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THE GENUS OF THE COXETER GRAPH

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ABSTRACT. In [1], Biggs stated that the Coxeter graph can be embedded in an orientable surface of genus 3. The purpose of this note is to point out that Biggs' embedding is in fact into a non-orientable surface. Further, it is shown that the orientable genus is 3 and the non-orientable genus is 6.

1. **Introduction.** The Coxeter graph is a distance-transitive cubic graph with 28 vertices and 42 edges. For further details, see [1, 2, 3].

The following notation is that used in [1]. There are three 7-cycles, $(c_1c_2c_3c_4c_5c_6c_7)$, $(d_1d_3d_5d_7d_2d_4d_6)$ and $(e_1e_4e_7e_3e_6e_2e_5)$. There are also seven vertices t_i $(i = 1,\ldots,7)$ with t_i adjacent to c_i, d_i and e_i.

The embedding shown in Figure 6 of [1] has six 7-gons, three 8-gons And one 18-gon whose boundary is $d_3d_1t_1e_1e_4t_4c_4c_3c_2t_2e_2e_6t_6c_6c_7t_7d_7d_5d_3$. Some edges in the border of Figure 6 of [1], *e.g.* d_4d_6, occur twice with the same direction. If the boundary of each face in the diagram is traversed in the clockwise direction, then such an edge is traversed twice in the same direction. This shows that the embedding is not orientable.

The Euler characteristic for an embedding of this graph is $f - 14$, where f denotes the number of faces. To identify the genus we seek embeddings with f as large as possible.

2. 12**-face embedding?** Since the girth of the graph is 7, each face of an embedding has at least seven sides. As there are 42 edges each separating two faces (or two parts of one face), there can be no more than 12 faces in an embedding. Moreover if there are 12 faces then each is a 7-gon. However, as noted in [1], it is not possible to have three 7-gons surrounding one vertex. Each pair of edges at the vertex would form part of the boundary of one 7-gon. For each pair there are two possible 7-cycles in the graph to choose from. See Figure 1, which shows all the 7-cycles that pass through c_1.

$$c_1c_2\begin{array}{c}c_3c_4c_5c_6\\ t_2d_2d_7t_7\end{array}c_7c_1 \qquad c_1c_7\begin{array}{c}c_6t_6d_6d_1\\ t_7e_7e_4e_1\end{array}t_1c_1 \qquad c_1t_1\begin{array}{c}d_1d_3t_3c_3\\ e_1e_5e_2t_2\end{array}c_2c_1$$

FIGURE 1

Now, whichever choice is made, it turns out that two of the 7-cycles have a path of length 2 in common. The third edge incident with the midpoint of such a path is then not on the boundary of a face. It follows that there cannot be more than eight 7-gons in an embedding.

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3. 10-**face embedding.** Figure 2 shows an embedding of the graph with ten faces in an orientable surface. There are three 7-gons, $c_1c_2c_3c_4c_5c_6c_7$, $d_1d_3d_5d_7d_2d_4d_6$ and $e_1e_5e_2e_6e_3e_7e_4$, and seven 9-gons. Each pair of faces that are not both 7-gons has a common edge. The 7-gons do not meet. Since the Euler characteristic is -4, the surface is a sphere with three handles. For an orientable embedding, the Euler characteristic has to be even, so the orientable genus has now been shown to be 3.

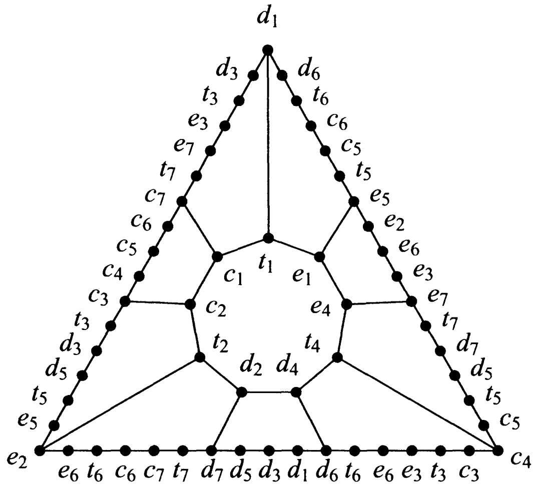

FIGURE 2

4. 11-**face embedding?** To find the non-orientable genus it is necessary to consider the possibility of an embedding with 11 faces. The average number of edges per face would have to be 84/11, which is less than 8. There would have to be many 7-gons and the following observations show that it is not possible to perform the construction. Not all the details are given, but the main argument used considers the unique 7-gons that could be adjacent to each edge of a given face. Distinct 7-gons whose boundaries include a common path of length 2 are incompatible. This shows that Biggs' embedding is an optimal one, and the non-orientable genus of the graph is 6.

A. All 7-cycles in the graph are isomorphic. The same is true for 8-, 9- and 10-cycles. In the case of 7- and 8-cycles all vertices are similar (*i.e.* the stabilizer of the cycle acts transitively on its vertices); for 9- and 10-cycles they are not. There are no 11-cycles.

B. There cannot be 7-gons on a pair of opposite edges of an 8-gon. So the total number of 7-gons adjacent to a given 8-gon is no more than four and the total number of vertices of the 8-gon that are on two 7-gons can be no more than three. (Biggs' embedding has four 7-gons on consecutive edges of each 8-gon.)

C. There can be no more than three 7-gons adjacent to a given 9-gon, and no more than two vertices of the 9-gon can be on two 7-gons.

D. There can be no more than five 7-gons adjacent to a given 10-gon, and no more than one vertex of the 10-gon can be on two 7-gons.

E. There cannot be two 8-gons and one 7-gon round one vertex.

F. Two 8-gons can share only one edge. The same is true for an 8-gon and a 9-gon. To see this, note that there are no adjacencies among the eight vertices that are adjacent to the vertices of an 8-gon.

The suitable partitions of 84 for the sizes of the faces of an 11-face embedding are: $(7^8, 8^2, 12)$, $(7^8, 8, 10^2)$, $(7^8, 9^2, 10)$; $(7^7, 8^2, 9, 10)$, $(7^7, 8, 9^3)$; $(7^6, 8^4, 10)$, $(7^6, 8^3, 9^2)$ (for all these cases it is not possible to find a sufficient number of vertices that are on two 7-gons), $(7^5, 8^5, 9)$ (the 9-gon can have only five 8-gons and three 7-gons on its boundary), $(7^4, 8^7)$ (a 7-gon can have no more than three 8-gons on its boundary).

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REFERENCES

1. N. Biggs, *Three remarkable graphs*, Canad. J. Math. **25**(1973), 397–411.

2. A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, New York, Berlin, Heidelberg, 1989, 382–383.

3. H. S. M. Coxeter, *My graph*, Proc. London Math. Soc. **46**(1983), 117-136.

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