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MONOTONIC PHINOMIAL COEFFICIENTS

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Abstract

We investigate the monotonic characteristics of the generalised binomial coefficients (phinomials) based upon Euler's totient function. We show, unconditionally, that the set of integers for which this sequence is unimodal is finite and, assuming the generalised Riemann hypothesis, we find all the exceptions.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers and, for a function $f : \mathbb{N} \to \mathbb{N}$, define the generalised binomial coefficients

$$\binom{n}{k}_f = \frac{f(1)f(2)\cdots f(n)}{(f(1)f(2)\cdots f(k))(f(1)f(2)\cdots f(n-k))}.$$

It is known that under some conditions on f, these generalised binomial coefficients are integers. For example, Knuth and Wilf [3] showed that when f is strongly divisible, that is, $\gcd(f(m), f(\ell)) = f(\gcd(m, \ell))$, for all m, ℓ in $\mathbb{N}$, then $\binom{n}{k}_f \in \mathbb{N}$. The same result holds for the Euler totient function [1] and for multiplicative functions that are divisible, that is, if $m|\ell$, then $f(m)|f(\ell)$ (see [2]). For convenience, we shall introduce the notion of an f-*actorial* of an integer n, to be $n!_f = f(1)f(2)\cdots f(n)$, and so the generalised binomial coefficients can be written as

$$\binom{n}{k}_f = \frac{n!_f}{k!_f(n-k)!_f}.$$

Here, we take the function $f = \phi$, Euler's totient function, and call the corresponding generalised binomial coefficients, $\binom{n}{k}_\phi$, the *phinomials*. We display below the first few rows for $1 \le n \le 14$ of the Pascal-like triangle of phinomials:

$$
\begin{array}{c}
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 2 \quad 2 \quad 1 \\
1 \quad 2 \quad 4 \quad 2 \quad 1 \\
1 \quad 4 \quad 8 \quad 8 \quad 4 \quad 1 \\
1 \quad 2 \quad 8 \quad 8 \quad 8 \quad 2 \quad 1 \\
1 \quad 6 \quad 12 \quad 24 \quad 24 \quad 12 \quad 6 \quad 1 \\
1 \quad 4 \quad 24 \quad 24 \quad 48 \quad 24 \quad 24 \quad 4 \quad 1 \\
1 \quad 6 \quad 24 \quad 72 \quad 72 \quad 72 \quad 72 \quad 24 \quad 6 \quad 1 \\
1 \quad 4 \quad 24 \quad 48 \quad 144 \quad 72 \quad 144 \quad 48 \quad 24 \quad 4 \quad 1 \\
1 \quad 10 \quad 40 \quad 120 \quad 240 \quad 360 \quad 360 \quad 240 \quad 120 \quad 40 \quad 10 \quad 1 \\
1 \quad 4 \quad 40 \quad 80 \quad 240 \quad 240 \quad 720 \quad 240 \quad 240 \quad 80 \quad 40 \quad 4 \quad 1 \\
1 \quad 12 \quad 48 \quad 240 \quad 480 \quad 720 \quad 1440 \quad 1440 \quad 720 \quad 480 \quad 240 \quad 48 \quad 12 \quad 1 \\
1 \quad 6 \quad 72 \quad 144 \quad 720 \quad 720 \quad 2160 \quad 1440 \quad 2160 \quad 720 \quad 720 \quad 144 \quad 72 \quad 6 \quad 1
\end{array}
$$

While it is obvious that each row is symmetric, it is not clear just what might be the monotonic properties of the phinomials, and that is the objective of this paper. Recall that a sequence $\{x_k\}_{1\le k\le n}$ is called *unimodal* if there exists an index k_0 such that $x_1 \le x_2 \le \cdots \le x_{k_0} \ge x_{k_0+1} \ge \cdots \ge x_n$. In spite of the fact that the initial data suggests that for most values of n, the sequence formed by the $\binom{n}{k}_\phi$ with $1 \le k \le n$ is unimodal, we will in fact show that the set of such n is finite and, assuming the generalised Riemann hypothesis (GRH), we find all these exceptions.

2. The results

The next lemma combines [2, Corollaries 7 and 11] and gives a connection between the phinomials and the classical binomial coefficients.

Proposition 2.1. *We have*

$$\binom{n}{k}_\phi = \binom{n}{k} \prod_{\substack{p\le n\\ p\text{ prime}}} \Big(\frac{p-1}{p}\Big)^{\epsilon^p_{0,n,k}} = \prod_{\substack{p\le n\\ p\text{ prime}}} (p-1)^{\epsilon^p_{0,n,k}} p^{\mathfrak{t}^p_{n,k}-\epsilon^p_{0,n,k}},$$

where $\epsilon^p_{i,n,k} = \lfloor n/p^{i+1}\rfloor - \lfloor k/p^{i+1}\rfloor - \lfloor (n-k)/p^{i+1}\rfloor$ *and* $\mathfrak{t}^p_{n,k} = \sum_i \epsilon^p_{i,n,k}$ *is the sum of carries in the base-*p *sum of* n *and* $n-k$.

