J. Austral. Math. Soc. (Series A) **46** (1989), 22–60

VARIETIES OF GROUPS AND ISOLOGISMS

N. S. HEKSTER

(Received 2 February 1987)

Communicated by H. Lausch

Abstract

In order to classify solvable groups Philip Hall introduced in 1939 the concept of isoclinism. Subsequently he defined a more general notion called isologism. This is so to speak isoclinism with respect to a certain variety of groups. The equivalence relation isologism partitions the class of all groups into families. The present paper is concerned with the internal structure of these families.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 20 E 10, 20 E 99, 20 F 14.
Keywords and phrases: variety of groups, isologism.

1. Introduction, preliminaries and notational conventions

It is a well-known fact that solvable groups are difficult to classify because of the abundance of normal subgroups. In a certain sense the class of groups of prime-power order is the simplest case to handle. But already here the classification is far from being complete. Up to now there are only a few classes of such prime-power order groups which have been adequately analysed. The first to create some order in the plethora of groups of prime-power order was Philip Hall. He observed that the notion of isomorphism of groups is really too strong to give rise to a satisfactory classification and that it had to be replaced by a weaker equivalence relation. Subsequently he discovered a suitable equivalence relation and called it *isoclinism of groups* (see [6]). It is this classification principle that

https://doi.org/10.1017/S1446788700030366 Published online by Cambridge University Press

underlies the famous monograph of M. Hall and J. K. Senior on the classification of 2-groups of order at most 64.

Shortly after the notion of isoclinism was defined, Hall generalized this to what he called $\mathfrak{V}$-*isologism*, where $\mathfrak{V}$ is some variety of groups. Isologism is so to speak isoclinism with respect to a certain variety. In this way for each variety an equivalence relation on the class of all groups arises. The larger the variety, the weaker this equivalence relation is. If $\mathfrak{V}$ is the variety of all abelian groups, $\mathfrak{V}$-isologism coincides with isoclinism. The groups in a variety $\mathfrak{V}$ fall into one single equivalence class: they are $\mathfrak{V}$-isologic to the trivial group.

For $\mathfrak{V}$ the variety of all nilpotent groups of class at most n ($n \geq 0$), the notion of $\mathfrak{V}$-isologism is nothing else but the notion of n-isoclinism (where it is understood that 1-isoclinism equals isoclinism). The structure of n-isoclinism classes was extensively studied in [8]. The aim of the present paper is to investigate which results on n-isoclinism (see [8]) can be generalized to $\mathfrak{V}$-isologism, and how this depends on the defining laws and internal structure of the variety $\mathfrak{V}$.

The paper is organized as follows.

Section 2 deals with the basic properties of verbal and marginal subgroups needed in the sequel. If N is a normal subgroup of the group G we introduce (following [11]) a subgroup of G denoted by $[NV^*G]$, depending on a variety $\mathfrak{V}$. This group $[NV^*G]$ plays a prominent role in this paper. For example, it turns up in calculating the verbal subgroup of a group which can be written as the product of a subgroup and a normal subgroup (see (2.4)). The group $[NV^*G]$ features also as the verbal subgroup of a group with respect to a certain product variety to be defined in Section 3. (We remark that there is still another place where the usefulness of the group $[NV^*G]$ is demonstrated. The definition of the so-called Schur-Baer multiplicator (see [11] and [1], Chapter IV, Section 7) involves this group. However we will not deal with the Schur-Baer multiplicator since our results are not established by cohomological means.)

There exist many ways in which a variety can be produced starting from two given ones. Section 3 deals with one of these fashions, namely a certain product variety enters the scene. The definition here stems from [11].

In Section 4 the definition of $\mathfrak{V}$-isologism is introduced and some of its elementary properties are derived.

A result of P. M. Weichsel and J. C. Bioch (see [21] and [2]) states that two n-isoclinic groups G_1 and G_2 have a common n-isoclinic ancestor G, that is, G_1 and G_2 can be realized as quotients of a group G, while G, G_1 and G_2 are in the same n-isoclinism class. This can directly be generalized to $\mathfrak{V}$-isologism. On the other hand, it was proved in [8] that any two isoclinic groups G_1 and G_2 have a common isoclinic descendant G, that is, G_1 and G_2 can be realized as subgroups of a group G, whereas G, G_1 and G_2 are isoclinic to each other. It is unknown whether this result allows a $\mathfrak{V}$-isologic generalization. However under a mild restriction on the variety $\mathfrak{V}$ a theorem similar to the isoclinic case can be derived. All this is the object of Section 5.

