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On the Local Lifting Properties of Operator Spaces

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Abstract. In this paper, we mainly study operator spaces which have the locally lifting property (LLP). The dual of any ternary ring of operators is shown to satisfy the strongly local reflexivity, and this is used to prove that strongly local reflexivity holds also for operator spaces which have the LLP. Several homological characterizations of the LLP and weak expectation property are given. We also prove that for any operator space V, V^{**} has the LLP if and only if V has the LLP and V^* is exact.

1 Introduction

If one wishes to prove that an operator space or a C^*-algebra has an approximate property, one begins by proving that an appropriate model (such as the second dual) has the corresponding exact property. One must then relate the exact property in the model to the approximate property in the original space. In the theory of operator spaces, this is often accomplished by using the principle of local reflexivity. The local reflexivity of operator spaces was introduced in [4], and was further studied in [5–8, 14]. All exact operator spaces are locally reflexive [11, Corollary 4.8]. In particular, all exact C^*-algebras are locally reflexive (also see [16]). On the other hand, it was shown in [4] that some C^*-algebras are not locally reflexive. In light of the fact that C^*-algebras need not be locally reflexive, it was thought the same would be true for their dual operator spaces. It therefore came as quite a surprise to find in [10] that all such dual spaces, as well as von Neumann algebraic preduals, are locally reflexive. This is one of the most surprising results in the theory of operator spaces. What is even more surprising is that these operator spaces are locally reflexive in the strong sense that is called strong local reflexivity, *i.e.,* we can assume that the approximations are close in the sense of the Pisier–Banach–Mazur distance for operator spaces. Ternary rings of operators (TROs) form a very interesting class of operator spaces. In many cases, TROs come out more naturally than C^*-algebras in the theory of operator spaces. It is natural to consider the above remarkable result for the case of TROs. In Section 2, we generalize the above result to TROs and show that if V is a TRO, then V^* is strongly locally reflexive. This is used to prove that if V is an operator space which has the LLP, then V is strongly locally reflexive. In Section 3, several homological characterizations of the LLP and weak expectation property (WEP) are given. These results are very similar to the homological characterization of nuclearity

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(see [9, Theorem 14.6.1]). This part is closely related to Kirchberg's Conjecture about QWEP: every separable C^*-algebra is a quotient of a C^*-algebra with the WEP, since Kirchberg's Conjecture about QWEP is equivalent to LLP $\Rightarrow$ WEP. In the last section, we discuss the relationship between the LLP of V, V^{**}. We show that V^{**} has the LLP if and only if V has the LLP and V^* is exact. The exactness of V^* is essential in this result.

The theory of operator spaces is a recently arising area in modern analysis, which is a natural non-commutative quantization of Banach space theory. For the convenience of the readers, we recall some of basic notations and terminology in operator spaces, the details can be found in [9, 19]. Given a Hilbert space $\mathcal{H}$, we let $\mathcal{B}(\mathcal{H})$ denote the space of all bounded linear operators on $\mathcal{H}$. For each natural number $n \in \mathbf{N}$, there is a canonical norm $\|\cdot\|_n$ on the $n \times n$ matrix space $M_n(\mathcal{B}(\mathcal{H}))$ given by identifying $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^n)$. We call this family of norms $\{\|\cdot\|_n\}$ an operator space matrix norm on $\mathcal{B}(\mathcal{H})$. An operator space V is a norm closed subspace of some $\mathcal{B}(\mathcal{H})$ equipped with the distinguished operator space matrix norm inherited from $\mathcal{B}(\mathcal{H})$. An abstract matrix norm characterization of operator spaces was given in [20]. The morphisms in the category of operator spaces are the completely bounded linear maps. Given operator spaces V and W, a linear map $\varphi\colon V \to W$ is completely bounded if the corresponding linear mappings $\varphi_n\colon M_n(V) \to M_n(W)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ are uniformly bounded, *i.e.,*

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbf{N}\} < \infty.$$

A map φ is completely contractive (respectively, completely isometric, a complete quotient) if $\|\varphi\|_{cb} \leq 1$ (respectively, for each $n \in \mathbf{N}, \varphi_n$ is an isometry, a quotient map). We denote by $CB(V, W)$ the space of all completely bounded maps from V into W. It is known that $CB(V, W)$ is an operator space with the operator space matrix norm given by identifying $M_n(CB(V, W)) = CB(V, M_n(W))$. In particular, if V is an operator space, then its dual space V^* is an operator space with operator space matrix norm given by the identification $M_n(V^*) = CB(V, M_n)$. Given operator spaces V and W, and a completely bounded mapping $\varphi\colon V \to W$, the corresponding adjoint mapping $\varphi^*\colon W^* \to V^*$ is completely bounded with $\|\varphi^*\|_{cb} = \|\varphi\|_{cb}$. Furthermore, $\varphi\colon V \to W$ is a completely isometric injection if and only if φ^* is a complete quotient mapping. On the other hand, if $\varphi\colon V \to W$ is a surjection, then φ is a complete quotient mapping if and only if φ^* is a completely isometric injection. We use the notations

$$V \check{\otimes} W, \quad V \hat{\otimes} W, \quad \text{and} \quad V \overset{h}{\otimes} W$$

for the injective, projective and Haagerup operator space tensor products (see [1,3]). The operator space tensor products share many of the properties of the Banach space analogues. For example, we have the natural complete isometries

$$(V \hat{\otimes} W)^* = CB(V, W^*), \qquad (V \hat{\otimes} W)^* = CB(W, V^*)$$

and the completely isometric injection $V^* \check{\otimes} W \hookrightarrow CB(V, W)$. The tensor product $\check{\otimes}$ is injective in the sense that if $\varphi\colon W \to Y$ is a completely isometric injection, then so is

$$\mathrm{id}_V \otimes \varphi\colon V \check{\otimes} W \to V \check{\otimes} Y.$$

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On the other hand, the tensor product $\hat{\otimes}$ is projective in the sense that if $\varphi\colon W \to Y$ is a complete quotient mapping, then so is

$$\mathrm{id}_V \otimes \varphi\colon V\hat{\otimes}W \to V\hat{\otimes}Y.$$

The Haagerup tensor product $\overset{h}{\otimes}$ satisfies the surprising property that it is both injective and projective.