We shall make use of the following estimate.

Lemma 2.2. *Let* $n \ge 210$ *and let* q *be the smallest prime not dividing* $n+1$. *Then* $q < 2\log n$.

Proof. Assume this is not so. Then

$$n+1 \ge \prod_{p\le 2\log n} p.$$

Taking logarithms,

$$\log(n+1) \geq \sum_{p \leq 2\log n} \log p := \theta(2\log n),$$

where θ is the Chebyshev function. Inequality [4, (3.16)] says that

$$\theta(x) > x(1 - 1/\log x) \quad \text{holds for all } x \geq 41.$$

Imposing the condition that $2\log n \geq 41$ (which holds for, say, $n \geq 8 \times 10^8$) yields

$$\log(n+1) \geq \theta(2\log n) \geq 2(\log n)\Big(1 - \frac{1}{\log(2\log n)}\Big),$$

and the right-most inequality is false for $n \geq 8 \times 10^8$. Hence, in fact $n \leq 8 \times 10^8$, so $2\log n \leq 41$. We now write

$$n + 1 = m \prod_{j=1}^{k} p_j,$$

where p_j is the jth prime, for all $k = 1, 2, \ldots, 13$ (note that $p_{13} = 41$), and $m \geq 1$ is a positive integer assumed not to be a multiple of p_{k+1} and test numerically the condition

$$p_{k+1} \geq 2 \sum_{j \leq k} \log p_j.$$

This yields $k \leq 4$. Then, for $k = 1, 2, 3, 4$, we test for the largest m such that

$$p_{k+1} - 2 \sum_{j=1}^{k} \log p_j \geq 2\log m.$$

This gives $m = 1$. Hence, the only numbers n failing the condition $q < 2\log n$ are the numbers of the form

$$n = \prod_{j=1}^{k} p_j - 1 \quad \text{for } k = 1, 2, 3, 4.$$

This completes the proof. □

We now state our main result.

THEOREM 2.3.

(i) *Let $j \geq 1$. There exists a positive integer n_j such that, if $n \geq n_j$, then the sequence of phinomials*

$$\left\{ \binom{n}{k}_{\phi} \right\}_{0 \leq k \leq n}$$

has at least j local maxima and j local minima.

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(ii) *Under the generalised Riemann hypothesis (GRH), the only values of n such that the sequence of phinomials*

$$\left\{\binom{n}{k}_{\phi}\right\}_{0\le k\le n}$$

is unimodal are

$$1,2,3,4,5,6,7,8,9,11,12,13,15,17,19,21,23,29,$$
$$31,35,41,43,53,59,71,83,89,119,161,209,239.$$

Proof. Let $n \ge 210$ be sufficiently large in a way that we will make precise later. Let k be such that $k+1 \le n/2$. The inequality $\binom{n}{k}_{\phi} < \binom{n}{k+1}_{\phi}$ is equivalent to

$$\phi(k+1) < \phi(n-k). \tag{2.1}$$

Let q be the smallest prime not dividing $n+1$. Then $q < 2\log n$ by Lemma 2.2. Let $k+1$ be any prime $p \equiv n+1 \pmod q$ in the interval

$$\mathcal{I} = \left[\frac{n}{2}+1-\frac{n}{10\log n}, \frac{n}{2}\right].$$

We shall show that for such k the inequality opposite of the inequality (2.1) holds, namely

$$\phi(k+1) > \phi(n-k). \tag{2.2}$$

Note that $n-k = (n+1)-p = qm$ for some positive integer m. Then the inequality (2.2) is implied by

$$p-1 = \phi(k+1) > \left(1-\frac{1}{q}\right)(n-k) = \left(1-\frac{1}{q}\right)(n+1-p),$$

which in turn is implied by

$$\frac{(p-1)/n}{1-(p-1)/n} \ge 1-\frac{1}{2\log n}.$$

The function $x \mapsto x/(1-x)$ is increasing for $x \in (0,1)$ and, since $p \in \mathcal{I}$, the inequality $(p-1)/n \ge 1/2 - 1/(10\log n)$ holds. It suffices that

$$\frac{1/2-1/(10\log n)}{1/2+1/(10\log n)} > 1-\frac{1}{2\log n}.$$

Algebraic manipulations show that the above inequality is satisfied for $\log n > 2$, which is certainly true in our range for n.