It was proved in [8] that an n-isoclinism between two groups yields certain n-isoclinisms for the corresponding lower commutator subgroups and for the corresponding upper central factor groups. In Section 6 this phenomenon will be illustrated from a varietal point of view. The product variety as defined in Section 3 comes into play here. The ceiling function $\lceil x \rceil$ of a real number x, defined as the smallest integer not smaller than x, has to be employed to describe the so-called induced isologisms. This is caused by a multiplicative property of the lower commutator subgroups (see [15], 5.1.11(ii)).

Section 7 is concerned with characterizations of groups being $\mathfrak{V}$-isologic to finitely generated groups of a certain type. The results obtained, extend a theorem in [8] on the characterization of groups being n-isoclinic to a finite group. Moreover they are related to solutions of Hall's problem on Schur pairs (see [14], Chapter 4, Section 2 and also [18]).

It was Philip Hall who coined the name *stemgroup* for groups having their center contained in its commutator subgroup. He proved their existence within an arbitrary isoclinism class and hence recognized the importance of stemgroups in classifying groups of prime-power order. The obvious generalization to a variety $\mathfrak{V}$ of a stemgroup — groups having their $\mathfrak{V}$-marginal subgroup contained in its $\mathfrak{V}$-verbal subgroup (see [1], Chapter IV, 7.25) — cannot be maintained, *if one requires the existence of such groups within* $\mathfrak{V}$*-isologism classes.* This is one of the matters brought up in Section 8. Here we define $\mathfrak{V}$-stemgroups as being groups with the property that its $\mathfrak{V}$-verbal subgroup contains the center of the whole group. This agrees with an earlier notion introduced in [8] in the n-isoclinic case. Finally the existence of $\mathfrak{V}$-stemgroups within a $\mathfrak{V}$-isologism class is proved for $\mathfrak{V}$ being the variety of all polynilpotent groups of some fixed class row.

As to the use of terminology and notational conventions in this paper, the following is relevant. The reader is referred to the book of Hanna Neumann [13] for the basic definitions and facts concerning varieties of groups. On the whole the notation of [15] will be adhered to. Capital Roman letters will denote groups and capital Gothic letters will denote varieties of groups. The following notation of special varieties will be in force throughout:

$\mathfrak{E}$: the variety of all trivial groups,

$\mathfrak{A}_m$: the variety of all abelian groups of exponent dividing m ($m \geq 0$),

$\mathfrak{A}$: the variety of all abelian groups,

$\mathfrak{N}_c$: the variety of all nilpotent groups of class at most c ($c \geq 0$),

$\mathfrak{S}_l$: the variety of all solvable groups of length at most l ($l \geq 0$),

$\mathfrak{N}_{c_1,\dots,c_l}$: the variety of all polynilpotent groups of class row $(c_1,\dots,c_l)$

$(c_i \geq 1, l \geq 1)$ (see [13], 14.66 and 21.52).

If G is a group and $\mathfrak{V}$ a variety, then $V(G)$ denotes the verbal subgroup and $V^*(G)$ the marginal subgroup of G with respect to $\mathfrak{V}$ (see also [11] and [15], Chapter 2, Section 3). The latter notations run according to the following rule: whenever a capital Gothic letter is used to denote a variety, the corresponding capital Roman letter is employed to denote the verbal and marginal subgroups. In some cases the verbal and marginal subgroups can be calculated explicitly: the terms of the lower and upper central series of a group G are denoted by

$$\gamma_1(G) = G \geq \gamma_2(G) \geq \gamma_3(G) \geq \cdots \quad \text{and} \quad \varsigma_0(G) = 1 \leq \varsigma_1(G) \leq \varsigma_2(G) \leq \cdots$$

respectively (see [15], Chapter 5, Section 1). These are the verbal, respectively the marginal subgroups of G with respect to $\mathfrak{N}_0$, $\mathfrak{N}_1$, $\mathfrak{N}_2,\dots$. Recall that the marginal subgroups are characteristic, while the verbal subgroups are even fully invariant.