A TRO between Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$ is a norm closed subspace V of $\mathcal{B}(\mathcal{K},\mathcal{H})$, which is closed under the triple product

$$(x, y, z) \in V \times V \times V \to xy^*z \in V.$$

A TRO $V \subseteq \mathcal{B}(\mathcal{K},\mathcal{H})$ is called a W^*-TRO if it is w^*-closed (equivalently, weak operator closed, or strong operator closed) in $\mathcal{B}(\mathcal{K},\mathcal{H})$. TROs were first introduced by Hestenes [13], and have been intensively studied by Harris [12], Zettl [22], Effros–Ozawa–Ruan [11] and Kaur–Ruan[15]. In general, a TRO can be identified with the off-diagonal corner of its linking C^*-algebra

$$A(V) = \begin{bmatrix} C(V) & V \\ V^* & D(V) \end{bmatrix},$$

where $C(V)$ and $D(V)$ are C^*-algebras generated by VV^* and V^*V. If we let $M(C)$ and $M(D)$ denote the multiplier C^*-algebras of $C(V)$ and $D(V)$, respectively, then we may identify V with the off-diagonal corner of the unital C^*-algebra

$$R(V) = \begin{bmatrix} M(C) & V \\ V^* & M(D) \end{bmatrix}.$$

If V is a W^*-TRO, then it is known from [15, Proposition 2.3] that $R(V)$ is a von Neumann algebra. In this case, we call $R(V)$ the linking von Neumann algebra of V. Without loss of generality, we may always assume that a TRO is non-degenerately represented on Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$, *i.e.*, $V\mathcal{K}$ is norm dense in $\mathcal{H}$ and $V^*\mathcal{H}$ is norm dense in $\mathcal{K}$. More details about TROs may be found in [11, 15].

2 Strong Local Reflexivity

Given operator spaces V and W, we say that a completely bounded mapping $\varphi\colon V^* \to W$ satisfies the weak* approximation property (W*AP) if there exists a net of finite rank weak*-continuous mappings $\varphi_\alpha\colon V^* \to W$ with $\|\varphi_\alpha\|_{cb} \le \|\varphi\|_{cb}$ that converges to φ in the point-norm topology.

Lemma 2.1 *Suppose that V, W are operator spaces and every complete contraction from V^* to W satisfies the W*AP. If $G \subseteq V$ is completely complemented in V, then every complete contraction from G^* to W satisfies the W*AP.*

Proof Let $P\colon V \to G$ be a completely contractive projection, and $\iota\colon G \hookrightarrow V$ the completely isometric inclusion. We have $P \circ \iota = \mathrm{id}_G$ and $\iota^* \circ P^* = \mathrm{id}_{G^*}$. For any complete contraction $\varphi\colon G^* \to W$, $\varphi \circ \iota^*$ is a complete contraction from V^* to W. It follows from the assumption that there exists a net of finite rank weak*-continuous complete contractions $\varphi_\alpha\colon V^* \to W$ which converges to $\varphi \circ \iota^*$ in the point-norm topology. Set $\psi_\alpha = \varphi_\alpha \circ P^*$. It is apparent that ψ_α is a net of finite rank weak*-continuous complete contractions from G^* to W. For any $g^* \in G^*$, we have

$$\psi_\alpha(g^*) = \varphi_\alpha(P^*(g^*)) \xrightarrow{\|\cdot\|} \varphi \circ \iota^*(P^*(g^*)) = \varphi(g^*).$$

This shows that $\{\psi_\alpha\}$ converges to φ in the point-norm topology and φ satisfies the W*AP. ■

Theorem 2.2 *Given non-degenerately TROs $V \subseteq \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $W \subseteq \mathcal{B}(\mathcal{K}', \mathcal{H}')$, then every complete contraction $\varphi\colon V^* \to W$ satisfies the W*AP.*

Proof The corresponding linking C^*-algebras

$$A(V) = \begin{bmatrix} C(V) & V \\ V^* & D(V) \end{bmatrix}, \qquad A(W) = \begin{bmatrix} C(W) & W \\ W^* & D(W) \end{bmatrix}.$$

Since V is non-degenerately represented on Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$, it is easy to see that the induced C^*-algebras $C(V)$ and $D(V)$ are non-degenerately represented on $\mathcal{H}$ and $\mathcal{K}$ respectively. Thus the identity operators $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ are contained in the multiplier C^*-algebras $M(C(V))$ and $M(D(V))$ of $C(V)$ and $D(V)$ respectively. If we let

$$e = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e^\perp = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{K}} \end{bmatrix},$$

then we may write $V = eA(V)e^\perp$. Similarly, if we let

$$f = \begin{bmatrix} I_{\mathcal{H}'} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad f^\perp = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{K}'} \end{bmatrix},$$

then we may write $W = fA(W)f^\perp$. Define $P\colon A(V) \to V = eA(V)e^\perp$ by $P(a) = eae^\perp$, for any $a \in A(V)$. Certainly P is a completely contractive projection from $A(V)$ onto V. This means that V is completely complemented in $A(V)$. The Junge Approximation Theorem(see [9, Theorem 15.1.1]) shows that every complete contraction from $A(V)^*$ into $A(W)$ satisfies the W*AP. It follows from Lemma 2.1 that every complete contraction $\varphi\colon V^* \to W \hookrightarrow A(W)$ can be approximated by a net of finite rank weak*-continuous complete contractions $\varphi_\alpha\colon V^* \to (W)$ in the point-norm topology. Set $\psi_\alpha(v^*) = f\varphi_\alpha(v^*)f^\perp$, for all $v^* \in V^*$. Thus $\{\psi_\alpha\}$ is a net of finite rank weak*-continuous complete contractions from V^* into W. Since $\varphi_\alpha(v^*) \xrightarrow{\|\cdot\|} \varphi(v^*)$ for any $v^* \in V^*$,

$$\psi_\alpha(v^*) = f\varphi_\alpha(v^*)f^\perp \xrightarrow{\|\cdot\|} f\varphi(v^*)f^\perp = \varphi(v^*).$$

This implies that any complete contraction $\varphi\colon V^* \to W$ satisfies the W*AP. ■

Definition 2.3 Given operator spaces V and W, we say that a completely bounded mapping $\varphi\colon V^* \to W^{**}$ satisfies the weak* reflexive property (or simply, W*RP) if there exists a net finite rank weak*-continuous mapping $\varphi_\alpha\colon V^* \to W$ with $\|\varphi_\alpha\|_{cb} \le \|\varphi\|_{cb}$ that converges to φ in the point-weak* topology.

It is obvious that W*RP implies W*AP.

