NOTE ON THE GAMMA FUNCTION

B. VAN DER POL

1. The gamma function $\Gamma(z+1) = \Pi(z)$ has been defined in different ways:[1]

(1) $$\Pi(z) = e^{-\gamma z} \prod_{k=1}^{\infty} \frac{e^{z/k}}{1 + z/k}$$ (Weierstrass)

(2) $$\Pi(z) = \prod_{k=1}^{\infty} \frac{(1 + 1/k)^z}{1 + z/k}$$ (Euler)

(3) $$\Pi(z) = \lim_{N\to\infty} \left\{ N^z \prod_{k=1}^{N} \frac{1}{1 + z/k} \right\}$$ (Gauss)

(4) $$\Pi(z) = \int_0^\infty e^{-s} s^z \, ds \quad (\Re z > -1),$$ (Euler)

(5) $$\Pi(z) = \exp \frac{\partial}{\partial s} \{\zeta(s, z+1) - \zeta(s, 1)\}_{s=0}$$ (Lerch)

where, for $\Re s > 1$, $\zeta(s, z)$ is given by

(5a) $$\zeta(s, z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^s}$$ $(z > 0)$

which is Hurwitz's generalization of Riemann's zeta function

(5b) $$\zeta(s) = \zeta(s, 1) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$ $(\Re s > 1)$.

The gamma function has also been defined by Harold Bohr and Mollerup (**2**, 154, 161–163) and Artin (**1**) as that solution of the difference equation

(6) $$\Gamma(x+1) = x\Gamma(x)$$

which is normalized through

$$\Gamma(1) = 1,$$

and which is logarithmic convex for $x > 0$.

Finally, we have the well-known Stirling formula

(7) $$\Pi(z) = \sqrt{2\pi z}\, z^z e^{-z} e^{\mu(z)}$$ (Stirling)

where

(7a) $$\mu(z) = \int_0^\infty \frac{[s] - s + \frac{1}{2}}{s + z} \, ds$$

or, asymptotically,

Received May 12, 1953.

[1]An up to date review of the gamma function is to be found in (**3**).

(7b) $$\mu(z) \approx \frac{B_2}{1\cdot 2z} + \frac{B_4}{3\cdot 4z^3} + \frac{B_6}{5\cdot 6z^5} + \dots .$$

In (7b), B_n are the Bernoulli numbers defined by the generating function

$$\frac{t}{e^t - 1} = \sum_0^\infty B_k \frac{t^k}{k!}\,.$$

Other expressions for $\mu(z)$ are

(7c) $$\mu(z) = \int_0^\infty e^{-zs}\left(\frac{1}{e^s - 1} - \frac{1}{s} + \frac{1}{2}\right)\frac{ds}{s} \quad (\Re\, z > 0)$$

(7d) $$\mu(z) = \sum_{k=0}^\infty \left\{(z + k + \tfrac{1}{2}) \log\left(1 + \frac{1}{k+z}\right) - 1\right\} \quad \text{(Gudermann)}$$

and

(7e) $$\mu(z) = 2\int_0^\infty \frac{1}{e^{2\pi s} - 1} \tan^{-1}\left(\frac{s}{z}\right) ds \quad (\Re\, z > 0) \qquad \text{(Binet)}.$$

The expressions (7a), (7b), and (7d) are valid in the whole z-plane cut from $-\infty$ to 0 along the negative real axis, the values being principal values.

To this list we now add the following formula which will be the subject of this note:

(8) $$\Pi(z) = z^z e^{-z} \prod_{k=0}^\infty \left(\frac{e_{k+z}}{e_k}\right) = z^z e^{-z} \lim_{N\to\infty} \frac{e_z\, e_{z+1}\, e_{z+2} \dots e_{z+N}}{e_0\, e_1\, e_2 \dots e_N},$$

where $e_\alpha = (1 + 1/\alpha)^\alpha$, and $e_0 = 1$.