The letter F will denote an (absolutely) free group of unspecified rank; we write F_s if the free group is of finite rank s. For the elements of F (or F_s) the small Roman letters x, x_1, x_2, $x_3,\dots$ will be used.

A variety is *nilpotent* if it is contained in $\mathfrak{N}_c$ for some c.

A variety is *finitely based* if it can be defined by a finite set of laws, and hence by one law.

Let $\mathscr{P}$ be a property of groups. A variety $\mathfrak{V}$ is *locally* $\mathscr{P}$ if the finitely generated members of $\mathfrak{V}$ satisfy the property $\mathscr{P}$.

A *$\mathfrak{V}$-splitting group* is a group $G \in \mathfrak{V}$ such that any extension of a group of $\mathfrak{V}$ by the group G is split. The relatively free groups $F/V(F)$, $F_s/V(F_s),\dots$ are called $\mathfrak{V}$-free groups. A $\mathfrak{V}$-free group is $\mathfrak{V}$-splitting (but not conversely, see [13], Chapter 4, Section 4).

The letters $\mathfrak{X}$ and $\mathfrak{Y}$ are reserved for denoting classes of groups. A class of groups is understood to contain a group of order 1 and to be closed under isomorphisms. The class of finite groups will be denoted by $\mathfrak{F}$. A group G is $\mathfrak{X}$-by-$\mathfrak{Y}$ if it contains a normal subgroup N such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$. The letter π will always denote a non-void set of prime numbers. By $\mathfrak{X}_\pi$ the subclass of π-groups of $\mathfrak{X}$ is meant.

The *Frattini subgroup* $\Phi(G)$ of an arbitrary group G is defined to be the intersection of all maximal subgroups of G, with the convention that $\Phi(G) = G$ in case G does not possess any maximal subgroups.

The *socle* $\mathrm{soc}(G)$ of an arbitrary group G is the group generated by the minimal normal subgroups of G, with the stipulation that $\mathrm{soc}(G) = 1$ if G is lacking minimal normal subgroups.

A group is *polycyclic* if it has a series with cyclic factors. A cyclic group of order n is denoted by C_n or $\mathbf{Z}/n\mathbf{Z}$.

If H is a subgroup of G we set $[H, {}_0G] = H$ and by induction $[H, {}_{n+1}G] = [[H, {}_nG], G]$ $(n \geq 0)$. Note that $[H, {}_{n+1}G]$ is a normal subgroup of G.

2. Basic properties of verbal and marginal subgroups

In this section we provide some preliminary properties and notions concerning verbal and marginal subgroups.

(2.1) DEFINITION. If $N \trianglelefteq G$ and $\mathfrak{V}$ is a variety, define $[NV^*G]$ to be the subgroup of G generated by

$$\{v(g_1, \ldots, g_{i-1}, g_i n, g_{i+1}, \ldots, g_s)(v(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_s))^{-1}: \\ 1 \leq i \leq s < \infty, v \in V(F_s), g_1, \ldots, g_s \in G, n \in N\}.$$

It is easily checked that $[NV^*G]$ is the smallest normal subgroup T of G contained in N, such that $N/T \subseteq V^*(G/T)$. In other words, $G/[NV^*G]$ is the largest quotient of G in which N becomes marginal. The following examples will be helpful in the rest of this paper: if $\mathfrak{V} = \mathfrak{E}$, then $[NV^*G] = N$; if $\mathfrak{V} = \mathfrak{N}_c$, then $[NV^*G] = [N, {}_cG]$. In Section 6 we will calculate $[NV^*G]$ for $\mathfrak{V} = \mathfrak{N}_{c_1,\ldots,c_l}$.

(2.2) LEMMA. *Let $N \trianglelefteq G$ and $\mathfrak{V}$ and $\mathfrak{W}$ varieties such that $\mathfrak{V} \subseteq [\mathfrak{W}, \mathfrak{E}]$. Then the following hold.*

(a) $[N, W(G)] \subseteq [NV^*G]$.

(b) $[V^*(G), W(G)] = 1$.

PROOF. (a) If $w \in W(F_s)$ is a law of $\mathfrak{W}$, then $[w, x_{s+1}]$ is a law of $\mathfrak{V}$. So if we put $v(x_1, \ldots, x_{s+1}) = [w(x_1, \ldots, x_s), x_{s+1}]$, then $[w(g_1, \ldots, g_s), n] = v(g_1, \ldots, g_s, n)v(g_1, \ldots, g_s, 1)^{-1}$ is an element of $[NV^*G]$ for all $n \in N$ and $g_1, \ldots, g_s \in G$. Hence $[N, W(G)] \subseteq [NV^*G]$.