Lemma 2.4 *Suppose that V, W are operator spaces and every complete contraction from V^* into W^{**} satisfies the W*RP. If Y is completely complemented in W, then every complete contraction from V^* into Y^{**} also satisfies the W*RP.*

Proof Let $P\colon W \to Y$ be a completely contractive projection and $\iota\colon Y \hookrightarrow W$ the completely isometric inclusion. We have $P \circ \iota = \mathrm{id}_Y$ and $\iota^* \circ P^* = \mathrm{id}_{Y^*}$. For any complete contraction $\varphi\colon V^* \to Y^{**}$, it follows from the assumption that the complete contraction $\iota^{**} \circ \varphi\colon V^* \to W^{**}$ can be approximated by a net finite rank weak*-continuous complete contractions $\varphi_\alpha\colon V^* \to W$ in the point-weak* topology. Define $\psi_\alpha\colon V^* \to Y$ by $\psi_\alpha(v^*) = P \circ \varphi_\alpha(v^*)$ for $v^* \in V^*$. Certainly $\{\psi_\alpha\}$ is a net of finite rank weak*-continuous complete contractions from V^* into Y. For any $v^* \in V^*$ and $y^* \in Y^*$, we have

$$\begin{aligned}
\langle \psi_\alpha(v^*), y^* \rangle &= \langle P \circ \varphi_\alpha(v^*), y^* \rangle \\
&= \langle \varphi_\alpha(v^*), P^*(y^*) \rangle \\
&\to \langle \iota^{**} \circ \varphi(v^*), P^*(y^*) \rangle \\
&= \langle \varphi(v^*), \iota^* \circ P^*(y^*) \rangle \\
&= \langle \varphi(v^*), y^* \rangle.
\end{aligned}$$

This shows that $\{\psi_\alpha\}$ converges to φ in the point-weak* topology and φ satisfies the W*RP. ■

The exposition and structure in the rest of this section follows closely those in [10]. The relationship between the completely nuclear, completely integral, and exactly integral mappings introduced in [6, 7, 14] play a fundamental role in this section. More details about them can be found in [10].

Proposition 2.5 *If W is a TRO and V is an operator space, then we have the isometric identification $\mathcal{I}^{\mathrm{ex}}(V, W) = \mathcal{I}(V, W)$.*

Proof Certainly we have $\iota^{\mathrm{ex}}(\varphi) \le \iota(\varphi)$, so it suffices to show $\iota(\varphi) \le \iota^{\mathrm{ex}}(\varphi)$. Let us assume that $\iota^{\mathrm{ex}}(\varphi) \le 1$. Then it follows from the definition of ι^{ex} that we can find a factorization

$$\begin{tikzcd}
\mathcal{B}(\mathcal{H}) \arrow[r, "M(\omega)"] & \mathcal{B}(\mathcal{K})^* \arrow[dr, "s"] & \\
V \arrow[u, "r"] \arrow[r, "\phi"] & W \arrow[r, hook, "\iota_W"] & W^{**}
\end{tikzcd}$$

where r, s are complete contractions and $\omega \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})^*$ is of norm one. Since W is a TRO, W is completely complemented in its linking C^*-algebra $A(W)$. The Junge Approximation Theorem (see [10, Theorem 15.1.1]) shows that every complete contraction from $\mathcal{B}(\mathcal{K})^*$ into $A(W)^{**}$ satisfies the W*RP. From Lemma 2.4, we may approximate s in the point-weak* topology by a net of finite rank weak*-continuous complete contractions $s_\alpha \colon \mathcal{B}(\mathcal{K})^* \to W$. If we fix α and let $\varphi_\alpha = s_\alpha \circ M(\omega) \circ r$, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{B}(\mathcal{H}) \arrow[r, "M(\omega)"] & \mathcal{B}(\mathcal{K})^* \arrow[d, "s_\alpha"] \arrow[dr, "\iota_W \circ s_\alpha"] & \\
V \arrow[u, "r"] \arrow[r, "\varphi_\alpha"] & W \arrow[r, hook, "\iota_W"] & W^{**}
\end{tikzcd}$$

where $\iota_W \circ s_\alpha \colon \mathcal{B}(\mathcal{K})^* \to W^{**}$ is a weak*-continuous complete contraction. It follows from [10, Corollary 4.6] that $\iota(\varphi_\alpha) \leq 1$. Since each of the mappings s_α and φ has its range in W, the net $\{\varphi_\alpha\}$ converges to φ in the point-weak topology.

Now given any finite dimensional subspace $L \subseteq V$, it follows that $\nu(\varphi_\alpha|_L) = \iota(\varphi_\alpha|_L) \leq 1$ and the net $\{\varphi_\alpha|_L\}$ converges to φ_L in the point-weak topology. From [9, Lemma 12.2.7], $\nu(\varphi|_L) \leq 1$. Thus $\iota(\varphi) \leq 1$ follows by the definition of ι, so $\iota(\varphi) = \iota^{\mathrm{ex}}(\varphi)$. ■

Theorem 2.6 *For any TRO W, W^* is a locally reflexive operator space.*

Proof From [10, Proposition 4.4], it suffices to show that $\mathcal{I}(W^*, F) = \mathcal{N}(W^*, F)$ for any finite dimensional operator space F. Given $\varphi \colon W^* \to F$, it is trivial from the definition that $\iota(\varphi) \leq \nu(\varphi)$. On the other hand, if we let $S(\varphi) = \varphi^*$, then the composition of the following mappings

$$\mathcal{I}(W^*, F) \xrightarrow{S} \mathcal{I}^{\mathrm{ex}}(F^*, W^{**}) \cong \mathcal{I}(F^*, W^{**}) \cong \mathcal{N}(F^*, W^{**}) \xrightarrow{S^{-1}} \mathcal{N}(W^*, F)$$

is a contraction (where S is contractive by [10, Lemma 5.1]. The first identification is proved in Proposition 2.5 and W^{**} is a W*-TRO, the second is trivial, and by [10, Lemma 3.2] S^{-1} is isometric between $\mathcal{N}(F^*, W^{**})$ and $\mathcal{N}(W^*, F)$). This means that $\nu(\varphi) \leq \iota(\varphi)$. Thus, $\nu(\varphi) = \iota(\varphi)$ and W^* is locally reflexive. ■

Corollary 2.7 *For any W*-TRO W, the predual W_* is locally reflexive.*

Theorem 2.8 *If W is a TRO, then W^* is strongly locally reflexive.*

Proof Since W^{**} is also a TRO, it follows from Theorem 2.2 that any completely isometric injection from W^{***} into $\mathcal{B}(\mathcal{H})$ satisfies the W*AP. Since W^* is locally reflexive from Theorem 2.6, the strongly local reflexivity of W^* follows from [10, Theorem 6.6]. ■

Corollary 2.9 *If W is a W*-TRO, then the predual W_* is strongly locally reflexive.*

Proof From Theorem 2.2, any complete contraction from $W^* = (W_*)^{**}$ into $\mathcal{B}(\mathcal{H})$ has the W*AP. Since W_* is locally reflexive from Corollary 2.7, the result follows from [10, Theorem 6.6]. ■

Corollary 2.10 *Suppose that V is an injective operator space, then V^* is locally reflexive, and, in fact, it is strongly locally reflexive.*

Proof If V is an injective operator space, it follows from [21, Theorem 4.5] that V has the form $e\mathcal{A}e^{\perp}$, where e is a projection in an injective C^*-algebra $\mathcal{A}$. In particular, V is a TRO. From Theorem 2.8, V^* is strongly locally reflexive, certainly it is locally reflexive. ■

Corollary 2.11 *Suppose that V is a dual injective operator space, then the predual V_* is strongly locally reflexive.*

Proof It follows from [11, Theorem 1.3] and Corollary 2.9 directly. ■

In [17], Kye and Ruan showed that the LLP implies the local reflexivity. The following result shows that the LLP implies the strong local reflexivity.

