Nagoya Math. J., **231** (2018), 1–22
DOI 10.1017/nmj.2017.8

ON VANISHING THEOREMS FOR LOCAL SYSTEMS ASSOCIATED TO LAURENT POLYNOMIALS

ALEXANDER ESTEROV AND KIYOSHI TAKEUCHI

Abstract. We prove some vanishing theorems for the cohomology groups of local systems associated to Laurent polynomials. In particular, we extend one of the results of Gelfand *et al.* [*Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. **84** (1990), 255–271] to various directions. In the course of the proof, some properties of vanishing cycles of perverse sheaves and twisted Morse theory are used.

§1. Introduction

The study of the cohomology groups of local systems is an important subject in algebraic geometry, hyperplane arrangements, topology and hypergeometric functions of several variables. Many mathematicians are interested in the conditions for which we have their concentrations in the middle degrees. (For a review of this subject, see, for example, [4, Section 6.4].) Here, let us consider this problem in the following situation. Let $B=\{b(1),b(2),\ldots,b(N)\}\subset\mathbb{Z}^{n-1}$ be a finite subset of the lattice $\mathbb{Z}^{n-1}$. Assume that the affine lattice generated by B in $\mathbb{Z}^{n-1}$ coincides with $\mathbb{Z}^{n-1}$. For $z=(z_1,\ldots,z_N)\in\mathbb{C}^N$, we consider Laurent polynomials $P(x)$ on the algebraic torus $T_0=(\mathbb{C}^*)^{n-1}$ defined by $P(x)=\sum_{j=1}^N z_j x^{b(j)}$ $(x=(x_1,\ldots,x_{n-1})\in T_0=(\mathbb{C}^*)^{n-1})$. Then, for $c=(c_1,\ldots,c_n)\in\mathbb{C}^n$, we obtain a possibly multivalued function $P(x)^{-c_n}x_1^{c_1-1}\cdots x_{n-1}^{c_{n-1}-1}$ on $W=T_0\setminus P^{-1}(0)$. It generates the rank-one local system

$$\mathcal{L}=\mathbb{C}_W P(x)^{-c_n}x_1^{c_1-1}\cdots x_{n-1}^{c_{n-1}-1} \tag{1.1}$$

on W. Under the nonresonance condition (see Definition 3.1) on $c\in\mathbb{C}^n$, Gelfand *et al.* [10] proved that we have the concentration

$$H^j(W;\mathcal{L})\simeq 0 \quad (j\neq n-1) \tag{1.2}$$

Received October 28, 2014. Revised February 21, 2016. Accepted May 7, 2016.
2010 Mathematics subject classification. 14M25, 32S22, 32S60, 33C70, 52C35.
First author was partially supported by RFBR grant 16-01-00409.

for nondegenerate Laurent polynomials $P(x)$. This result was obtained as a byproduct of their study on the integral representations of A-hypergeometric functions in [10]. Since their proof of this concentration heavily relies on the framework of the $\mathcal{D}$-module theory, it is desirable to prove it more directly. In this paper, by applying the twisted Morse theory to perverse sheaves, we extend the result of Gelfand–Kapranov–Zelevinsky to various directions.

First, in Theorem 3.3, we relax the nondegeneracy condition on $P(x)$ by replacing it with a weaker one (see Definition 3.2). We thus extend the result of [10] to the case where the hypersurface $P^{-1}(0) \subset T_0$ may have isolated singular points in T_0. In fact, in Theorem 3.3, we relax also the condition that B generates $\mathbb{Z}^{n-1}$ to a weaker one that the dimension of the convex hull $\Delta \subset \mathbb{R}^{n-1}$ of B in $\mathbb{R}^{n-1}$ is $n-1$. In Theorem 3.11, we extend these results to more general local systems associated to several Laurent polynomials. Namely, we obtain a vanishing theorem for arrangements of toric hypersurfaces with isolated singular points. Our proofs of Theorems 3.3 and 3.11 are very natural and are obtained by taking (possibly singular) "minimal" toric compactifications of T_0. In order to work on such singular varieties, we use our previous idea in the proof of [7, Lemma 4.2]. See Section 3 for the details. Moreover, in Theorem 5.1 (assuming the nondegeneracy of Gelfand *et al.* [10] for Laurent polynomials), we relax the nonresonance condition of $c \in \mathbb{C}^n$ in Theorem 3.11 by replacing it with the much weaker one $c \notin \mathbb{Z}^n$. To prove Theorem 5.1, we first perturb Laurent polynomials by multiplying monomials. Then, we apply the twisted Morse theory to the real-valued functions associated to them by using some standard properties of vanishing cycles of perverse sheaves. See Sections 4 and 5 for the details. In the course of the proof of Theorem 5.1, we obtain also the following result which might be of independent interest. Let $Q_1, \ldots, Q_l$ be Laurent polynomials on $T = (\mathbb{C}^*)^n$, and for $1 \leqslant i \leqslant l$ denote by $\Delta_i \subset \mathbb{R}^n$ the Newton polytope $NP(Q_i)$ of Q_i. Set $\Delta = \Delta_1 + \cdots + \Delta_l$.

THEOREM 1.1. *Let $\mathcal{L}$ be a nontrivial local system of rank one on $T = (\mathbb{C}^*)^n$. Assume that for any $1 \leqslant i \leqslant l$ we have $\dim \Delta_i = n$, and the subvariety*

$$Z_i = \{x \in T \mid Q_1(x) = \cdots = Q_i(x) = 0\} \subset T \tag{1.3}$$

of T is a nondegenerate complete intersection. Then, for any $1 \leqslant i \leqslant l$, we have the concentration

$$H^j(Z_i; \mathcal{L}) \simeq 0 \quad (j \neq n-i). \tag{1.4}$$

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

Moreover, we have

$$\dim H^{n-i}(Z_i;\mathcal{L}) = \sum_{\substack{m_1,\dots,m_i\geqslant 1\\ m_1+\cdots+m_i=n}} \mathrm{Vol}_{\mathbb{Z}}(\underbrace{\Delta_1,\dots,\Delta_1}_{m_1\text{-times}},\dots,\underbrace{\Delta_i,\dots,\Delta_i}_{m_i\text{-times}}), \tag{1.5}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\underbrace{\Delta_1,\dots,\Delta_1}_{m_1\text{-times}},\dots,\underbrace{\Delta_i,\dots,\Delta_i}_{m_i\text{-times}})\in\mathbb{Z}$ *is the normalized n-dimensional mixed volume with respect to the lattice $\mathbb{Z}^n\subset\mathbb{R}^n$.*

Note that this result can be considered as a refinement of the classical Bernstein–Khovanskii–Kushnirenko theorem (see [13]). On the other hand, Matusevich *et al.* [21] and Saito *et al.* [26] studied the condition on the parameter vector $c\in\mathbb{C}^n$, for which the corresponding local system of A-hypergeometric functions is non-rank-jumping. They also relaxed the nonresonance condition of $c\in\mathbb{C}^n$. It would be an interesting problem to study the relationship between Theorem 5.1 and their results.

§2. Preliminary results

In this section, we recall basic notions and results which are used in this paper. In this paper, we essentially follow the terminology of [4], [12], etc. For example, for a topological space X, we denote by $\mathbf{D}^b(X)$ the derived category whose objects are bounded complexes of sheaves of $\mathbb{C}_X$-modules on X. We denote by $\mathbf{D}^b_c(X)$ the full subcategory of $\mathbf{D}^b(X)$ consisting of constructible objects. Let $\Delta\subset\mathbb{R}^n$ be a lattice polytope in $\mathbb{R}^n$. For an element $u\in\mathbb{R}^n$ of (the dual vector space of) $\mathbb{R}^n$, we define the supporting face $\gamma_u\prec\Delta$ of u in Δ by

$$\gamma_u = \left\{ v\in\Delta \mid \langle u,v\rangle = \min_{w\in\Delta}\langle u,w\rangle \right\}, \tag{2.1}$$

where for $u=(u_1,\dots,u_n)$ and $v=(v_1,\dots,v_n)$ we set $\langle u,v\rangle=\sum_{i=1}^n u_iv_i$. For a face γ of Δ, we set

$$\sigma(\gamma) = \overline{\{u\in\mathbb{R}^n \mid \gamma_u=\gamma\}}\subset\mathbb{R}^n. \tag{2.2}$$

Then, $\sigma(\gamma)$ is an $(n-\dim\gamma)$-dimensional rational convex polyhedral cone in $\mathbb{R}^n$. Moreover, the family $\{\sigma(\gamma)\mid\gamma\prec\Delta\}$ of cones in $\mathbb{R}^n$ thus obtained is a subdivision of $\mathbb{R}^n$. We call it the dual subdivision of $\mathbb{R}^n$ by Δ. If $\dim\Delta=n$, it satisfies the axiom of fans (see [8] and [22], etc.). We call it the dual fan of Δ.

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

Let $\Delta_1, \dots, \Delta_p \subset \mathbb{R}^n$ be lattice polytopes in $\mathbb{R}^n$, and let $\Delta = \Delta_1 + \cdots + \Delta_p \subset \mathbb{R}^n$ be their Minkowski sum. For a face $\gamma \prec \Delta$ of Δ, by taking a point $u \in \mathbb{R}^n$ in the relative interior of its dual cone $\sigma(\gamma)$, we define the supporting face $\gamma_i \prec \Delta_i$ of u in Δ_i. Then, it is easy to see that $\gamma = \gamma_1 + \cdots + \gamma_p$. Now, we recall Bernstein–Khovanskii–Kushnirenko's theorem [13].