2. In order to derive (8), we first write (1) as

$$\log \Pi(z) + \gamma z = \sum_{k=1}^\infty \left\{\frac{z}{k} - \log\left(1 + \frac{z}{k}\right)\right\}.$$

Next we express the sum as a Stieltjes integral

$$\log \Pi(z) + \gamma z = \int_{1-0}^\infty \left\{\frac{z}{s} - \log\left(1 + \frac{z}{s}\right)\right\} d[s]$$

and transform the latter as follows:

$$= \int_{1-0}^\infty \left\{\frac{z}{s} - \log\left(1 + \frac{z}{s}\right)\right\} d\,([s] - s) + \int_1^\infty \left\{\frac{z}{s} - \log\left(1 + \frac{z}{s}\right)\right\} ds.$$

(9) Hence

$$\log \Pi(z) + \gamma z = \int_{1-0}^\infty \left\{\frac{z}{s} - \log\left(1 + \frac{z}{s}\right)\right\} d([s] - s) + (z+1)\log(z+1) - z.$$

We further have for the Euler constant:

$$\gamma = \int_{1-0}^\infty \frac{1}{s}\, d([s] - s),$$

which enables us to simplify (9) to

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$$\log \Pi(z) = (z+1)\log(z+1) - z - \int_{1-0}^{\infty} \log\left(1+\frac{z}{s}\right) d([s]-s).$$

The Stieltjes integral can now be integrated by parts, the integrated part being $-\log(z+1)$. Hence we are left with

$$\log \Pi(z) = z\log(z+1) - z + \int_{1}^{\infty}([s]-s)\left(\frac{1}{s+z}-\frac{1}{s}\right) ds$$

(10) $$= z\log(z+1) - z + \sum_{k=1}^{\infty}\int_{k}^{k+1}(k-s)\left(\frac{1}{s+z}-\frac{1}{s}\right) ds$$

$$= z\log(z+1) - z + \sum_{k=1}^{\infty}\left\{(k+z)\log\left(1+\frac{1}{k+z}\right) - k\log\left(1+\frac{1}{k}\right)\right\}.$$

Taking exponentials we have

(11) $$\Pi(z) = (z+1)^z e^{-z}\prod_{k=1}^{\infty}\left(\frac{e_{k+z}}{e_k}\right),$$

where $e_\alpha = (1+1/\alpha)^\alpha$. If we further define $e_0 = 1$, we obtain as our final expression

(8) $$\Pi(z) = z^z e^{-z}\prod_{k=0}^{\infty}\left(\frac{e_{k+z}}{e_k}\right)$$

which is valid in the whole z-plane cut from $-\infty$ to 0 along the negative real axis, the values being principal values.

3. Comparing (8) with Stirling's formula (7) we note that both contain the factor $z^z e^{-z}$. Hence after splitting off this factor we arrive at the interesting relation

(12) $$\sqrt{2\pi z}\, e^{\mu(z)} = \prod_{k=0}^{\infty}\left(\frac{e_{k+z}}{e_k}\right).$$

4. It is easy to show that our expression (8) satisfies the difference equation

$$\Pi(z+1) = (z+1)\Pi(z).$$

To this end we rewrite (8) with z replaced by $z+1$ and transform as follows:

$$\Pi(z+1) = (z+1)^{z+1} e^{-(z+1)} \lim_{N\to\infty}\frac{e_{z+1}\,e_{z+2}\ldots e_{z+N+1}}{e_0\,e_1\ldots e_N}$$

$$= (z+1)^{z+1} e^{-(z+1)}\frac{1}{e_z}\lim_{N\to\infty}\frac{e_z\,e_{z+1}\ldots e_{z+N}}{e_0\,e_1\ldots e_N}\, e_{z+N+1}$$

$$= (z+1)\, z^z e^{-(z+1)}\lim_{N\to\infty}\frac{e_z\,e_{z+1}\ldots e_{z+N}}{e_0\,e_1\ldots e_N}\lim e_{z+N+1}$$