Theorem 2.12 *Suppose that V is an operator space which has the LLP, then V is strongly locally reflexive.*

Proof From [17, Theorem 5.5], V^* is injective. Corollary 2.11 implies that V is strongly locally reflexive. ■

In general, even nuclear C^*-algebras do not satisfy the strong local reflexivity. The following corollary shows that V^* satisfies the strong local reflexivity for any nuclear operator space V.

Corollary 2.13 *Suppose that V is a nuclear operator space, then V^* is strongly locally reflexive.*

Proof It follows from [11, Theorem 4.5] that V^* has the LLP. Theorem 2.12 implies that V^* satisfies the strongly local reflexivity. ■

3 Characterizations of the LLP and WEP

We say that a diagram of operator spaces and complete contractions

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0$$

is 1-exact if φ is a complete isometry, ψ is a complete quotient mapping, and $\ker\psi = \operatorname{Im}\varphi$.

The following result was discussed by Pisier [18], who attributed the result to Kirchberg and Valliant.

Lemma 3.1 *An operator space V is nuclear if and only if it has the following property. For any 1-exact sequence of operator spaces*

$$0 \longrightarrow X \xhookrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0,$$

it follows that

$$0 \longrightarrow X\check{\otimes}V \xrightarrow{\varphi\otimes\mathrm{id}} Y\check{\otimes}V \xrightarrow{\psi\otimes\mathrm{id}} Z\check{\otimes}V \longrightarrow 0$$

is 1-exact.

We say an operator space $V \subseteq \mathcal{B}(\mathcal{H})$ has the weak expectation property (or simply, WEP) if there exists a completely contractive projection P from $\mathcal{B}(\mathcal{H})$ onto V^{**} such that $P(v) = v$ for any $v \in V$. In the following, we will give some similar characterizations about the LLP and WEP.

Lemma 3.2 *Suppose that X, Y, Z are operator spaces, $X \subseteq Y$ and $\pi\colon Y \to Y/X$ is the canonical complete quotient mapping. If for any finite dimensional operator subspace F of Z, the mapping $\pi \otimes \mathrm{id}_F\colon Y\check{\otimes}F \to Y/X\check{\otimes}F$ is a complete quotient mapping, then* $\ker(\pi \otimes \mathrm{id}_Z\colon Y\check{\otimes}Z \to Y/X\check{\otimes}Z) = X\check{\otimes}Z$.

Proof Suppose that $u \in Y\check{\otimes}Z$ satisfies $(\pi\check{\otimes}\,\mathrm{id}_Z)(u) = 0$. Then given $\epsilon > 0$, we may choose an element $u_0 = \sum_{i=1}^n h_i \otimes v_i \in Y \otimes_\vee Z$ such that $\|u - u_0\| < \epsilon$. It follows that $u_0 \in Y\check{\otimes}F$, where F is the finite dimensional subspace of Z spanned by $v_1, \dots, v_n$. Since the obvious mapping $Y/X\check{\otimes}F \to Y/X\check{\otimes}Z$ is isometric, we have

$$\begin{aligned}
\|(\pi \otimes \mathrm{id}_F)(u_0)\| &= \|(\pi \otimes \mathrm{id}_Z)(u_0)\| \\
&\le \|(\pi \otimes \mathrm{id}_Z)(u_0) - (\pi \otimes \mathrm{id}_Z)(u)\| + \|(\pi \otimes \mathrm{id}_Z)(u)\| \\
&= \|(\pi \otimes \mathrm{id}_Z)(u_0 - u)\| + 0 \le \|u_0 - u\| < \epsilon.
\end{aligned}$$

From the hypothesis, $\pi\otimes\mathrm{id}_F\colon Y\check{\otimes}F \to Y/X\check{\otimes}F$ is a quotient mapping, and thus there is an element $u_1 \in Y\check{\otimes}F$ with $\|u_1\| < \epsilon$ and $(\pi \otimes \mathrm{id}_F)(u_1) = (\pi \otimes \mathrm{id}_F)(u_0)$. We have $\|u-(u_0-u_1)\| \le \|u-u_0\|+\|u_1\| < 2\epsilon$, where $u_0-u_1 \in \ker \pi\otimes\mathrm{id}_F = X\check{\otimes}F \subseteq X\check{\otimes}Z$ and thus $\mathrm{dist}(u, X\check{\otimes}Z) < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $u \in X\check{\otimes}Z$. The converse inclusion is obvious. ■

Theorem 3.3 *For any operator space, the following are equivalent.*

(i) *V has the LLP;*

(ii) *for any finite dimensional operator space F and any 1-exact sequence of operator spaces*

$$0 \longrightarrow X \xhookrightarrow{\iota} Y \xrightarrow{\pi} V \longrightarrow 0 \tag{3.1}$$

it follows that

$$0 \longrightarrow X\check{\otimes}F \xrightarrow{\iota\otimes\mathrm{id}_F} Y\check{\otimes}F \xrightarrow{\pi\otimes\mathrm{id}_F} V\check{\otimes}F \longrightarrow 0$$

is 1-exact;