Definition 2.1. Let $g(x) = \sum_{v \in \mathbb{Z}^n} c_v x^v$ be a Laurent polynomial on the algebraic torus $T = (\mathbb{C}^*)^n$ $(c_v \in \mathbb{C})$.

(1) We call the convex hull of $\mathrm{supp}(g) := \{v \in \mathbb{Z}^n \mid c_v \neq 0\} \subset \mathbb{Z}^n \subset \mathbb{R}^n$ in $\mathbb{R}^n$ the Newton polytope of g and denote it by $NP(g)$.
(2) For a face $\gamma \prec NP(g)$ of $NP(g)$, we define the γ-part g^γ of g by $g^\gamma(x) := \sum_{v \in \gamma} c_v x^v$.

Definition 2.2. (see [14], [23], etc.) Let $g_1, g_2, \dots, g_p$ be Laurent polynomials on $T = (\mathbb{C}^*)^n$. Set $\Delta_i = NP(g_i)$ $(i = 1, \dots, p)$ and $\Delta = \Delta_1 + \cdots + \Delta_p$. Then, we say that the subvariety $Z = \{x \in T = (\mathbb{C}^*)^n \mid g_1(x) = g_2(x) = \cdots = g_p(x) = 0\}$ of $T = (\mathbb{C}^*)^n$ is a nondegenerate complete intersection if for any face $\gamma \prec \Delta$ of Δ the p-form $dg_1^{\gamma_1} \wedge dg_2^{\gamma_2} \wedge \cdots \wedge dg_p^{\gamma_p}$ does not vanish on $\{x \in T = (\mathbb{C}^*)^n \mid g_1^{\gamma_1}(x) = \cdots = g_p^{\gamma_p}(x) = 0\}$.

Definition 2.3. Let $\Delta_1, \dots, \Delta_n$ be lattice polytopes in $\mathbb{R}^n$. Then, their normalized n-dimensional mixed volume $\mathrm{Vol}_{\mathbb{Z}}(\Delta_1, \dots, \Delta_n) \in \mathbb{Z}$ is defined by the formula

$$\mathrm{Vol}_{\mathbb{Z}}(\Delta_1, \dots, \Delta_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{I \subset \{1, \dots, n\} \\ \sharp I = k}} \mathrm{Vol}_{\mathbb{Z}} \left(\sum_{i \in I} \Delta_i \right), \tag{2.3}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\cdot) = n!\, \mathrm{Vol}(\cdot) \in \mathbb{Z}$ is the normalized n-dimensional volume with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Theorem 2.4. [13] *Let $g_1, g_2, \dots, g_p$ be Laurent polynomials on $T = (\mathbb{C}^*)^n$. Assume that the subvariety $Z = \{x \in T = (\mathbb{C}^*)^n \mid g_1(x) = g_2(x) = \cdots = g_p(x) = 0\}$ of $T = (\mathbb{C}^*)^n$ is a nondegenerate complete intersection. Set $\Delta_i = NP(g_i)$ $(i = 1, \dots, p)$. Then, we have*

$$\chi(Z) = (-1)^{n-p} \sum_{\substack{m_1, \dots, m_p \geqslant 1 \\ m_1 + \cdots + m_p = n}} \mathrm{Vol}_{\mathbb{Z}}(\underbrace{\Delta_1, \dots, \Delta_1}_{m_1\text{-times}}, \dots, \underbrace{\Delta_p, \dots, \Delta_p}_{m_p\text{-times}}), \tag{2.4}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\underbrace{\Delta_1, \dots, \Delta_1}_{m_1\text{-times}}, \dots, \underbrace{\Delta_p, \dots, \Delta_p}_{m_p\text{-times}}) \in \mathbb{Z}$ is the normalized n-dimensional mixed volume with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

§3. A vanishing theorem for local systems

Let $B = \{b(1), b(2), \ldots, b(N)\} \subset \mathbb{Z}^{n-1}$ be a finite subset of the lattice $\mathbb{Z}^{n-1}$. Let $\Delta \subset \mathbb{R}^{n-1}$ be the convex hull of B in $\mathbb{R}^{n-1}$. Assume that $\dim \Delta = n-1$. For $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, we define a Laurent polynomial $P(x)$ on $T_0 = (\mathbb{C}^*)^{n-1}$ by $P(x) = \sum_{j=1}^N z_j x^{b(j)}$ $(x = (x_1, \ldots, x_{n-1}) \in T_0 = (\mathbb{C}^*)^{n-1})$. Then, for $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, the possibly multivalued function $P(x)^{-c_n} x_1^{c_1-1} \cdots x_{n-1}^{c_{n-1}-1}$ on $W = T_0 \setminus P^{-1}(0)$ generates the local system

$$\mathcal{L} = \mathbb{C}_W P(x)^{-c_n} x_1^{c_1-1} \cdots x_{n-1}^{c_{n-1}-1}. \tag{3.1}$$

Set $a(j) = (b(j), 1) \in \mathbb{Z}^n$ $(1 \leqslant j \leqslant N)$ and $A = \{a(1), a(2), \ldots, a(N)\} \subset \mathbb{Z}^n$. Then, $K = \mathbb{R}_+ A \subset \mathbb{R}^n$ is an n-dimensional closed convex polyhedral cone in $\mathbb{R}^n$. For a face $\Gamma \prec K$ of K, let $\mathrm{Lin}(\Gamma) \simeq \mathbb{C}^{\dim \Gamma} \subset \mathbb{C}^n$ be the $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ generated by Γ.

Definition 3.1. (Gelfand *et al.* [10, page 262]) We say that the parameter vector $c \in \mathbb{C}^n$ is nonresonant (with respect to A) if for any face $\Gamma \prec K$ of K such that $\dim \Gamma = n-1$ we have $c \notin \{\mathbb{Z}^n + \mathrm{Lin}(\Gamma)\}$.

The following definition is essentially weaker than the usual (Kouchnirenko) nondegeneracy (see [14], [23], etc.).

Definition 3.2. We say that the Laurent polynomial $P(x) = \sum_{j=1}^N z_j x^{b(j)}$ is "weakly" nondegenerate if for any face γ of Δ such that $\dim \gamma < \dim \Delta = n-1$, the hypersurface

$$\left\{ x \in T_0 = (\mathbb{C}^*)^{n-1} \mid P^\gamma(x) = \sum_{j : b(j) \in \gamma} z_j x^{b(j)} = 0 \right\} \subset T_0 \tag{3.2}$$

is smooth and reduced.

Let $\iota : W = T_0 \setminus P^{-1}(0) \hookrightarrow T_0$ be the inclusion map, and set $\mathcal{M} = R\iota_* \mathcal{L} \in \mathbf{D}_c^b(T_0)$. Then, the following theorem generalizes one of the results in Gelfand *et al.* [10] to the case where the hypersurface $P^{-1}(0) \subset T_0$ may have isolated singular points.

Theorem 3.3. *Assume that* $\dim \Delta = n-1$, *the parameter vector* $c \in \mathbb{C}^n$ *is nonresonant and the Laurent polynomial* $P(x)$ *is weakly nondegenerate. Then, there exists an isomorphism*

$$H_c^j(T_0; \mathcal{M}) \simeq H^j(T_0; \mathcal{M}) \simeq H^j(W; \mathcal{L}) \tag{3.3}$$

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

for any $j \in \mathbb{Z}$. Moreover, we have the concentration

$$H^j(W; \mathcal{L}) \simeq 0 \quad (j \neq n-1). \tag{3.4}$$

Proof. Let Σ_0 be the dual fan of Δ in $\mathbb{R}^{n-1}$, and let X be the (possibly singular) toric variety associated to it. Then, there exists a natural action of T_0 on X whose orbits are parametrized by the faces of Δ. For a face γ of Δ, denote by $X_\gamma \simeq (\mathbb{C}^*)^{\dim \gamma}$ the T_0-orbit associated to γ. Note that $X_\Delta \simeq T_0$ is the unique open dense T_0-orbit in X and its complement $X \setminus X_\Delta$ is the union of X_γ for $\gamma \prec \Delta$ such that $\dim \gamma < n-1$. Let $i: X_\Delta \simeq T_0 \hookrightarrow X$ be the inclusion map. Then, by the weak nondegeneracy of $P(x)$, the closure $S = \overline{i(P^{-1}(0))} \subset X$ of the hypersurface $i(P^{-1}(0)) \subset i(T_0)$ in X intersects T_0-orbits X_γ in $X \setminus X_\Delta$ transversely. Moreover, by the nonresonance of $c \in \mathbb{C}^n$, for any $\gamma \prec \Delta$ such that $\dim \gamma = n-2$, the monodromy of the local system $\mathcal{L}$ around the codimension-one T_0-orbit $X_\gamma \subset X$ in X is nontrivial. Indeed, let $\gamma \prec \Delta$ be such a facet of Δ. We denote by Γ the facet of the cone $K = \mathbb{R}_+ A$ generated by $\gamma \times \{1\} \subset K$. Let $\nu \in \mathbb{Z}^{n-1} \setminus \{0\}$ be the primitive inner conormal vector of the facet γ of $\Delta \subset \mathbb{R}^{n-1}$, and set

$$m = \min_{v \in \Delta} \langle \nu, v \rangle = \min_{v \in \gamma} \langle \nu, v \rangle \in \mathbb{Z}. \tag{3.5}$$