Now let

$$\mathcal{J} = \left[\frac{n}{2}+1, \frac{n}{2}+\frac{n}{10\log n}\right]$$

and let r be a prime in $\mathcal{J}$. On putting $n-k=r$, we have $k+1 = n-r+1 \in \mathcal{I}$, so

$$\phi(k+1) < k+1 \le n-k-1 = r-1 = \phi(n-k).$$

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Thus, for such values of k, namely those of the form $n - r$ with $r \in \mathcal{J}$ prime, the inequality (2.1) holds. Let us look at the numbers

$$p_1 < p_2 < \cdots < p_s \quad \text{and} \quad n - r_t + 1 < n - r_{t-1} + 1 < \cdots < n - r_1 + 1,$$

where $p_1 < \cdots < p_s$ are all the primes in $\mathcal{I}$ which are congruent to $n + 1 \pmod{q}$ and $r_1 < \cdots < r_t$ are all the primes in $\mathcal{J}$. As we have seen, if $k + 1$ is a number of the form p_i for some $1 \le i \le s$, then (2.2) holds, whereas if $k + 1$ is of the form $n - r_i + 1$ for some $1 \le i \le t$, then inequality (2.1) holds.

Now we are ready to prove (i) and (ii).

For (i), we use the Siegel–Walfitz theorem, which states that for all $A > 0$ there is a constant $C := C(A)$ such that whenever a and k are positive integers with $\gcd(a, k) = 1$ and $k < (\log x)^A$,

$$\pi(x; a, k) = \frac{\mathrm{Li}(x)}{\phi(k)} + O\Big(\frac{x}{\exp(C\sqrt{\log x})}\Big).$$

Here, as usual, $\pi(x; a, k)$ counts the number of prime numbers $p \le x$ with $p \equiv a \pmod{k}$ and

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

Imposing the condition that $n \ge 30$ yields

$$q \le 2\log n \le (\log(n/2 - n/(10\log n)))^2;$$

that is, the inequality $q \le (\log x)^A$ holds with $A = 2$ and for all $x \ge n/2 - n/(10\log n)$. Now put

$$y := \frac{n}{\exp(0.5C\sqrt{\log n})}$$

and let $x \in \mathcal{I}$. Then

$$\begin{aligned}\pi(x + y; n + 1, q) - \pi(x; n + 1, q) &= \frac{1}{\phi(q)} \int_x^{x+y} \frac{dt}{\log t} + O\Big(\frac{n}{\exp(C\sqrt{\log n})}\Big)\\ &> \frac{y}{3(\log n)^2}\end{aligned}$$

uniformly for $x \in \mathcal{I}$ provided that n is sufficiently large. In particular, every interval of length y in $\mathcal{I}$ contains one of the p_i. The same is true for intervals of length y containing one of the numbers of the form $n - r_i + 1$ because every interval of length y in $\mathcal{J}$ will contain at least $y/(2\log n)$ primes if n is sufficiently large. Hence, in the interval $\mathcal{I}$, the phinomial coefficients have at least

$$\frac{|\mathcal{I}|}{y} \ge \exp(0.25C\sqrt{\log n})$$

local maxima and local minima if n is sufficiently large, which proves (i). From the above proof, we conclude that an acceptable value of n_j is

$$n_j = \exp(C_1(\log j)^2)$$

with some absolute constant C_1.

To show our second claim (ii), we shall use Winckler's result [6, Theorem 1.2], which states that, assuming that the generalised Riemann hypothesis is true for the Riemann zeta function ζ_L of L, then, for all $x \geq 2$,

$$\Big|\pi_C(x) - \frac{|C|}{|G|}\mathrm{Li}(x)\Big| \leq \frac{|C|}{|G|}\sqrt{x}\Big(\Big(32 + \frac{181}{\log x}\Big)\log|d_L| + \Big(28\log x + 330 + \frac{1655}{\log x}\Big)n_L\Big), \tag{2.3}$$

where L/K is a Galois extension of number fields with Galois group G and conjugacy class C, d_L is the absolute discriminant of L, $n_L = [L : K]$ is the degree of L over K and $\pi_C(x)$ counts the number of prime ideals of K of norm $\leq x$ which do not ramify over L and whose Frobenius lifts to L are in C.

We will apply Winckler's result with $K = \mathbb{Q}$, $L = \mathbb{Q}(e^{2\pi i/q})$ and so $n_L = q - 1$. The Galois group G is cyclic and so $|C| = 1$. In fact, every conjugacy class C corresponds to a residue class c coprime to q. Furthermore, since q is prime, the only ramified prime in L is q itself. Thus, $\pi_C(x) = \pi(x; c, q) - 1$ for $x > q$. In addition, $d_L = (-1)^{(q-1)/2}q^{q-2}$ (see [5, Proposition 2.7]).