(b) As $[V^*(G)V^*G] = 1$, it follows from (a) that $[V^*(G), W(G)] = 1$.

(2.3) PROPOSITION. *Let $N \trianglelefteq G$ and $\mathfrak{V}$ a variety. Then the following properties hold.*

(a) $V(V^*(G)) = 1$ *and* $V^*(G/V(G)) = G/V(G)$.

(b) $V(G) = 1 \Leftrightarrow V^*(G) = G \Leftrightarrow G \in \mathfrak{V}$.

(c) $[NV^*G] = 1 \Leftrightarrow N \subseteq V^*(G)$.

(d) $V(G/N) = V(G)N/N$ *and* $V^*(G/N) \supseteq V^*(G)N/N$.

(e) $V(N) \subseteq [NV^*G] \subseteq N \cap V(G)$. *In particular* $V(G) = [GV^*G]$.

(f) *If* $N \cap V(G) = 1$, *then* $N \subseteq V^*(G)$ *and* $V^*(G/N) = V^*(G)/N$.

(g) *If* $[G, N] \subseteq V^*(G)$, *then* $[V(G), N] = 1$. *In particular* $[V(G), V^*(G)] = 1$.

PROOF. (a), (b), (c) and (d) are clear from the definitions.

(e) Let $n_1, \dots, n_s \in N$ and $v \in V(F_s)$. Then $v(n_1, \dots, n_s) =$

$$v(n_1, \dots, n_s)v(1, n_2, \dots, n_s)^{-1} \cdot v(1, n_2, \dots, n_s)v(1, 1, n_3, \dots, n_s)^{-1} \\ \cdots v(1, \dots, 1, n_{s-1}, n_s)v(1, \dots, 1, 1, n_s)^{-1} \cdot v(1, \dots, 1, n_s)$$

and this is clearly an element of $[NV^*G]$. We conclude that $V(N) \subseteq [NV^*G]$. The inclusion $[NV^*G] \subseteq N \cap V(G)$ is immediate.

(f) The first part of the assertion follows from (c) and (e). From (d) we have $V^*(G)/N \subseteq V^*(G/N)$. Put $V^*(G/N) = M/N$, so that $M \trianglelefteq G$ and $[MV^*G] \subseteq N$. From (e) we see that $[MV^*G] \subseteq V(G)$. Hence $[MV^*G] = 1$ and by virtue of (c) we get $M \subseteq V^*(G)$. Whence $V^*(G)/N = V^*(G/N)$.

(g) Let $v \in V(F_s), g_1, \dots, g_s \in G$ and $n \in N$. Then

$$\begin{aligned} [v(g_1, \dots, g_s), n] &= v(g_1, \dots, g_s)^{-1}v(n^{-1}g_1n, \dots, n^{-1}g_sn) \\ &= v(g_1, \dots, g_s)^{-1}v(g_1[g_1, n], \dots, g_s[g_s, n]) \\ &= v(g_1, \dots, g_s)^{-1}v(g_1, \dots, g_s) = 1. \end{aligned}$$

We conclude that N centralizes $V(G)$.

We remark that Phillip Hall called the assertion (2.3)(g) the Permutability Theorem (see [7]). Next we turn to the computation of a verbal subgroup of the group G, when G is given as a product of certain subgroups.

(2.4) THEOREM. *Let $H \le G$ and $N \trianglelefteq G$ such that $G = HN$. Let $\mathfrak{V}$ be a variety. Then $V(G) = V(H)[NV^*G]$.*