Then, the primitive inner conormal vector $\widetilde{\nu} \in \mathbb{Z}^n \setminus \{0\}$ of the facet Γ of $K \subset \mathbb{R}^n$ is explicitly given by the formula

$$\widetilde{\nu} = \begin{pmatrix} \nu \\ -m \end{pmatrix} \in \mathbb{Z}^n \setminus \{0\}, \tag{3.6}$$

and the condition $c = (c_1, \dots, c_{n-1}, c_n) \notin \{\mathbb{Z}^n + \mathrm{Lin}(\Gamma)\}$ is equivalent to the one

$$m(\gamma) := \left\langle \nu, \begin{pmatrix} c_1 - 1 \\ \vdots \\ c_{n-1} - 1 \end{pmatrix} \right\rangle - m \cdot c_n \quad \notin \mathbb{Z}. \tag{3.7}$$

We can easily see that the order of the (multivalued) function $P(x)^{-c_n} x_1^{c_1 - 1} \cdots x_{n-1}^{c_{n-1}-1}$ along the codimension-one T_0-orbit $X_\gamma \subset X$ in X is equal to $m(\gamma) \notin \mathbb{Z}$. Then, by constructing suitable distance functions as in the proof of [7, Lemma 4.2], we can show that for the open embedding $i: T_0 \hookrightarrow X$ we have

$$(Ri_* \mathcal{M})_p \simeq 0 \quad \text{for any } p \in X \setminus i(T_0), \tag{3.8}$$

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

as follows. Let us first assume that the point $p \in X \setminus i(T_0)$ lies in a 0-dimensional T_0-orbit X_γ. Let $U_\gamma \subset X$ be an $(n-1)$-dimensional affine toric variety containing $\{p\} = X_\gamma$, and regard it as a subvariety of $\mathbb{C}^l_\zeta$ for some l. Let $a = (a_1, \ldots, a_{n-1}) \in \mathbb{Z}^{n-1}$ be the coordinate of the vertex γ of Δ, and define a (nontrivial) rank-one local system $\widetilde{\mathcal{L}}$ on T_0 by

$$\widetilde{\mathcal{L}} = \mathbb{C}_{T_0} x_1^{c_1 - c_n a_1 - 1} \cdots x_{n-1}^{c_{n-1} - c_n a_{n-1} - 1}. \tag{3.9}$$

Then, on a neighborhood of the point p in $U_\gamma \subset \mathbb{C}^l_\zeta$, $Ri_*\mathcal{M}$ is isomorphic to $Ri_*\widetilde{\mathcal{L}}$. Next, as in the proof of [7, Lemma 4.2], we construct a real-valued function φ on $\mathbb{C}^l_\zeta$ whose level sets $\Omega_t = \{\zeta \in \mathbb{C}^l \mid \varphi(\zeta) < t\}$ $(t \in \mathbb{R}_{>0})$ satisfy the conditions $\bigcap_{t>0} \Omega_t = \{p\} = X_\gamma$ and $(\bigcup_{t>0} \Omega_t) \cap T_0 = T_0$, and use it to show the isomorphism

$$0 \simeq R\Gamma(T_0; \widetilde{\mathcal{L}}) \xrightarrow{\sim} (Ri_*\widetilde{\mathcal{L}})_p \tag{3.10}$$

by the twisted Morse theory. We thus obtain the isomorphism $(Ri_*\mathcal{M})_p \simeq 0$. When the point $p \in X \setminus i(T_0)$ lies in a T_0-orbit X_γ such that $\dim X_\gamma = \dim \gamma > 0$, by taking a normal slice of X_γ in X, we can reduce the problem to the case where $\dim X_\gamma = 0$. We thus obtain an isomorphism $i_!\mathcal{M} \simeq Ri_*\mathcal{M}$ in $\mathbf{D}^b_c(X)$. Applying the functor $R\Gamma_c(X; \cdot) = R\Gamma(X; \cdot)$ to it, we obtain the desired isomorphisms

$$H^j_c(T_0; \mathcal{M}) \simeq H^j(T_0; \mathcal{M}) \simeq H^j(W; \mathcal{L}) \tag{3.11}$$

for $j \in \mathbb{Z}$. Now, recall that T_0 is an affine variety, and $\mathcal{M} \in \mathbf{D}^b_c(T_0)$ is a perverse sheaf on it (up to some shift). Then, by Artin's vanishing theorem for perverse sheaves over affine varieties (see [4, Corollaries 5.2.18 and 5.2.19], etc.), we have

$$H^j_c(T_0; \mathcal{M}) \simeq 0 \quad \text{for } j < \dim T_0 = n-1 \tag{3.12}$$

and

$$H^j(T_0; \mathcal{M}) \simeq 0 \quad \text{for } j > \dim T_0 = n-1, \tag{3.13}$$

from which the last assertion immediately follows. This completes the proof. ∎

By Theorem 2.4, we obtain the following corollary of Theorem 3.3.

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

COROLLARY 3.4. *In the situation of Theorem 3.3, let $p_1, \dots, p_r \in P^{-1}(0)$ be the (isolated) singular points of $P^{-1}(0) \subset T_0$, and for $1 \leqslant i \leqslant r$, let $\mu_i > 0$ be the Milnor number of $P^{-1}(0)$ at p_i. Then, we have*

$$\dim H^{n-1}(W; \mathcal{L}) = \mathrm{Vol}_{\mathbb{Z}}(\Delta) - \sum_{i=1}^{r} \mu_i. \tag{3.14}$$

Proof. By multiplying a monomial x^a ($a \in \mathbb{Z}^{n-1}$) to $P(x)$, we may assume that the Newton polytope Δ of P contains the origin $0 \in \mathbb{R}^{n-1}$. Then, by Sard's theorem, the generic fiber $P^{-1}(t) \subset T_0$ ($t \neq 0$) of the map $P : T_0 \longrightarrow \mathbb{C}$ is a nondegenerate hypersurface of T_0 in the sense of Definition 2.2. Hence, it follows from Theorem 2.4 that its Euler characteristic $\chi(P^{-1}(t))$ is equal to $(-1)^{n-1-1}\,\mathrm{Vol}_{\mathbb{Z}}(\Delta) = (-1)^{n-2}\,\mathrm{Vol}_{\mathbb{Z}}(\Delta)$. It is also well known that we have

$$\chi(P^{-1}(0)) = \chi(P^{-1}(t)) - (-1)^{n-2} \sum_{i=1}^{r} \mu_i. \tag{3.15}$$

For the open set $W = T_0 \setminus P^{-1}(0)$ of T_0, by $\chi(T_0) = 0$, we thus obtain the equality

$$(-1)^{n-1}\chi(W) = \mathrm{Vol}_{\mathbb{Z}}(\Delta) - \sum_{i=1}^{r} \mu_i. \tag{3.16}$$

Moreover, by applying the Mayer–Vietoris argument to the rank-one local system $\mathcal{L}$, we have $\chi(W) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j(W; \mathcal{L})$. Then, the assertion follows immediately from Theorem 3.3. □

We can generalize Theorem 3.3 to the case where the hypersurface $S = \overline{i(P^{-1}(0))} \subset X$ has (stratified) isolated singular points p also in T_0-orbits $X_\gamma \subset X \setminus i(T_0)$ as follows. For such a point $p \in S \cap X_\gamma$ of S, let us show that we have the vanishing $(Ri_*\mathcal{M})_p \simeq 0$ in general. First, consider the case where the codimension of X_γ in X is one. The question being local, it suffices to consider the case where $X = \mathbb{C}^{n-1}_y \supset X_\gamma = \{y_{n-1} = 0\}$, $S = \{f(y) = 0\} \ni p = 0$, $T_0 = \mathbb{C}^{n-1} \setminus \{y_{n-1} = 0\}$, $i : \mathbb{C}^{n-1} \setminus \{y_{n-1} = 0\} \hookrightarrow \mathbb{C}^{n-1}$ and

$$\mathcal{L} = \mathbb{C}_{\mathbb{C}^{n-1} \setminus \{f(y) \cdot y_{n-1} = 0\}} f(y)^{\alpha} y_{n-1}^{\beta} \tag{3.17}$$

for $\alpha = -c_n$ and some $\beta \in \mathbb{C}$. (By the notation in the proof of Theorem 3.3, we have $\beta = m(\gamma)$.) Here, $f(y)$ is a polynomial on $\mathbb{C}^{n-1}$ such that $S = f^{-1}(0)$ has a (stratified) isolated singular point at $p = 0 \in S \cap X_\gamma$. Moreover, for the inclusion map $\iota : \mathbb{C}^{n-1} \setminus \{f(y) \cdot y_{n-1} = 0\} \hookrightarrow \mathbb{C}^{n-1} \setminus \{y_{n-1} = 0\}$, we have $\mathcal{M} \simeq R\iota_* \mathcal{L}$. By the nonresonance of $c \in \mathbb{C}^n$, we have $\beta = m(\gamma) \notin \mathbb{Z}$, and there exists an isomorphism

$$i_!(\mathbb{C}_{\mathbb{C}^{n-1}\setminus\{y_{n-1}=0\}} y_{n-1}^{\beta}) \xrightarrow{\sim} Ri_*(\mathbb{C}_{\mathbb{C}^{n-1}\setminus\{y_{n-1}=0\}} y_{n-1}^{\beta}). \tag{3.18}$$

Set $\mathcal{N} = i_!(\mathbb{C}_{\mathbb{C}^{n-1}\setminus\{y_{n-1}=0\}} y_{n-1}^{\beta})$. Then, $\mathcal{N}$ is a perverse sheaf (up to some shift) on $X = \mathbb{C}^{n-1}$ and satisfies the condition $\psi_f(\mathcal{N})_p \simeq \phi_f(\mathcal{N})_p$ (using Equation (3.18)), where

$$\psi_f, \phi_f : \mathbf{D}_c^b(X) \longrightarrow \mathbf{D}_c^b(\{f = 0\}) \tag{3.19}$$

are the nearby and vanishing cycle functors associated to f respectively (see [4], etc.). By the t-exactness of the functor ϕ_f, the constructible sheaf $\phi_f(\mathcal{N})$ on $S = f^{-1}(0)$ is perverse (up to some shift). Moreover, by our assumption, its support is contained in the point $\{p\} = \{0\} \subset X = \mathbb{C}^{n-1}$. This implies that we have the concentration

$$H^j \psi_f(\mathcal{N})_p \simeq H^j \phi_f(\mathcal{N})_p \simeq 0 \quad (j \neq n-2). \tag{3.20}$$