(iii) *for any operator space Z and* 1*-exact sequence*(3.1)*, it follows that*

$$0 \to X \check{\otimes} Z \xrightarrow{\iota \otimes \mathrm{id}_Z} Y \check{\otimes} Z \xrightarrow{\pi \otimes \mathrm{id}_Z} V \check{\otimes} Z \to 0$$

is 1*-exact.*

Proof (i) $\Rightarrow$ (ii). Since F is a finite dimensional operator space, it is easy to show that $\ker \pi \otimes \mathrm{id}_F = X \check{\otimes} F$. From the injectivity of $\check{\otimes}$, $\iota \otimes \mathrm{id}_F$ is a complete isometry. So it suffices to show that $\pi \otimes \mathrm{id}_F$ is a complete quotient mapping. For any $\varphi \in CB(T_n(F^*), V)$ with $\|\varphi\|_{cb} < 1$ and any $\epsilon > 0$ with $(1+\epsilon)\|\varphi\|_{cb} < 1$, since V has the LLP there exists a completely bounded linear mapping $\tilde{\varphi} \colon \varphi(T_n(F^*)) \to Y$ such that $\|\tilde{\varphi}\|_{cb} < 1 + \epsilon$ and $\pi \circ \tilde{\varphi} = \mathrm{id}_V |_{\varphi(T_n(F^*))}$, *i.e.*, the following diagram commutes:

$$\begin{tikzcd}
 & & & Y \arrow[d, "\pi"] \\
T_n(F^*) \arrow[r, "\varphi"] & \varphi(T_n(F^*)) \arrow[r, hook] \arrow[urr, "\tilde{\varphi}"] & V \arrow[r, "\mathrm{id}_V"] & V = Y/X
\end{tikzcd}$$

Set $\psi = \tilde{\varphi} \circ \varphi \colon T_n(F^*) \to Y$. We have $\pi \circ \psi = \pi \circ \tilde{\varphi} \circ \varphi = \varphi$ and

$$\|\psi\|_{cb} \le \|\tilde{\varphi}\|_{cb} \cdot \|\varphi\|_{cb} < (1+\epsilon)\|\varphi\|_{cb} < 1.$$

Thus the top row of the following commutative diagram is a quotient mapping:

$$\begin{tikzcd}
CB(T_n(F^*), Y) \arrow[r] \arrow[d, equal] & CB(T_n(F^*), V) \arrow[d, equal] \\
M_n(F) \check{\otimes} Y \arrow[r] \arrow[d, equal] & M_n(F) \check{\otimes} V \arrow[d, equal] \\
M_n(Y \check{\otimes} F) \arrow[r, "(\pi \otimes \mathrm{id}_F)_n"] & M_n(V \check{\otimes} F)
\end{tikzcd}$$

This implies that the bottom row $(\pi \otimes \mathrm{id}_F)_n$ is also a quotient mapping and thus $\pi \otimes \mathrm{id}_F$ is a complete quotient mapping.

(ii) $\Rightarrow$ (iii). It suffices to prove that the kernel $\pi \otimes \mathrm{id}_Z$ is $X \check{\otimes} Z$, and $\pi \otimes \mathrm{id}_Z$ is a complete quotient mapping. From the hypothesis of (ii), the mapping

$$\pi \otimes \mathrm{id}_F \colon Y \check{\otimes} F \to V \check{\otimes} F = Y / X \check{\otimes} F$$

is a complete quotient mapping for any finite dimensional operator space F. It follows from Lemma 3.2 that $\ker(\pi \otimes \mathrm{id}_Z \colon Y \check{\otimes} Z \to V \check{\otimes} Z) = X \check{\otimes} Z$.

To show the quotient condition, it suffices to show that $\pi \otimes \mathrm{id}_Z$ maps $(Y \otimes_\vee Z)_{\|\cdot\|<1}$ onto a dense subset of $(V \otimes_\vee Z)_{\|\cdot\|<1}$. Given an element $\tilde{u}$ in the latter set,

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there exists a finite dimensional subspace $F \subseteq Z$ with $\tilde{u} \in V \otimes_\vee F \subseteq V \otimes_\vee Z$. From (ii) there exists an element $u \in Y \check{\otimes} F$ with $\|u\| < 1$ and $(\pi \otimes \mathrm{id}_F)(u) = \tilde{u}$, and since we may regard u as an element of $Y \check{\otimes} Z$, we have $(\pi \otimes \mathrm{id}_Z)(u) = \tilde{u}$. Thus we have the desired result and $\pi \otimes \mathrm{id}_Z$ is a quotient mapping. The following commutative diagram implies that $\pi \otimes \mathrm{id}_Z$ is always a complete quotient mapping, where the bottom row is a quotient mapping by the above proof:

$$\begin{tikzcd} M_n(Y \check{\otimes} Z) \arrow[r] \arrow[d, equal] & M_n(V \check{\otimes} Z) \arrow[d, equal] \\ Y \check{\otimes} M_n(Z) \arrow[r] & V \check{\otimes} M_n(Z) \end{tikzcd}$$

(iii) $\Rightarrow$ (ii). It is obvious.

(ii) $\Rightarrow$ (i). It is known from [2, Corollary 3.2] that every operator space V is a complete quotient space of some $\mathcal{T}_I$ space. Let $\pi\colon \mathcal{T}_I \to V$ denote the complete quotient mapping from $\mathcal{T}_I$ onto V and $W = \ker \pi$. Then we have a 1-exact sequence of operator spaces

$$0 \longrightarrow W \overset{\iota}{\hookrightarrow} \mathcal{T}_I \xrightarrow{\pi} V \longrightarrow 0.$$

For any finite dimensional operator space E, it follows from (ii) that the top sequence of the following commutative diagram is 1-exact,

$$\begin{tikzcd} 0 \arrow[r] & W \check{\otimes} E^* \arrow[r, "\iota \otimes \mathrm{id}_{E^*}"] \arrow[d, equal] & \mathcal{T}_I \check{\otimes} E^* \arrow[r, "\pi \otimes \mathrm{id}_{E^*}"] \arrow[d, equal] & V \check{\otimes} E^* \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & CB(E, W) \arrow[r] & CB(E, \mathcal{T}_I) \arrow[r] & CB(E, V) \arrow[r] & 0, \end{tikzcd}$$

and this implies that the bottom sequence is also 1-exact. From [17, Theorem 3.2], V has the LLP. ∎

In the following result, we use the projective tensor product to characterize the LLP. This result can be seen as the "dual" result of Lemma 3.1.