We need to estimate the number of primes in the intervals $\mathcal{I}, \mathcal{J}$ used in the first part of our proof. We assume that $n \geq 210$. Then $n/2 + 1 - n/(10\log n) > 2\log n$; therefore, by Lemma 2.2, $n/2 + 1 - n/(10\log n) > q$. Using Winckler's inequality (2.3) twice (for $x_1 := n/2$ and for $x_2 := n/2 + 1 - n/(10\log n)$), we find that under the GRH, the number of primes in $\mathcal{I}$ which are congruent to $n + 1 \pmod q$ is

$$\begin{aligned}
&\pi\Big(\frac{n}{2}; n+1, q\Big) - \pi_C\Big(\frac{n}{2} + 1 - \frac{n}{10\log n}; n+1, q\Big)\\
&\quad\geq \frac{1}{q-1}\int_{n/2+1-n/10\log n}^{n/2}\frac{dt}{\log t} - B_1 - B_2 \geq \frac{n - 10\log n}{20\log^3 n} - B_1 - B_2\\
&\quad\geq \frac{n - 10\log n}{20\log^3 n} - \sqrt{2n}\Big(\Big(32 + \frac{181}{\log n}\Big)\log(2\log n) + 28\log n + 330 + \frac{1655}{\log n}\Big),
\end{aligned}$$

where we used the inequality $1/\log t \geq 1/\log n$ valid for all $t \in \mathcal{I}$ and the fact that $q < 2\log n$ by Lemma 2.2. Here,

$$\begin{aligned}
B_i &:= \sqrt{x_i}\Big(\Big(32 + \frac{181}{\log x_i}\Big)\frac{(q-2)\log q}{q-1} + \Big(28\log x_i + 330 + \frac{1655}{\log x_i}\Big)\Big)\\
&\leq \sqrt{x_i}\Big(\Big(32 + \frac{181}{\log x_i}\Big)\log(2\log n) + 28\log x_i + 330 + \frac{1655}{\log x_i}\Big)\\
&\leq \sqrt{\frac{n}{2}}\Big(\Big(32 + \frac{181}{\log n}\Big)\log(2\log n) + 28\log n + 330 + \frac{1655}{\log n}\Big)
\end{aligned}$$

(because the B_i are increasing functions of x_i in our range for n). Therefore, we need

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$$\frac{n - 10\log n}{20\log^3 n} \geq \sqrt{2n}\Big(\Big(32 + \frac{181}{\log n}\Big)\log(2\log n) + 28\log n + 330 + \frac{1655}{\log n}\Big),$$

which happens if $n \geq 1.1 \cdot 2^{64}$.

For the second interval $\mathcal{J}$, a similar argument with $\lambda_n := n/2 + n/(10\log n)$ yields the inequality

$$\frac{n - 10\log n}{20\log^3 n} \geq 2\sqrt{\lambda_n}\Big(\Big(32 + \frac{181}{\log \lambda_n}\Big)\log(2\log n) + 28\log \lambda_n + 330 + \frac{1655}{\log \lambda_n}\Big),$$

which again holds if $n \geq 1.1 \cdot 2^{64}$. In particular, we conclude that both $\mathcal{I}$ and $\mathcal{J}$ contain primes of the forms required at the beginning of our argument provided that $n > 2^{65}$.

Next, we need to cover computationally the range $n \leq 2^{65}$. For $k \leq 65$, we cover each interval $(2^{k-1}, 2^k]$ with subintervals of the form $\mathcal{L}_i := [2^{k-1} + iM + 1, 2^{k-1} + (i+1)M]$ for $i = 0, 1, \ldots,$ where we take $M := \lfloor 2^{k-1}/(20(k-1)\log 2)\rfloor$. Let $n \in [2^{k-1}, 2^k)$. Since both $\mathcal{I}$ and $\mathcal{J}$ have length $n/(10\log n) - 1$ and

$$\frac{n}{10\log n} - 1 > \frac{2^{k-1}}{10\log 2^{k-1}} - 1 = \frac{2^{k-1}}{10(k-1)\log 2} - 1 \geq 2(M-1) + 1,$$

it follows easily that for each such n both $\mathcal{I}$ and $\mathcal{J}$ contain some subinterval of the form $\mathcal{L}_i$ for some $i \geq 0$. We then checked that in each such subinterval $\mathcal{L}_i$, for all primes $q \leq 43$ and all residues $a \pmod q$ coprime to q, there is a prime in $\mathcal{L}_i$ with $p \equiv a \pmod q$. This worked well until $k = 26$, when it started to fail, so we adopted a different 'brute-force' approach. For all $n \leq 2^{26}$, we checked that for such n the inequality $\phi(j+1) > \phi(n-j)$ holds for some j close to the index of the middle binomial coefficient (prompted by computational data, we chose $\lfloor n/2\rfloor - 30 \leq j \leq \lfloor n/2\rfloor - 1$; in fact, it seems that even fewer values of j are needed to test for nonunimodality). In any case, we found that the only values of n for which the phinomial sequence is unimodal are the ones listed in the theorem. □

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