Namely, for the Milnor fiber F_p of f at $p = 0 \in \mathbb{C}^{n-1}$, we have

$$H^j(F_p; \mathcal{N}) \simeq H^j \psi_f(\mathcal{N})_p \simeq 0 \quad (j \neq n-2). \tag{3.21}$$

Let $B(p;\varepsilon) \subset \mathbb{C}^{n-1}$ be a small open ball in $\mathbb{C}^{n-1}$ centered at $p = 0$, and for $0 < \eta \ll \varepsilon$ set

$$G = \{y \in \overline{B(p;\varepsilon)} \mid 0 < |f(y)| < \eta\}. \tag{3.22}$$

Then, in order to show the vanishing $(Ri_*\mathcal{M})_p \simeq 0$, it suffices to prove the one $R\Gamma(G; Ri_*\mathcal{M}) \simeq 0$ for the constructible sheaf

$$(Ri_*\mathcal{M})|_G \simeq (\mathcal{N}|_G) \otimes_{\mathbb{C}_G} (f|_G)^{-1}\mathcal{L}' \tag{3.23}$$

on G, where $\mathcal{L}'$ is the rank-one local system on the punctured disk $D_\eta^* = \{t \in \mathbb{C} \mid 0 < |t| < \eta\} \subset \mathbb{C}$ generated by the function t^α. By the projection formula, we have

$$R\Gamma(G; Ri_*\mathcal{M}) \simeq R\Gamma(D_\eta^*; R(f|_G)_*(\mathcal{N}|_G) \otimes_{\mathbb{C}_{D_\eta^*}} \mathcal{L}'). \tag{3.24}$$

Note that $H^j R(f|_G)_*(\mathcal{N}|_G) \simeq 0$ $(j \neq n-2)$, and $H^{n-2}R(f|_G)_*(\mathcal{N}|_G)$ is a local system on D_η^* whose stalks are isomorphic to $H^{n-2}(F_p; \mathcal{N}) \simeq$

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

$H^{n-2}\psi_f(\mathcal{N})_p$. Hence, in order to show the vanishing $(Ri_*\mathcal{M})_p \simeq 0$, it suffices to prove that the monodromy operator $\Phi : H^{n-2}\psi_f(\mathcal{N})_p \xrightarrow{\sim} H^{n-2}\psi_f(\mathcal{N})_p$ does not have the eigenvalue $\exp(-2\pi i\alpha)$. For this purpose, we use the results in [18, Section 5]. Let $\Gamma_+(f) \subset \mathbb{R}_+^{n-1}$ be the convex hull of $\bigcup_{v\in \mathrm{supp}(f)}(v + \mathbb{R}_+^{n-1})$ in $\mathbb{R}_+^{n-1}$. We call it the Newton polyhedron of f at the origin $p = 0 \in \mathbb{C}^{n-1}$.

DEFINITION 3.5. (see [14], [23], etc.) We say that f is Newton nondegenerate at the origin $p = 0 \in \mathbb{C}^{n-1}$ if for any compact face $\gamma \prec \Gamma_+(f)$ of $\Gamma_+(f)$ the hypersurface $\{y \in (\mathbb{C}^*)^{n-1} \mid f^\gamma(y) = 0\}$ of $(\mathbb{C}^*)^{n-1}$ is smooth and reduced.

For each subset $I \subset \{1, 2, \ldots, n-1\}$, we set

$$\mathbb{R}_+^I = \{v = (v_1, \ldots, v_{n-1}) \in \mathbb{R}_+^{n-1} \mid v_i = 0 \text{ for any } i \notin I\} \simeq \mathbb{R}_+^{\sharp I}. \tag{3.25}$$

Let $\gamma_1^I, \ldots, \gamma_{n(I)}^I \prec \Gamma_+(f) \cap \mathbb{R}_+^I$ be the compact facets of $\Gamma_+(f) \cap \mathbb{R}_+^I$. For $1 \leqslant i \leqslant n(I)$, denote by $d_i^I \in \mathbb{Z}_{>0}$ the lattice distance of γ_i^I from the origin $0 \in \mathbb{R}_+^I$, and let $u_i^I = (u_{i,1}^I, \ldots, u_{i,n-1}^I) \in \mathbb{R}_+^I \cap \mathbb{Z}^{n-1}$ be the unique (nonzero) primitive vector which takes its minimum exactly on γ_i^I. For simplicity, we set $\delta_i^I := u_{i,n-1}^I$. Finally, we define a finite subset $E_p \subset \mathbb{C}$ of $\mathbb{C}$ by

$$E_p = \bigcup_{I : I \ni n-1} \bigcup_{i=1}^{n(I)} \{\lambda \in \mathbb{C} \mid \lambda^{d_i^I} = \exp(2\pi\sqrt{-1}\beta \cdot \delta_i^I)\}. \tag{3.26}$$

Then, the following result is a special case of [18, Theorem 5.5].

PROPOSITION 3.6. *In the above situation, assume moreover that f is Newton nondegenerate at the origin $p = 0 \in \mathbb{C}^{n-1}$. Then, the set of the eigenvalues of the monodromy operator $\Phi : H^{n-2}\psi_f(\mathcal{N})_p \xrightarrow{\sim} H^{n-2}\psi_f(\mathcal{N})_p$ is contained in E_p.*

COROLLARY 3.7. *Assume that $\dim \Delta = n-1$, $c \in \mathbb{C}^n$ is nonresonant, $\exp(-2\pi\sqrt{-1}\alpha) = \exp(2\pi\sqrt{-1}c_n) \notin E_p$ and f is Newton nondegenerate at the origin $p = 0 \in \mathbb{C}^{n-1}$. Then, we have $(Ri_*\mathcal{M})_p \simeq 0$.*

In fact, by [18, Theorem 5.5], we can generalize this corollary to the case where the codimension of the T_0-orbit X_γ in $X_\gamma \subset X \setminus i(T_0)$ containing the (stratified) isolated singular point p of S is larger than one. We leave the precise formulation to the reader and omit the details here. In this way, our Theorem 3.3 can be generalized to the case where S has (stratified) isolated

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

singular points p also in T_0-orbits $X_\gamma \subset X \setminus i(T_0)$. In particular, we have the following result. For a face γ of Δ, let $L_\gamma \simeq \mathbb{R}^{\dim \gamma}$ be the linear subspace of $\mathbb{R}^{n-1}$ parallel to the affine span of γ in $\mathbb{R}^{n-1}$, and consider the γ-part P^γ of P as a function on $T_\gamma = \mathrm{Spec}(\mathbb{C}[L_\gamma \cap \mathbb{Z}^{n-1}]) \simeq (\mathbb{C}^*)^{\dim \gamma}$.

THEOREM 3.8. *Assume that* $\dim \Delta = n-1$, *and for any face* γ *of* Δ, *the hypersurface* $(P^\gamma)^{-1}(0) \subset T_\gamma$ *of* T_γ *has only isolated singular points. Then, for generic parameter vectors* $c \in \mathbb{C}^n$, *we have the concentration*

$$H^j(W; \mathcal{L}) \simeq 0 \quad (j \neq n-1). \tag{3.27}$$

From now on, let us generalize Theorem 3.3 to the following more general situation. For $0 < k < n$, let $B_i = \{b_i(1), b_i(2), \ldots, b_i(N_i)\} \subset \mathbb{Z}^{n-k}$ $(1 \leqslant i \leqslant k)$ be k finite subsets of the lattice $\mathbb{Z}^{n-k}$, and set $N = N_1 + N_2 + \cdots + N_k$. For $1 \leqslant i \leqslant k$ and $(z_{i1}, \ldots, z_{iN_i}) \in \mathbb{C}^{N_i}$, we define a Laurent polynomial $P_i(x)$ on $T_0 = (\mathbb{C}^*)^{n-k}$ by $P_i(x) = \sum_{j=1}^{N_i} z_{ij} x^{b_i(j)}$ $(x = (x_1, \ldots, x_{n-k}) \in T_0 = (\mathbb{C}^*)^{n-k})$. Let us set $W = T_0 \setminus \bigcup_{i=1}^k P_i^{-1}(0)$. Then, for $c = (c_1, \ldots, c_{n-k}, \widetilde{c_1}, \ldots, \widetilde{c_k}) \in \mathbb{C}^n$, the possibly multivalued function

$$P_1(x)^{-\widetilde{c_1}} \cdots P_k(x)^{-\widetilde{c_k}} x_1^{c_1 - 1} \cdots x_{n-k}^{c_{n-k}-1} \tag{3.28}$$

on W generates the local system

$$\mathcal{L} = \mathbb{C}_W P_1(x)^{-\widetilde{c_1}} \cdots P_k(x)^{-\widetilde{c_k}} x_1^{c_1 - 1} \cdots x_{n-k}^{c_{n-k}-1}. \tag{3.29}$$

Let $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^k$ $(1 \leqslant i \leqslant k)$ be the standard basis of $\mathbb{Z}^k$, and set $a_i(j) = (b_i(j), e_i) \in \mathbb{Z}^{n-k} \times \mathbb{Z}^k = \mathbb{Z}^n$ $(1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant N_i)$ and

$$A = \{a_1(1), \ldots, a_1(N_1), \ldots\ldots, a_k(1), \ldots, a_k(N_k)\} \subset \mathbb{Z}^n. \tag{3.30}$$

For $1 \leqslant i \leqslant k$, let $\Delta_i \subset \mathbb{R}^{n-k}$ be the convex hull of B_i in $\mathbb{R}^{n-k}$. Denote by $\Delta \subset \mathbb{R}^{n-k}$ their Minkowski sum $\Delta_1 + \cdots + \Delta_k$. Assume that $\dim \Delta = n-k$. Then, by using the n-dimensional closed convex polyhedral cone $K = \mathbb{R}_+ A \subset \mathbb{R}^n$ generated by A in $\mathbb{R}^n$, we can define the nonresonance of the parameter $c \in \mathbb{C}^n$ as in Definition 3.1. For a face $\gamma \prec \Delta$ of Δ, let $\gamma_i \prec \Delta_i$ be the faces of Δ_i $(1 \leqslant i \leqslant k)$ canonically associated to γ such that $\gamma = \gamma_1 + \cdots + \gamma_k$.