Theorem 3.4 *Suppose that V is an operator space, the following are equivalent.*

(i) *V has the LLP;*

(ii) *for any 1-exact sequence of operator spaces*

$$0 \longrightarrow X \overset{\iota}{\hookrightarrow} Y \xrightarrow{\pi} Z \longrightarrow 0$$

it follows that

$$0 \longrightarrow X \hat{\otimes} V \xrightarrow{\iota \otimes \mathrm{id}_V} Y \hat{\otimes} V \xrightarrow{\pi \otimes \mathrm{id}_V} Z \hat{\otimes} V \longrightarrow 0$$

is 1-exact;

(iii) *for any* 1*-exact sequence of finite dimensional operator spaces*

$$0 \longrightarrow E \xhookrightarrow{\iota} F \xrightarrow{\pi} G \longrightarrow 0$$

it follows that

$$0 \longrightarrow E\hat{\otimes}V \xrightarrow{\iota\otimes \mathrm{id}_V} F\hat{\otimes}V \xrightarrow{\pi\otimes \mathrm{id}_V} G\hat{\otimes}V \longrightarrow 0$$

is 1*-exact.*

Proof (i) $\Rightarrow$ (ii). Suppose that V has the LLP, it follows from [17, Theorem 5.5] that V^* is injective, and [9, Lemma 4.1.7] shows that the bottom restrictive mapping is a complete quotient mapping

$$\begin{array}{ccc} (Y\hat{\otimes}V)^* & \xrightarrow{(\iota\otimes \mathrm{id}_V)^*} & (X\hat{\otimes}V)^* \\ \| & & \| \\ CB(Y,V^*) & \longrightarrow & CB(X,V^*) \end{array}$$

thus $(\iota \otimes \mathrm{id}_V)^*$ is also a complete quotient mapping and $\iota \otimes \mathrm{id}_V$ is a completely isometric injection. It follows from [9, Proposition 7.1.7] that $\ker(\pi \otimes \mathrm{id}_V)$ is equal to the closure of $X \otimes V$ in $Y\hat{\otimes}V$. Since $\iota \otimes \mathrm{id}_V$ is a complete isometry, we have $\ker(\pi \otimes \mathrm{id}_V) = X\hat{\otimes}V$. The projectivity of $\hat{\otimes}$ implies that $\pi \otimes \mathrm{id}_V$ is a complete quotient mapping. Thus the sequence

$$0 \longrightarrow X\hat{\otimes}V \xrightarrow{\iota\otimes \mathrm{id}_V} Y\hat{\otimes}V \xrightarrow{\pi\otimes \mathrm{id}_V} Z\hat{\otimes}V \longrightarrow 0$$

(ii) $\Rightarrow$ (iii). It is obvious.

(iii) $\Rightarrow$ (i). For any 1-exact sequence of finite dimensional operator spaces

$$0 \longrightarrow E \xhookrightarrow{\iota} F \xrightarrow{\pi} F/E \longrightarrow 0,$$

it follows from the hypothesis of (iii) that $\iota \otimes \mathrm{id}_V : E\hat{\otimes}V \to F\hat{\otimes}V$ is completely isometric, and thus $(\iota\otimes\mathrm{id}_V)^*$ is a complete quotient mapping. The following commutative diagram implies that the bottom restrictive mapping is also a complete quotient

$$\begin{array}{ccc} (F\hat{\otimes}V)^* & \xrightarrow{(\iota\otimes \mathrm{id}_V)^*} & (E\hat{\otimes}V)^* \\ \| & & \| \\ CB(F,V^*) & \longrightarrow & CB(E,V^*) \end{array}$$

This means that V^* is finitely injective. From [17, Corollary 4.4, Theorem 5.5], V^* is injective and V has the LLP. ∎

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Proposition 3.5 *For any operator space V and 1-exact sequence of operator spaces*

$$0 \longrightarrow X \xhookrightarrow{\iota} Y \xrightarrow{\pi} Z \longrightarrow 0$$

it follows that

$$0 \longrightarrow X \overset{h}{\otimes} V \xrightarrow{\iota\otimes \mathrm{id}_V} Y \overset{h}{\otimes} V \xrightarrow{\pi\otimes \mathrm{id}_V} Z \overset{h}{\otimes} V \longrightarrow 0$$

is 1*-exact.*

Proof We may suppose that X is an operator subspace of Y and Z is the quotient Y/X. Since the Haagerup tensor product is both injective and projective, $\iota \otimes \mathrm{id}_V$ is a complete isometry and $\pi \otimes \mathrm{id}_V$ is a complete quotient mapping. It suffices to show $\ker \pi \otimes \mathrm{id}_V = X \overset{h}{\otimes} V$. Suppose that $u \in Y \overset{h}{\otimes} V$ satisfies $(\pi \otimes \mathrm{id}_V)(u) = 0$. Then given $\epsilon > 0$, we may choose an element $u_0 = \sum_{i=1}^n y_i \otimes v_i \in Y \otimes_h V$ such that $\|u - u_0\| < \epsilon$. It follows that $u_0 \in Y \overset{h}{\otimes} L$, where L is the finite dimensional subspace of V spanned by $v_1, \dots, v_n$. Since the obvious mapping $Z \overset{h}{\otimes} L \to Z \overset{h}{\otimes} V$ is isometric,

$$\begin{aligned}
\|(\pi \otimes \mathrm{id}_L)(u_0)\| &= \|(\pi \otimes \mathrm{id}_V)(u_0)\| \\
&\leq \|(\pi \otimes \mathrm{id}_V)(u_0) - (\pi \otimes \mathrm{id}_V)(u)\| + \|(\pi \otimes \mathrm{id}_V)(u)\| \\
&= \|(\pi \otimes \mathrm{id}_V)(u_0 - u)\| + 0 \leq \|u_0 - u\| < \epsilon.
\end{aligned}$$

Since $\overset{h}{\otimes}$ is projective, $\pi \otimes \mathrm{id}_L : Y \overset{h}{\otimes} L \to Z \overset{h}{\otimes} L$ is a complete quotient mapping. Thus there exists an element $u_1 \in Y \overset{h}{\otimes} L$ with $\|u_1\| < \epsilon$ and $(\pi \otimes \mathrm{id}_L)(u_1) = (\pi \otimes \mathrm{id}_L)(u_0)$. We have $\|u - (u_0 - u_1)\| \leq \|u - u_0\| + \|u_1\| < 2\epsilon$, where

$$u_0 - u_1 \in \ker \pi \otimes \mathrm{id}_L = X \overset{h}{\otimes} L \subseteq X \overset{h}{\otimes} V$$

and thus $\mathrm{dist}(u, X \overset{h}{\otimes} Z) < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $u \in X \overset{h}{\otimes} Z$. The converse inclusion is obvious. ■