$$= (z+1)\, z^z e^{-(z+1)}\lim_{N\to\infty}\frac{e_z\,e_{z+1}\ldots e_{z+N}}{e_0\,e_1\ldots e_N}\, e$$

$$= (z+1)\,\Pi(z).$$

5. A check of (8) can be obtained from (11). The latter expression can be written

$$\Pi(z) = (z+1)^z e^{-z} \lim_{N\to\infty} \frac{\left(\frac{z+2}{z+1}\right)^{z+1}\left(\frac{z+3}{z+2}\right)^{z+2}\cdots\left(\frac{z+N+1}{z+N}\right)^{z+N}}{\left(\frac{2}{1}\right)^{1}\quad\left(\frac{3}{2}\right)^{2}\quad\cdots\quad\left(\frac{N+1}{N}\right)^{N}}$$

$$= e^{-z} \lim_{N\to\infty} \frac{(z+N+1)^z}{\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{N}\right)} \lim_{N\to\infty}\left(1+\frac{z}{N+1}\right)^N$$

$$= e^{-z} \lim_{N\to\infty} \frac{N^z}{\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{N}\right)} e^z$$

$$= \lim_{N\to\infty}\left\{N^z \prod_{k=1}^{N}\left(\frac{1}{1+\frac{z}{k}}\right)\right\}$$

this being Gauss's formula (3).

6. The infinite product in (8) is convergent only as it stands, whereas the infinite products occurring in the numerator and denominator separately would diverge. This however is simply overcome if we rewrite (10) as follows:

$$(13)\quad \log \Pi(z) = z\log(z+1) - z + \sum_{k=1}^{\infty}\left\{(k+z)\log\left(1+\frac{1}{k+z}\right) - 1 + \frac{1}{2k}\right\}$$
$$- \sum_{k=1}^{\infty}\left\{k\log\left(1+\frac{1}{k}\right) - 1 + \frac{1}{2k}\right\},$$

where now each of the two series converges by itself. The second, numerical, series can at once be evaluated. One finds

$$(13a)\qquad \sum_{k=1}^{\infty}\left\{k\log\left(1+\frac{1}{k}\right) - 1 + \frac{1}{2k}\right\} = 1 + \tfrac{1}{2}(\gamma - \log 2\pi).$$

Combining (13) and (13a), and taking exponentials, we now obtain, instead of (8):

$$(14)\qquad \Pi(z) = \sqrt{2\pi}\,(z+1)^z e^{-(z+1)} e^{-\frac{1}{2}\gamma} \prod_{k=1}^{\infty}\left(\frac{e_{k+z}}{e} e^{1/2k}\right).$$

Using again for γ the expression

$$\gamma = \lim_{N\to\infty}\left\{\sum_{1}^{N}\frac{1}{k} - \log N\right\}$$

we obtain from (14)

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$$\Pi(z) = \sqrt{2\pi}\,(z+1)^z e^{-(z+1)} \lim_{N\to\infty} \left\{\sqrt{N} \prod_{k=1}^{N} \left(\frac{e_{k+z}}{e}\right)\right\}$$

$$= \sqrt{2\pi}\,(z+1)^z e^{-(z+1)} \frac{e}{e_z} \lim_{N\to\infty} \left\{\sqrt{N} \prod_{k=0}^{N} \left(\frac{e_{k+z}}{e}\right)\right\}$$

or

$$\Pi(z) = \sqrt{2\pi}\, z^z e^{-z} \lim_{N\to\infty} \left\{\sqrt{N} \prod_{k=0}^{N} \left(\frac{e_{k+z}}{e}\right)\right\}, \tag{15}$$

which is another expression for the gamma function and which can be compared with (8).

Finally, from (15) and using Stirling's formula (7) we obtain as companion relation to (12):

$$\sqrt{z}\, e^{\mu(z)} = \lim_{N\to\infty} \left\{\sqrt{N} \prod_{k=0}^{N} \left(\frac{e_{k+z}}{e}\right)\right\}. \tag{16}$$

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