DEFINITION 3.9. We say that the k-tuple of the Laurent polynomials $(P_1, \ldots, P_k)$ is "weakly" (resp. "strongly") nondegenerate if for any face γ of Δ such that $\dim \gamma < \dim \Delta = n-k$ (resp. $\dim \gamma \leqslant \dim \Delta = n-k$) and

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

nonempty subset $J \subset \{1, 2, \dots, k\}$, the subvariety

$$\{x \in T_0 = (\mathbb{C}^*)^{n-k} \mid P_i^{\gamma_i}(x) = 0 \ (i \in J)\} \subset T_0 \tag{3.31}$$

is a nondegenerate complete intersection.

REMARK 3.10. Denote the convex hull of $\bigcup_{i=1}^k (\Delta_i \times \{e_i\}) \subset \mathbb{R}^{n-k} \times \mathbb{R}^k = \mathbb{R}^n$ in $\mathbb{R}^n$ by $\Delta_1 * \cdots * \Delta_k$. Then, $\Delta_1 * \cdots * \Delta_k$ is naturally identified with the Newton polytope of the Laurent polynomial $R(x, t) = \sum_{i=1}^k P_i(x) t_i$ on $\widetilde{T_0} := T_0 \times (\mathbb{C}^*)_t^k \simeq (\mathbb{C}^*)_{x,t}^n$. In [10], the authors considered the condition that for any face γ of $\Delta_1 * \cdots * \Delta_k$, the hypersurface $\{(x, t) \in \widetilde{T_0} \mid R^\gamma(x, t) = 0\} \subset \widetilde{T_0}$ of $\widetilde{T_0}$ is smooth and reduced. It is easy to see that our strong nondegeneracy of the k-tuple $(P_1, \dots, P_k)$ in Definition 3.9 is equivalent to their condition.

Let $\iota : W = T_0 \setminus \bigcup_{i=1}^k P_i^{-1}(0) \longrightarrow T_0$ be the inclusion map, and set $\mathcal{M} = R\iota_* \mathcal{L} \in \mathbf{D}_c^b(T_0)$.

THEOREM 3.11. *Assume that* $\dim \Delta = n - k$, *the parameter vector* $c \in \mathbb{C}^n$ *is nonresonant and* $(P_1, \dots, P_k)$ *is weakly nondegenerate. Then, there exists an isomorphism*

$$H_c^j(T_0; \mathcal{M}) \simeq H^j(T_0; \mathcal{M}) \simeq H^j(W; \mathcal{L}) \tag{3.32}$$

for any $j \in \mathbb{Z}$. *Moreover, we have the concentration*

$$H^j(W; \mathcal{L}) \simeq 0 \quad (j \neq n - k). \tag{3.33}$$

Proof. The proof is similar to that of Theorem 3.3. Let Σ_0 be the dual fan of Δ in $\mathbb{R}^{n-k}$, and let X be the (possibly singular) toric variety associated to it. For a face γ of Δ, we denote by $X_\gamma \simeq (\mathbb{C}^*)^{\dim \gamma}$ the T_0-orbit associated to γ. Let $i : X_\Delta \simeq T_0 \hookrightarrow X$ be the inclusion map. Then, by the weak nondegeneracy of the k-tuple $(P_1, \dots, P_k)$, for any T_0-orbits X_γ in $X \setminus X_\Delta$ and the closure $S = \overline{i(\bigcup_{i=1}^k P_i^{-1}(0))} \subset X$ of the hypersurface $i(\bigcup_{i=1}^k P_i^{-1}(0)) \subset i(T_0)$ in X, their intersection $S \cap X_\gamma \subset X_\gamma$ is a normal crossing divisor in X_γ. In fact, S itself is normal crossing on a neighborhood of such X_γ, and any irreducible component of it intersects X_γ transversely. Moreover, by the nonresonance of $c \in \mathbb{C}^n$, for any $\gamma \prec \Delta$ such that $\dim \gamma = n - k - 1$, the monodromy of the local system $\mathcal{L}$ around the codimension-one T_0-orbit $X_\gamma \subset X$ in X is nontrivial. Indeed, let $\gamma \prec \Delta$ be such a facet of Δ, and let $\gamma_i \prec \Delta_i$ be the faces of Δ_i $(1 \leqslant i \leqslant k)$ associated to γ

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

such that $\gamma = \gamma_1 + \cdots + \gamma_k$. We denote the convex hull of $\bigcup_{i=1}^k (\Delta_i \times \{e_i\})$ (resp. $\bigcup_{i=1}^k (\gamma_i \times \{e_i\})) \subset \mathbb{R}^{n-k} \times \mathbb{R}^k = \mathbb{R}^n$ in $\mathbb{R}^n$ by $\Delta_1 * \cdots * \Delta_k$ (resp. $\gamma_1 * \cdots * \gamma_k$). Then, $\Delta_1 * \cdots * \Delta_k$ is the join of $\Delta_1, \ldots, \Delta_k$ and $\gamma_1 * \cdots * \gamma_k$ is its facet. We denote by Γ the facet of the cone $K = \mathbb{R}_+ A$ generated by $\gamma_1 * \cdots * \gamma_k \subset K$. Let $\nu \in \mathbb{Z}^{n-k} \setminus \{0\}$ be the primitive inner conormal vector of the facet γ of $\Delta \subset \mathbb{R}^{n-k}$, and for $1 \leqslant i \leqslant k$ set

$$m_i = \min_{v \in \Delta_i} \langle \nu, v \rangle = \min_{v \in \gamma_i} \langle \nu, v \rangle \in \mathbb{Z}. \tag{3.34}$$

Then, the primitive inner conormal vector $\widetilde{\nu} \in \mathbb{Z}^n \setminus \{0\}$ of the facet Γ of $K \subset \mathbb{R}^n$ is explicitly given by the formula

$$\widetilde{\nu} = \begin{pmatrix} \nu \\ -m_1 \\ \vdots \\ -m_k \end{pmatrix} \in \mathbb{Z}^n \setminus \{0\}, \tag{3.35}$$

and the condition $c = (c_1, \ldots, c_{n-k}, \widetilde{c_1}, \ldots, \widetilde{c_k}) \notin \{\mathbb{Z}^n + \mathrm{Lin}(\Gamma)\}$ is equivalent to the one

$$m(\gamma) := \left\langle \nu, \begin{pmatrix} c_1 - 1 \\ \vdots \\ c_{n-k} - 1 \end{pmatrix} \right\rangle - \sum_{i=1}^k m_i \cdot \widetilde{c_i} \quad \notin \mathbb{Z}. \tag{3.36}$$

Moreover, we can easily see that the order of the (multivalued) function

$$P_1(x)^{-\widetilde{c_1}} \cdots P_k(x)^{-\widetilde{c_k}} x_1^{c_1 - 1} \cdots x_{n-k}^{c_{n-k} - 1} \tag{3.37}$$

along the codimension-one T_0-orbit $X_\gamma \subset X$ in X is equal to $m(\gamma) \notin \mathbb{Z}$. Finally, by constructing suitable distance functions as in the proof of [7, Lemma 4.2], we can show that

$$(Ri_* \mathcal{M})_p \simeq 0 \quad \text{for any } p \in X \setminus T_0. \tag{3.38}$$

Namely, there exists an isomorphism $i_! \mathcal{M} \simeq Ri_* \mathcal{M}$ in $\mathbf{D}_c^b(X)$. Applying the functor $R\Gamma_c(X; \cdot) = R\Gamma(X; \cdot)$ to it, we obtain the desired isomorphisms

$$H_c^j(T_0; \mathcal{M}) \simeq H^j(T_0; \mathcal{M}) \simeq H^j(W; \mathcal{L}) \tag{3.39}$$

for $j \in \mathbb{Z}$. Then, the remaining assertion can be proved as in the proof of Theorem 3.3. This completes the proof. ∎

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

In the situation of Theorem 3.11, for any $1 \leqslant i \leqslant k$, the hypersurface $P_i^{-1}(0) \subset T_0$ has only isolated singular points. Assume moreover that the hypersurface $\bigcup_{i=1}^{k} P_i^{-1}(0) \subset T_0$ is normal crossing outside them. Then, as in Corollary 3.4, by Theorem 2.4, we can also express the dimension of $H^{n-k}(W; \mathcal{L})$ in terms of some mixed volumes of the polytopes $\Delta_1, \ldots, \Delta_k$ and the Milnor numbers of the isolated singular points. Since the statement of this result is involved, we leave its precise formulation to the reader.