Theorem 3.6 *For any operator space V, the following are equivalent.*

(i) *V has the WEP;*

(ii) *for any finite dimensional operator space F and any 1-exact sequence of operator spaces*

$$0 \longrightarrow V \xhookrightarrow{\iota} X \xrightarrow{\pi} Y \longrightarrow 0 \tag{3.2}$$

it follows that

$$0 \longrightarrow V \hat{\otimes} F \xrightarrow{\iota\otimes \mathrm{id}_F} X \hat{\otimes} F \xrightarrow{\pi\otimes \mathrm{id}_F} Y \hat{\otimes} F \longrightarrow 0$$

is 1*-exact;*

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(iii) *for any operator space Z and any* 1*-exact sequence* (3.2) *it follows that*

$$0 \longrightarrow V\hat{\otimes}Z \xrightarrow{\iota\otimes \mathrm{id}_Z} X\hat{\otimes}Z \xrightarrow{\pi\otimes \mathrm{id}_Z} Y\hat{\otimes}Z \longrightarrow 0$$

is 1*-exact.*

Proof (i) ⇒ (ii). We may suppose that $V \subseteq X \subseteq \mathcal{B}(\mathcal{H})$. Since V has the WEP, there exists a completely contractive mapping $P\colon \mathcal{B}(\mathcal{H}) \to V^{**}$ such that $P(v) = v$ for any $v \in V$. Thus the composition of complete contractions $\iota \otimes \mathrm{id}_F$ and $P|_X \otimes \mathrm{id}_F$

$$V\hat{\otimes}F \xrightarrow{\iota\otimes \mathrm{id}_F} X\hat{\otimes}F \xrightarrow{P|_X\otimes \mathrm{id}_F} V^{**}\hat{\otimes}F$$

is

$$(P|_X \otimes \mathrm{id}_F) \circ (\iota \otimes \mathrm{id}_F) = (P|_X \circ \iota) \otimes \mathrm{id}_F\,. \tag{3.3}$$

From the definition P, $P|_X \circ \iota$ is the canonical inclusion from V into V^{**}. It follows from [9, (7.1.28)] that $(P|_X \circ \iota) \otimes \mathrm{id}_F$ is a completely isometric injection from $V\hat{\otimes}F$ into $V^{**}\hat{\otimes}F$. Hence the first mapping $\iota \otimes \mathrm{id}_F$ of (3.3) is completely isometric. From the projectivity of $\hat{\otimes}$, $\pi \otimes \mathrm{id}_F$ is a complete quotient mapping. It was shown in [9, Proposition 7.1.7] that $\ker \pi \otimes \mathrm{id}_F$ is equal to the closure of $V \otimes F$ in $X\hat{\otimes}F$. Since $\iota \otimes \mathrm{id}_F$ is a complete isometry, we have $\ker \pi \otimes \mathrm{id}_F = V\hat{\otimes}F$.

(ii) ⇒ (iii). We may suppose that V is an operator subspace of X and Y is the operator quotient X/V. In this case ι is the canonical inclusion and π is the canonical complete quotient mapping. Similarly, we only need to prove that $\iota \otimes \mathrm{id}_Z$ is completely isometric. Let u be an element in $V \otimes Z$, we can regard u as an element of $X \otimes Z$. If $\|u\|_{X\hat{\otimes}Z} < 1$, then there exists a representation $u = \alpha(x \otimes z)\beta$, where $\alpha \in M_{1,pq}$, $x = [x_{ij}] \in M_p(X)$, $z = [z_{kl}] \in M_q(Z)$, and $\beta \in M_{pq,1}$ with norm less than 1. Let $F = \mathrm{span}\{z_{kl}\}$ be the finite dimensional subspace of Z spanned by $\{z_{kl} : k, l = 1, \dots, q\}$. Then we can regard u as an element of $X\otimes F$ with $\|u\|_{X\hat{\otimes}F} < 1$. It is easy to see (by choosing a basis for F) that u is in fact contained in $V \otimes F$, thus $u = (\iota \otimes \mathrm{id}_F)(u)$. From the hypothesis of (ii), $\iota \otimes \mathrm{id}_F$ is an isometry. Hence we have

$$\|u\|_{V\hat{\otimes}Z} \leq \|u\|_{V\hat{\otimes}F} = \|(\iota \otimes \mathrm{id}_F)(u)\|_{X\hat{\otimes}F} = \|u\|_{X\hat{\otimes}F} < 1.$$

This shows that $\iota \otimes \mathrm{id}_Z$ is an isometry. In the following, we will prove that $\iota \otimes \mathrm{id}_Z$ is a complete isometry.

For any $n \in \mathbf{N}$, it follows from [9, (7.1.18), Theorem 4.1.8] that $T_n(\iota)\colon T_n(V) \to T_n(X)$ is a complete isometry. Thus it is easy to see that the sequence of operator spaces

$$0 \longrightarrow T_n(V) \xrightarrow{T_n(\iota)} T_n(X) \xrightarrow{T_n(\pi)} T_n(Y) \longrightarrow 0$$

is 1-exact. It follows from (ii) that for any finite dimensional operator space F the sequence

$$0 \longrightarrow T_n(V)\hat{\otimes}F \longrightarrow T_n(X)\hat{\otimes}F \longrightarrow T_n(Y)\hat{\otimes}F \longrightarrow 0$$

is also 1-exact. Thus by similar discussion as above, we can deduce that for any operator space Z, the top row of the commutative diagram

$$\begin{tikzcd} T_n(V)\hat{\otimes}Z \arrow[r] \arrow[d, equal] & T_n(X)\hat{\otimes}Z \arrow[d, equal] \\ T_n(V\hat{\otimes}Z) \arrow[r, "T_n(\iota\otimes \mathrm{id}_Z)"] & T_n(X\hat{\otimes}Z) \end{tikzcd}$$

is an isometry, thus $T_n(\iota \otimes \mathrm{id}_Z)$ is also isometric. As [9, Theorem 4.1.8] implies that $(\iota \otimes \mathrm{id}_Z)_n\colon M_n(V\hat{\otimes}Z) \to M_n(X\hat{\otimes}Z)$ is an isometry, this means that $\iota \otimes \mathrm{id}_Z$ is a complete isometry.

(iii) $\Rightarrow$ (i). For the 1-exact sequence

$$0 \longrightarrow V \xhookrightarrow{\iota} \mathcal{B}(\mathcal{H}) \xrightarrow{\pi} \mathcal{B}(\mathcal{H})/V \longrightarrow 0,$$

and we let $Z = V^*$, then it follows from (iii) that

$$0 \longrightarrow V\hat{\otimes}V^* \xrightarrow{\iota\otimes \mathrm{id}_{V^*}} \mathcal{B}(\mathcal{H})\hat{\otimes}V^* \xrightarrow{\pi\otimes \mathrm{id}_{V^*}} (\mathcal{B}(\mathcal{H})/V)\hat{\otimes}V^* \longrightarrow 0$$

is 1-exact. This implies that the top row of the commutative diagram

$$\begin{tikzcd} (\mathcal{B}(\mathcal{H})\hat{\otimes}V^*)^* \arrow[r, "(\iota\otimes \mathrm{id}_{V^*})^*"] \arrow[d, equal] & (V\hat{\otimes}V^*)^* \arrow[d, equal] \\ CB(\mathcal{B}(\mathcal{H}), V^{**}) \arrow[r] & CB(V, V^{**}) \end{tikzcd}$$

is a complete quotient mapping and thus the same is true for the bottom row. Hence the identity mapping $\mathrm{id}_V \in CB(V,V) \hookrightarrow CB(V,V^{**})$ has a completely contractive extension $P \in CB(\mathcal{B}(\mathcal{H}), V^{**})$ such that $P(v) = v$ for any $v \in V$. This shows that V has the WEP. ■

4 Weak* Locally Lifting Property (W*-LLP)

We know that an operator space V has the LLP if and only if there exist diagrams of complete contractions

$$\begin{tikzcd} & T_{n(\alpha)} \arrow[dr, "t_\alpha"] & \\ V \arrow[ur, "s_\alpha"] \arrow[rr, "\mathrm{id}_V"] & & V \end{tikzcd}$$

which approximately commute in the point-norm topology.

