ *are bi-free with respect to* τ_d.

(2) *If* $\{s_{\ell,i,j}\}_{i,j=1}^d$ *and* $\{s_{r,i,j}\}_{i,j=1}^d$ *are elements of* $\mathcal{A}$ *with mean 0 and variance 1 such that* $s_{\ell,i,i}$ *and* $s_{r,i,i}$ *are semicircular elements for all* i, $s_{\ell,i,j}$ *and* $s_{r,i,j}$ *are circular elements for all* i, j, $s_{\ell,i,j}^* = s_{\ell,j,i}$ *and* $s_{r,i,j}^* = s_{r,j,i}$ *for all* i, j, *and*

$$\{(D_\ell, D_r)\} \cup \{(s_{\ell,i,i}, 1)\}_{i=1}^d \cup \{(1, s_{r,i,i})\}_{i=1}^d \cup \{(\{s_{\ell,i,j}, s_{\ell,i,j}^*\}, 1)\}_{1\leq i<j\leq d} \cup \{(1, \{s_{r,i,j}, ss_{r,i,j}^*)\}_{1\leq i<j\leq d}$$

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are bi-free, then

$$S_\ell = \sum_{i,j=1}^{d} s_{\ell,i,j} \otimes E_{i,j} \otimes I_d \quad \text{and} \quad S_r = \sum_{i,j=1}^{d} s_{r,i,j} \otimes I_d \otimes E_{i,j}$$

are semicircular elements of mean 0 and variance d such that $(S_\ell, 1)$, $(1, S_r)$, *and* $(D_\ell \otimes M_d(\mathbb{C}) \otimes I_d, D_r \otimes I_d \otimes M_d(\mathbb{C}))$ *are bi-free with respect to* τ_d.

Note that the proof of Lemma 11.3 is obtained by first generalizing [8, Lemma 5.4], which clearly holds due to the bi-free cumulant characterization of the conjugate variables. The proof then proceeds via constructing the appropriate conjugate variables for either (C_ℓ, C_r) or (S_ℓ, S_r) using the techniques from Section 9.

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