As in the case where $k = 1$, we have the following result. For a face γ of Δ, let $L_\gamma \simeq \mathbb{R}^{\dim \gamma}$ be the linear subspace of $\mathbb{R}^{n-k}$ parallel to the affine span of γ in $\mathbb{R}^{n-k}$, and for $1 \leqslant i \leqslant k$, consider the γ_i-part $P_i^{\gamma_i}$ of P_i as a function on $T_\gamma = \operatorname{Spec}(\mathbb{C}[L_\gamma \cap \mathbb{Z}^{n-k}]) \simeq (\mathbb{C}^*)^{\dim \gamma}$.

THEOREM 3.12. *Assume that* $\dim \Delta = n - k$, *and for any* $1 \leqslant i \leqslant k$, *the hypersurface* $P_i^{-1}(0) \subset T_0$ *of* T_0 *has only isolated singular points. Assume moreover that for any face* γ *of* Δ *such that* $\dim \gamma < \dim \Delta = n - k$ *and nonempty subset* $J \subset \{1, 2, \ldots, k\}$, *the* k*-tuple of the Laurent polynomials* $(P_1, \ldots, P_k)$ *satisfies the following condition.*

> *If* $J = \{i\}$ *for some* $1 \leqslant i \leqslant k$ *and* $\dim \gamma_i = \dim \gamma = \dim \Delta - 1 = n - k - 1$, *the hypersurface* $(P_i^{\gamma_i})^{-1}(0) \subset T_\gamma$ *of* T_γ *has only isolated singular points. Otherwise, the subvariety*
>
> $$\{x \in T_0 = (\mathbb{C}^*)^{n-k} \mid P_i^{\gamma_i}(x) = 0 \ (i \in J)\} \subset T_0 \tag{3.40}$$
>
> *of* T_0 *is a nondegenerate complete intersection.*

Then, for generic parameter vectors $c \in \mathbb{C}^n$, *we have the concentration*

$$H^j(W; \mathcal{L}) \simeq 0 \quad (j \neq n - k). \tag{3.41}$$

Proof. Let Σ_0 be the dual fan of Δ in $\mathbb{R}^{n-k}$, and let X be the (possibly singular) toric variety associated to it. Then, our assumptions imply that for any $1 \leqslant i \leqslant k$, the hypersurface $S_i = \overline{i(P_i^{-1}(0))} \subset X$ has only stratified isolated singular points in X, and we can prove the assertion following the proofs of Theorems 3.8 and 3.11. $\square$

For a face γ of Δ and $1 \leqslant i \leqslant k$ such that $\dim \gamma_i < \dim \gamma \leqslant n - k - 1$, the hypersurface $(P_i^{\gamma_i})^{-1}(0) \subset T_\gamma$ of T_γ is smooth or has nonisolated singularities. In the latter case, we cannot prove the concentration in Theorem 3.12 by our methods. This is the reason why we do not allow such cases in our assumptions of Theorem 3.12. However, in the very special case where the

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

Newton polytopes $\Delta_1, \ldots, \Delta_k$ are similar to each other, we do not have this problem and obtain the following simpler result.

THEOREM 3.13. *Assume that* $\dim \Delta = n - k$*, the Newton polytopes* $\Delta_1, \ldots, \Delta_k$ *are similar to each other, and for any face* γ *of* Δ *and* $1 \leqslant i \leqslant k$*, the hypersurface* $(P_i^{\gamma_i})^{-1}(0) \subset T_\gamma$ *of* T_γ *has only isolated singular points. Assume moreover that for any face* γ *of* Δ *such that* $\dim \gamma < \dim \Delta = n - k$ *and any subset* $J \subset \{1, 2, \ldots, k\}$ *such that* $\sharp J \geqslant 2$*, the subvariety*

$$\{x \in T_0 = (\mathbb{C}^*)^{n-k} \mid P_i^{\gamma_i}(x) = 0 \ (i \in J)\} \subset T_0 \tag{3.42}$$

of T_0 *is a nondegenerate complete intersection. Then, for generic parameter vectors* $c \in \mathbb{C}^n$*, we have the concentration*

$$H^j(W; \mathcal{L}) \simeq 0 \quad (j \neq n - k). \tag{3.43}$$

§4. Some results on the twisted Morse theory

In this section, we prepare some auxiliary results on the twisted Morse theory which will be used in Section 5. The following proposition is a refinement of the results in [5, page 10]. See also [7, Proposition 7.1].

PROPOSITION 4.1. *Let* T *be an algebraic torus* $(\mathbb{C}^*)^n_x$*, and let* $T = \sqcup_\alpha Z_\alpha$ *be its algebraic stratification. In particular, we assume that each stratum* Z_α *in it is smooth. Let* $h(x)$ *be a Laurent polynomial on* $T = (\mathbb{C}^*)^n_x$ *such that the hypersurface* $\{h = 0\} \subset T$ *intersects* Z_α *transversely for any* α*. For* $a \in \mathbb{C}^n$*, consider the (possibly multivalued) function* $g_a(x) := h(x)x^{-a}$ *on* T*. Then, there exists a nonempty Zariski open subset* $\Omega \subset \mathbb{C}^n$ *of* $\mathbb{C}^n$ *such that the restriction* $g_a|_{Z_\alpha} : Z_\alpha \longrightarrow \mathbb{C}$ *of* g_a *to* Z_α *has only isolated nondegenerate (i.e., Morse type) critical points for any* $a \in \Omega \subset \mathbb{C}^n$ *and* α*.*

Proof. We may assume that each stratum Z_α is connected. We fix a stratum Z_α and set $k = \dim Z_\alpha$. For a subset $I \subset \{1, 2, \ldots, n\}$ such that $|I| = k = \dim Z_\alpha$, denote by $\pi_I : T = (\mathbb{C}^*)^n_x \longrightarrow (\mathbb{C}^*)^k$ the projection associated to I. We also denote by $Z_{\alpha,I} \subset Z_\alpha$ the maximal Zariski open subset of Z_α such that the restriction of π_I to it is locally biholomorphic. By the implicit function theorem, the variety Z_α is covered by such open subsets $Z_{\alpha,I}$. For simplicity, let us consider the case where $I = \{1, 2, \ldots, k\} \subset \{1, 2, \ldots, n\}$. Then, we may regard $g_a|_{Z_\alpha}$ locally as a function $g_{a,\alpha,I}(x_1, \ldots, x_k)$ on the Zariski open subset $\pi_I(Z_{\alpha,I}) \subset (\mathbb{C}^*)^k$ of the form

$$g_{a,\alpha,I}(x_1, \ldots, x_k) = \frac{h_{a,\alpha,I}(x_1, \ldots, x_k)}{x_1^{a_1} \cdots x_k^{a_k}}. \tag{4.1}$$

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

By our assumption, the hypersurface $\{h_{a,\alpha,I}=0\}\subset\pi_I(Z_{\alpha,I})\subset(\mathbb{C}^*)^k$ is smooth. Then, as in the proof of [7, Proposition 7.1], we can show that there exists a nonempty Zariski open subset $\Omega_{\alpha,I}\subset\mathbb{C}^n$ such that the (possibly multivalued) function $g_{a,\alpha,I}(x_1,\dots,x_k)$ on $\pi_I(Z_{\alpha,I})\subset(\mathbb{C}^*)^k$ has only isolated nondegenerate (i.e., Morse type) critical points for any $a\in\Omega_{\alpha,I}\subset\mathbb{C}^n$. This completes the proof. ∎

COROLLARY 4.2. *In the situation of Proposition 4.1, assume moreover that for the Newton polytope $NP(h)\subset\mathbb{R}^n$ of h, we have* $\dim NP(h)=n$. *Then, there exists $a\in\operatorname{Int}NP(h)$ such that the restriction $g_a|_{Z_\alpha}:Z_\alpha\longrightarrow\mathbb{C}$ of g_a to Z_α has only isolated nondegenerate (i.e., Morse type) critical points for any α.*

Now, let $Q_1,\dots,Q_l$ be Laurent polynomials on $T=(\mathbb{C}^*)^n$, and for $1\leqslant i\leqslant l$, denote by $\Delta_i\subset\mathbb{R}^n$ the Newton polytope $NP(Q_i)$ of Q_i. Set $\Delta=\Delta_1+\cdots+\Delta_l$. Then, by Corollary 4.2, we obtain the following result which might be of independent interest.