Definition 4.1 We say that a dual operator space V has the weak* locally lifting property (or simply W*-LLP) if there exist diagrams of weak*-continuous complete contractions

$$\begin{tikzcd} & T_{n(\alpha)} \arrow[dr, "t_\alpha"] & \\ V \arrow[ur, "s_\alpha"] \arrow[rr, "\mathrm{id}_V"] & & V \end{tikzcd}$$

which approximately commute in the point-weak* topology.

We recall that a dual operator space V is called semi-discrete if there exist diagrams of weak*-continuous complete contractions

$$\begin{tikzcd} & M_{n(\alpha)} \arrow[dr, "\psi_\alpha"] & \\ V \arrow[ur, "\varphi_\alpha"] \arrow[rr, "\mathrm{id}_V"] & & V \end{tikzcd}$$

which approximately commute in the point-weak* topology. From these definitions, we can see that there are close relationship between nuclearity, semi-discreteness, LLP and W*-LLP. For any operator space V, it follows from the definitions and the standard convexity argument that

$$V^* \text{is semi-discrete } \Leftrightarrow V \text{ has the LLP,}$$

$$V^* \text{has the W}^*\text{-LLP} \Leftrightarrow V \text{is nuclear.}$$

Theorem 4.5 in [11] shows that

$$\begin{aligned} V \text{is nuclear} &\Leftrightarrow V^{**} \text{ is semi-discrete and } V \text{ is locally reflexive} \\ &\Leftrightarrow V^* \text{ has the LLP and } V \text{ is locally reflexive.} \end{aligned}$$

So we have the following relationship between LLP and W*-LLP of V^*.

$$V^* \text{ has the W}^*\text{-LLP } \Leftrightarrow V^* \text{ has the LLP and } V \text{ is locally reflexive.}$$

As pointed out [11], local reflexivity is an essential condition in this result. Turning to the second dual V^{**} of V, we will find in the following result that it is in striking contrast to the situation for the dual space V^*. Theorem 4.5 in [11] shows the relationship between the nuclearity of V and the semi-discreteness of V^{**}. By analogy, we will consider the relationship between the LLP of V and the W*-LLP of V^{**}.

Theorem 4.2 *Suppose that V is an operator space, then the following are equivalent.*

(i) *V^{**} has the W*-LLP;*
(ii) *V^{**} has the LLP;*

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(iii) *V has the LLP and V^* is exact.*

Proof (i) $\Rightarrow$ (ii). Suppose that V^{**} has the W*-LLP. From the definitions and the standard convexity argument, V^* is nuclear. It follows from [11, Theorem 4.5] that V^{**} has the LLP.

(ii) $\Rightarrow$ (iii). Suppose that V^{**} has the LLP. It follows from [17, Theorem 5.5] that V^{***} is injective. Let $\iota\colon V \hookrightarrow V^{**}$ be the canonical inclusion. It is easy to see that the adjoint ι^* is a completely contractive projection from V^{***} onto V^*. Since V^{***} is injective, [9, Proposition 4.1.6] implies that V^* is also injective. It follows from [11, Theorem 1.3] that V^* is a TRO. Thus from [15, Theorem 6.5], the injectivity of V^{***} implies that V^* is nuclear and V^* is exact. For any finite operator spaces $E \subseteq F$, it follows from Lemma 3.1 that the top row of the commutative diagram

$$\begin{tikzcd} F^*\check{\otimes}V^* \arrow[r] \arrow[d, equal] & E^*\check{\otimes}V^* \arrow[d, equal] \\ CB(F, V^*) \arrow[r] & CB(E, V^*) \end{tikzcd}$$

is a complete quotient mapping, and the same is true for the bottom row. In other words, any mapping $\varphi\colon E \to V^*$ with $\|\varphi\|_{cb} < 1$ can be extended to a mapping $\psi\colon F \to V^*$, *i.e.*, V^* is finitely injective. Corollary 4.4 and Theorem 5.5 in [17] imply that V^* is injective and V has the LLP.

(iii) $\Rightarrow$ (i). Suppose that V has the LLP and V^* is exact. Given a finite dimensional subspace $L \subseteq V^*$ and $\epsilon > 0$, it follows from the exactness of V^* that we may choose an $n \in \mathbf{N}$, a subspace $G \subseteq M_n$, and a linear isomorphism $r\colon L \to G$ with $\|r\|_{cb} = 1$ and $\|r^{-1}\|_{cb} < 1 + \epsilon$. Since V^* is injective from [17, Theorem 5.5], we may find a corresponding extension $s\colon M_n \to V^*$ of r^{-1} with $\|s\|_{cb} < 1 + \epsilon$. We thus obtain a diagram

$$\begin{tikzcd} & M_n \arrow[dr, "s"] & \\ L \arrow[ur, "t_0"] \arrow[r, "r"] & G \arrow[u, phantom, "\cup\!\mid"] \arrow[r, "r^{-1}"] & V^* \end{tikzcd}$$

in which $t_0\colon L \to M_n$ is just the inclusion mapping composed with r. We may extend t_0 to a complete contraction $t\colon V^* \to M_n$. From this construction it is evident that V^* is nuclear and V^{**} has the W*-LLP. ■

Exactness of V^* is an essential condition in this result. For example, let $V = \mathcal{T}(l^2)$. We know that $\mathcal{T}(l^2)$ has the LLP, but V^{**} does not have the LLP since $V^{***} = \mathcal{B}(l^2)^{**}$ is not injective.

Corollary 4.3 *Suppose that V^* is a nuclear dual operator space. Then V is strongly locally reflexive.*

Proof Since V^* is nuclear, V^{**} has the W*-LLP. From Theorem 4.2, V has the LLP. Theorem 2.12 implies that V is strongly locally reflexive. ■

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