THEOREM 4.3. *Let $\mathcal{L}$ be a nontrivial local system of rank one on $T=(\mathbb{C}^*)^n$. Assume that for any $1\leqslant i\leqslant l$, we have* $\dim\Delta_i=n$, *and the subvariety*

$$Z_i=\{x\in T\mid Q_1(x)=\cdots=Q_i(x)=0\}\subset T \tag{4.2}$$

of T is a nondegenerate complete intersection. Then, for any $1\leqslant i\leqslant l$, we have the concentration

$$H^j(Z_i;\mathcal{L})\simeq 0\quad (j\neq n-i). \tag{4.3}$$

Moreover, we have

$$\dim H^{n-i}(Z_i;\mathcal{L})=\sum_{\substack{m_1,\dots,m_i\geqslant 1\\ m_1+\cdots+m_i=n}}\operatorname{Vol}_{\mathbb{Z}}(\underbrace{\Delta_1,\dots,\Delta_1}_{m_1\text{-times}},\dots,\underbrace{\Delta_i,\dots,\Delta_i}_{m_i\text{-times}}). \tag{4.4}$$

Proof. We prove the assertion by induction on i. For $i=0$, we have $Z_i=T$ and the assertion is obvious. Since $Z_i\subset T$ is affine, by Artin's vanishing theorem, we have the concentration

$$H^j(Z_i;\mathcal{L})\simeq 0\quad (j>n-i=\dim Z_i). \tag{4.5}$$

On the other hand, by Corollary 4.2, there exists $a_i\in\operatorname{Int}NP(Q_i)\subset\mathbb{R}^n$ such that the real-valued function

$$g_i:Z_{i-1}\longrightarrow\mathbb{R},\qquad x\longmapsto|Q_i(x)x^{-a_i}| \tag{4.6}$$

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

has only isolated nondegenerate (Morse type) critical points. Note that the Morse index of g_i at each critical point is $\dim Z_{i-1} = n - i + 1$. Let Σ_0 be the dual fan of the n-dimensional polytope Δ in $\mathbb{R}^n$, and let Σ be its smooth subdivision. We denote by X_Σ the toric variety associated to Σ. Then, X_Σ is a smooth compactification of T such that $D = X_\Sigma \setminus T$ is a normal crossing divisor in it. By our assumption, the closures $\overline{Z_{i-1}}, \overline{Z_i} \subset X_\Sigma$ of Z_{i-1}, Z_i in X_Σ are smooth. Moreover, they intersect D, etc. transversely. Let U be a sufficiently small tubular neighborhood of $\overline{Z_i} \cap D$ in $\overline{Z_{i-1}}$. Then, by [28, Section 3.5] (see also [16]), for any $t \in \mathbb{R}_+$, there exist isomorphisms

$$H^j(\{g_i < t\}; \mathcal{L}) \simeq H^j(\{g_i < t\} \setminus U; \mathcal{L}) \quad (j \in \mathbb{Z}). \tag{4.7}$$

Moreover, the level set $g_i^{-1}(t) \cap (Z_{i-1} \setminus U)$ of g_i in $Z_{i-1} \setminus U$ is compact in Z_{i-1} and intersects ∂U transversely for any $t \in \mathbb{R}_+$. Hence, for $t \gg 0$, we have isomorphisms

$$H^j(\{g_i < t\}; \mathcal{L}) \simeq H^j(Z_{i-1}; \mathcal{L}) \quad (j \in \mathbb{Z}). \tag{4.8}$$

Moreover, for $0 < t \ll 1$, we have isomorphisms

$$H^j(\{g_i < t\}; \mathcal{L}) \simeq H^j(Z_i; \mathcal{L}) \quad (j \in \mathbb{Z}). \tag{4.9}$$

When $t \in \mathbb{R}$ decreases passing through one of the critical values of g_i, only the dimensions of $H^{n-i+1}(\{g_i < t\}; \mathcal{L})$ and $H^{n-i}(\{g_i < t\}; \mathcal{L})$ may change, and the other cohomology groups $H^j(\{g_i < t\}; \mathcal{L})$ $(j \neq n - i + 1, n - i)$ remain the same. Then, by our induction hypothesis for $i - 1$ and Equation (4.5), we obtain the desired concentration

$$H^j(Z_i; \mathcal{L}) \simeq 0 \quad (j \neq n - i). \tag{4.10}$$

Moreover, the last assertion follows from Theorem 2.4. This completes the proof. $\square$

From now on, assume also that the l-tuple $(Q_1, \ldots, Q_l)$ is strongly nondegenerate, and $\dim \Delta_l = n$. Let $T = \bigsqcup_\alpha Z_\alpha$ be the algebraic stratification of T associated to the hypersurface $S = \bigcup_{i=1}^{l-1} Q_i^{-1}(0) \subset T$, and set $M = T \setminus S$. Then, by Corollary 4.2, there exists $a \in \operatorname{Int}(\Delta_l)$ such that the restriction of the (possibly multivalued) function $Q_l(x)x^{-a}$ to Z_α has only isolated nondegenerate (i.e., Morse type) critical points for any α. In particular, it has only stratified isolated singular points. We fix such $a \in \operatorname{Int}(\Delta_l)$ and define a real-valued function $g: T \longrightarrow \mathbb{R}_+$ by $g(x) = |Q_l(x)x^{-a}|$. For $t \in \mathbb{R}_+$, we set also

$$M_t = \{x \in M = T \setminus S \mid g(x) < t\} \subset M. \tag{4.11}$$

Then, we have the following result.

LEMMA 4.4. *Let $\mathcal{L}$ be a local system on $M = T \setminus S$. Then, for any $c > 0$, there exists a sufficiently small $0 < \varepsilon \ll 1$ such that we have the concentration*

$$H^j(M_{c+\varepsilon}, M_{c-\varepsilon}; \mathcal{L}) \simeq 0 \quad (j \neq n). \tag{4.12}$$

Proof. Let Σ_0 be the dual fan of the n-dimensional polytope Δ in $\mathbb{R}^n$, and let Σ be its smooth subdivision. We denote by X_Σ the toric variety associated to Σ. Then, X_Σ is a smooth compactification of T such that $D = X_\Sigma \setminus T$ is a normal crossing divisor in it. By the strong nondegeneracy of $(Q_1, \ldots, Q_l)$, the hypersurface $\overline{Q_l^{-1}(0)} \subset X_\Sigma$ intersects D, etc. transversely. Let U be a sufficiently small tubular neighborhood of $\overline{Q_l^{-1}(0)} \cap D$ in X_Σ, and for $t \in \mathbb{R}_+$, set $M'_t = M_t \setminus U$. Then, by [28, Section 3.5], for any $t \in \mathbb{R}_+$, there exist isomorphisms

$$H^j(M_t; \mathcal{L}) \simeq H^j(M'_t; \mathcal{L}) \quad (j \in \mathbb{Z}). \tag{4.13}$$

Moreover, the level set $g^{-1}(t) \cap (T \setminus U)$ of g in $T \setminus U$ is compact in T and intersects ∂U transversely for any $t \in \mathbb{R}_+$. For $c > 0$, let $p_1, \ldots, p_r \in T \setminus g^{-1}(0) = T \setminus Q_l^{-1}(0)$ be the stratified isolated singular points of the function $h(x) = Q_l(x)x^{-a}$ in T such that $g(p_i) = |h(p_i)| = c$. Note that we have

$$g(x) = |h(x)| = \exp[\mathrm{Re}\{\log h(x)\}]. \tag{4.14}$$

Then, there exist small open balls B_i centered at p_i in T and $0 < \varepsilon \ll 1$ such that we have isomorphisms

$$H^j(M'_{c+\varepsilon}, M'_{c-\varepsilon}; \mathcal{L}) \simeq \bigoplus_{i=1}^{r} H^j(B_i \cap M_{c+\varepsilon}, B_i \cap M_{c-\varepsilon}; \mathcal{L}) \quad (j \in \mathbb{Z}). \tag{4.15}$$

For $1 \leqslant i \leqslant r$, by taking a local branch $\log h$ of the logarithm of the function $h \neq 0$ on a neighborhood of $p_i \in T \setminus h^{-1}(0)$, we set $f_i = \log h - \log h(p_i)$. Then, f_i has also a stratified isolated singular point at p_i. Let $F_i \subset B_i$ be the Milnor fiber of f_i at $p_i \in f_i^{-1}(0)$. Then, for any $1 \leqslant i \leqslant r$, by shrinking

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

B_i if necessary, we can easily prove the isomorphisms

$$H^j(B_i \cap M_{c+\varepsilon}, B_i \cap M_{c-\varepsilon}; \mathcal{L}) \simeq H^j(B_i \setminus S, F_i \setminus S; \mathcal{L}) \quad (j \in \mathbb{Z}). \tag{4.16}$$

Let $j : M = T \setminus S \hookrightarrow T$ be the inclusion. Since the Milnor fibers $F_i \subset B_i$ intersect each stratum Z_α transversely, we have also isomorphisms

$$H^j(B_i \setminus S, F_i \setminus S; \mathcal{L}) \simeq H^{j-1}\phi_{f_i}(Rj_*\mathcal{L})_{p_i} \quad (j \in \mathbb{Z}), \tag{4.17}$$

where ϕ_{f_i} are Deligne's vanishing cycle functors. Hence, by (the proof of) [4, Proposition 6.1.1], the assertion follows from

$$\operatorname{supp} \phi_{f_i}(Rj_*\mathcal{L}) \subset \{p_i\} \quad (1 \leqslant i \leqslant r) \tag{4.18}$$

and the fact that $Rj_*\mathcal{L}$ and $\phi_{f_i}(Rj_*\mathcal{L})$ are perverse sheaves (up to some shifts). This completes the proof. ∎

§5. A new vanishing theorem

Now, let $P_1, \ldots, P_k$ be Laurent polynomials on $T_0 = (\mathbb{C}^*)^{n-k}$, and for $1 \leqslant i \leqslant k$, denote by $\Delta_i \subset \mathbb{R}^{n-k}$ the Newton polytope $NP(P_i)$ of P_i. Set $\Delta = \Delta_1 + \cdots + \Delta_k$. Let us set $W = T_0 \setminus \bigcup_{i=1}^k P_i^{-1}(0)$, and for $(c, \widetilde{c}) = (c_1, \ldots, c_{n-k}, \widetilde{c_1}, \ldots, \widetilde{c_k}) \in \mathbb{C}^n$ consider the local system

$$\mathcal{L} = \mathbb{C}_W P_1(x)^{\widetilde{c_1}} \cdots P_k(x)^{\widetilde{c_k}} x_1^{c_1} \cdots x_{n-k}^{c_{n-k}} \tag{5.1}$$

on W.

THEOREM 5.1. *Assume that the k-tuple of the Laurent polynomials $(P_1, \ldots, P_k)$ is strongly nondegenerate, $(c, \widetilde{c}) = (c_1, \ldots, c_{n-k}, \widetilde{c_1}, \ldots, \widetilde{c_k}) \notin \mathbb{Z}^n$, and for any $1 \leqslant i \leqslant k$, we have $\dim \Delta_i = n - k$. Then, we have the concentration*

$$H^j(W; \mathcal{L}) \simeq 0 \quad (j \neq n-k). \tag{5.2}$$

Proof. Set $T = T_0 \times (\mathbb{C}^*)^k_{t_1,\ldots,t_k} \simeq (\mathbb{C}^*)^n_{x,t}$, and consider the Laurent polynomials

$$\widetilde{P_i}(x,t) = t_i - P_i(x) \quad (1 \leqslant i \leqslant k) \tag{5.3}$$

on T. For $1 \leqslant i \leqslant k$, we set also

$$Z_i = \{(x,t) \in T \mid \widetilde{P_1}(x,t) = \cdots = \widetilde{P_i}(x,t) = 0\}. \tag{5.4}$$

https://doi.org/10.1017/nmj.2017.8 Published online by Cambridge University Press

We define a local system $\widetilde{\mathcal{L}}$ on T by

$$\widetilde{\mathcal{L}} = \mathbb{C}_T x_1^{c_1} \cdots x_{n-k}^{c_{n-k}} t_1^{\widetilde{c_1}} \cdots t_k^{\widetilde{c_k}}. \tag{5.5}$$

Then, $Z_k \simeq W$, and we have isomorphisms

$$H^j(W; \mathcal{L}) \simeq H^j(Z_k; \widetilde{\mathcal{L}}) \quad (j \in \mathbb{Z}). \tag{5.6}$$

First, let us consider the case where $\widetilde{c} = (\widetilde{c_1}, \ldots, \widetilde{c_k}) \notin \mathbb{Z}^k$. In this case, without loss of generality, we may assume that $\widetilde{c_k} \notin \mathbb{Z}$. Then, by the Künneth formula, for $i = 1, 2, \ldots, k-1$, we have the vanishings

$$H^j(Z_i; \widetilde{\mathcal{L}}) \simeq 0 \quad (j \in \mathbb{Z}). \tag{5.7}$$

Moreover, we can naturally identify $Z_{k-1} \subset T$ with $(T_0 \setminus \bigcup_{i=1}^{k-1} P_i^{-1}(0)) \times \mathbb{C}^*_{t_k}$. Consider $\widetilde{P_k}$ as a Laurent polynomial on $T_1 = T_0 \times \mathbb{C}^*_{t_k} \simeq (\mathbb{C}^*)^{n-k+1}$. Note that we have $\dim NP(\widetilde{P_k}) = n - k + 1 = \dim T_1$. By taking a sufficiently generic

$$(a_1, \ldots, a_{n-k}, a_{n-k+1}) \in \mathrm{Int}\, NP(\widetilde{P_k}) \subset \mathbb{R}^{n-k+1}, \tag{5.8}$$

we define a real-valued function g on $T_1 = T_0 \times \mathbb{C}^*_{t_k}$ by

$$g(x, t_k) = \left| \widetilde{P_k}(x, t_k) \times x_1^{-a_1} \cdots x_{n-k}^{-a_{n-k}} t_k^{-a_{n-k+1}} \right|. \tag{5.9}$$

Then, by applying Lemma 4.4 to the Morse function $g : T_1 = T_0 \times \mathbb{C}^* \longrightarrow \mathbb{R}$ and arguing as in the proof of Theorem 4.3, we obtain the desired concentration

$$H^j(Z_k; \widetilde{\mathcal{L}}) \simeq 0 \quad (j \neq n-k). \tag{5.10}$$

The proof for the remaining case where $(c, \widetilde{c}) = (c_1, \ldots, c_{n-k}, \widetilde{c_1}, \ldots, \widetilde{c_k}) \notin \mathbb{Z}^n$ and $\widetilde{c} = (\widetilde{c_1}, \ldots, \widetilde{c_k}) \in \mathbb{Z}^k$ is similar. In this case, $Z_1 \subset T$ is isomorphic to the product $Z_1' \times (\mathbb{C}^*)^{k-1}$ for a hypersurface Z_1' in $T_0 \times \mathbb{C}^*_{t_1}$, and $\widetilde{\mathcal{L}}$ is isomoprhic to the pullback of a local system on $T_0 \times \mathbb{C}^*_{t_1}$. Hence, by the Künneth formula and the proof of Theorem 4.3, we obtain the concentration

$$H^j(Z_1; \widetilde{\mathcal{L}}) \simeq 0 \quad (j \neq n-k, \ldots, n-1). \tag{5.11}$$

Repeating this argument with the help of Lemma 4.4 and the proof of Theorem 4.3, we obtain also

$$H^j(W; \mathcal{L}) \simeq H^j(Z_k; \widetilde{\mathcal{L}}) \simeq 0 \quad (j \neq n-k, \ldots, n-1). \tag{5.12}$$

Then, the assertion is obtained by applying Artin's vanishing theorem to the $(n-k)$-dimensional affine variety $Z_k \subset T$. This completes the proof. $\square$

Acknowledgments. We express our hearty gratitude to Professor N. Takayama for drawing our attention to this problem. Moreover, some discussions with Professor M. Yoshinaga were very useful during the preparation of this paper. We also thank the referee for giving us many valuable suggestions.

References

[1] A. Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), 269–290.
[2] K. Ando, A. Esterov and K. Takeuchi, *Monodromies at infinity of confluent A-hypergeometric functions*, Adv. Math. **272** (2015), 1–19.
[3] K. Aomoto and M. Kita, *Theory of Hypergeometric Functions*, Monographs in Mathematics, Springer, Tokyo, Dordrecht, Heidelberg, London, New York, 2011.
[4] A. Dimca, *Sheaves in Topology*, Universitext, Springer, Berlin, 2004.
[5] A. Esterov, *Index of a real singular point and its Newton diagram*, Moscow Univ. Math. Bull. **58** (2003), 7–11.
[6] A. Esterov and K. Takeuchi, *Motivic Milnor fibers over complete intersection varieties and their virtual Betti numbers*, Int. Math. Res. Not. IMRN **2012**(15) (2012), 3567–3613.
[7] A. Esterov and K. Takeuchi, *Confluent A-hypergeometric functions and rapid decay homology cycles*, Amer. J. Math. **137** (2015), 365–409.
[8] W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, Princeton, 1993.
[9] I. M. Gelfand, M. Kapranov and A. Zelevinsky, *Hypergeometric functions and toral manifolds*, Funct. Anal. Appl. **23** (1989), 94–106.
[10] I. M. Gelfand, M. Kapranov and A. Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. **84** (1990), 255–271.
[11] I. M. Gelfand, M. Kapranov and A. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, New York, 1994.
[12] R. Hotta, K. Takeuchi and T. Tanisaki, *D-modules, Perverse Sheaves and Representation Theory*, Progress in Mathematics, Birkhäuser, Boston, 2008.
[13] A. G. Khovanskii, *Newton polyhedra and toroidal varieties*, Funct. Anal. Appl. **11** (1978), 289–296.
[14] A. G. Khovanskii, *Newton polyhedra and the genus of complete intersections*, Funct. Anal. Appl. **12** (1978), 38–46.
[15] A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), 1–31.
[16] A. Libgober and S. Sperber, *On the zeta function of monodromy of a polynomial map*, Compos. Math. **95** (1995), 287–307.
[17] Y. Matsui and K. Takeuchi, *A geometric degree formula for A-discriminants and Euler obstructions of toric varieties*, Adv. Math. **226** (2011), 2040–2064.
[18] Y. Matsui and K. Takeuchi, *Milnor fibers over singular toric varieties and nearby cycle sheaves*, Tohoku Math. J. **63** (2011), 113–136.
[19] Y. Matsui and K. Takeuchi, *Monodromy at infinity, Newton polyhedra and constructible sheaves*, Math. Z. **268** (2011), 409–439.
[20] Y. Matsui and K. Takeuchi, *Monodromy at infinity of polynomial maps and Newton polyhedra, with Appendix by C. Sabbah*, Int. Math. Res. Not. IMRN **2013**(8) (2013), 1691–1746.

[21] F. Matusevich, E. Miller and U. Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), 919–941.
[22] T. Oda, *Convex Bodies and Algebraic Geometry: An Introduction to The Theory of Toric Varieties*, Springer, Berlin, Heidelberg, 1988.
[23] M. Oka, *Non-Degenerate Complete Intersection Singularity*, Hermann, Paris, 1997.
[24] A. Pajitnov, *Circle-Valued Morse Theory*, de Gruyter Studies in Mathematics **32**, Walter de Gruyter, Berlin, 2006.
[25] F. Pham, *Le descente des cols par les onglets de Lefschetz, avec vues sur Gauss–Manin*, Astérisque **130** (1985), 14–47.
[26] M. Saito, B. Sturmfels and N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Springer, Berlin, Heidelberg, New York, 2000.
[27] K. Takeuchi, *Monodromy at infinity of A-hypergeometric functions and toric compactifications*, Math. Ann. **348** (2010), 815–831.
[28] A. Zaharia, *On the bifurcation set of a polynomial function and Newton boundary II*, Kodai Math. J. **19** (1996), 218–233.

Alexander Esterov
National Research University Higher School of Economics
Faculty of Mathematics
6 Usacheva
199048 Moscow
Russia
aesterov@hse.ru

Kiyoshi Takeuchi
Institute of Mathematics
University of Tsukuba
1-1-1, Tennodai
Tsukuba
Ibaraki 305-8571
Japan
takemicro@nifty.com