PROOF. By (2.3)(e) we have $[NV^*G] \subseteq V(G)$. Also $V(H) \subseteq V(G)$. Hence $V(H)[NV^*G] \subseteq V(G)$. Next we claim that $V(H)[NV^*G]$ is a normal subgroup of G. Indeed, $[NV^*G] \trianglelefteq G$ and H normalizes $V(H)$, so that it suffices to show that N normalizes $V(H)[NV^*G]$, as $G = HN$. Let $n \in N$ and $v \in V(H)$. Then $n^{-1}vn = v[v, n] \in V(H)[V(H), N] \subseteq V(H)[V(G), N] \subseteq V(H)[NV^*G]$, by (2.2)(a). This proves the claim. By virtue of (2.3)(e) we have $V(G) = [GV^*G]$. So, let us consider an element

$$v(g_1, \dots, g_{i-1}, g_ig, g_{i+1}, \dots, g_s)v(g_1, \dots, g_i, \dots, g_s)^{-1}$$

where $g, g_1, \dots, g_s \in G$ and $v \in V(F_s)$. Now write $g = hn$ and $g_i = h_in_i$, with $h, h_i \in H$ and $n, n_i \in N$ $(i = 1, \dots, s)$. Then, if bar $\bar{\ }$ denotes reduction modulo $V(H)[NV^*G]$, we get

$$\begin{aligned} &v(\bar{g}_1, \dots, \bar{g}_{i-1}, \bar{g}_i\bar{g}, \bar{g}_{i+1}, \dots, \bar{g}_s)v(\bar{g}_1, \dots, \bar{g}_i, \dots, \bar{g}_s)^{-1} \\ &\quad = v(\bar{h}_1\bar{n}_1, \dots, \bar{h}_{i-1}\bar{n}_{i-1}, \bar{h}_i\bar{h} \cdot \bar{h}^{-1}\bar{n}_i\bar{h} \cdot \bar{n}, \\ &\qquad\qquad \bar{h}_{i+1}\bar{n}_{i+1}, \dots, \bar{h}_s\bar{n}_s)v(\bar{h}_1\bar{n}_1, \dots, \bar{h}_s\bar{n}_s)^{-1} \\ &\quad = v(\bar{h}_1, \dots, \bar{h}_{i-1}, \bar{h}_i\bar{h}, \bar{h}_{i+1}, \dots, \bar{h}_s)v(\bar{h}_1, \dots, \bar{h}_s)^{-1} = \bar{1}. \end{aligned}$$

https://doi.org/10.1017/S1446788700030366 Published online by Cambridge University Press

Note that the second equality sign follows from the fact that $\overline{N}$ is marginal in $\overline{G}$ (see (2.1)). Hence it follows that $V(G)/V(H)[NV^*G] = \overline{1}$. So $V(G) \subseteq V(H)[NV^*G]$. We obtain $V(G) = V(H)[NV^*G]$, as required.

A part of the next lemma can be found in [22], Lemma 6.4 and also, in case of finitely based varieties, in [18], Lemma 1(d). For the convenience of the reader we provide a short proof.

(2.5) LEMMA. *Let $H \leq G$ and $\mathfrak{V}$ be a variety. Suppose that $G = HV^*(G)$. Then the following properties hold.*
(a) $V^*(H) = V^*(G) \cap H$.
(b) $V(H) = V(G)$.
(c) $V(H) \cap V^*(H) = V(G) \cap V^*(G)$.

PROOF. (a) By definition it is clear that $V^*(G) \cap H \subseteq V^*(H)$. Now $V^*(G) = V^*(HV^*(G)) = V^*(H)V^*(G)$, so that $V^*(H) \subseteq V^*(G)$. Hence $V^*(H) \subseteq V^*(G) \cap H$.
(b) By (2.4) we have $V(G) = V(H)[V^*(G)V^*G]$. But (2.3)(c) tells us that $[V^*(G)V^*G] = 1$.
(c) This is a consequence of (a) and (b).

(2.6) PROPOSITION. *Let $\mathfrak{V}$ be any variety. Then $V^*(G) \cap V(G) \subseteq \Phi(G)$.*

PROOF. If G does not have maximal subgroups, there is nothing to prove. So let M be a maximal subgroup of G. Then either $V^*(G) \subseteq M$ or $V^*(G)M = G$. From (2.5)(c) it follows that in each case $V^*(G) \cap V(G) \subseteq M$.

3. A product variety

There are several ways to construct a variety out of two given varieties $\mathfrak{U}$ and $\mathfrak{V}$. An obvious way to do this is taking the intersection, $\mathfrak{U} \wedge \mathfrak{V}$, of the varieties, or defining the variety $\mathfrak{U} \vee \mathfrak{V}$ as the variety generated by $\mathfrak{U}$ and $\mathfrak{V}$ (see [13], page 20). More sophisticated, Hanna Neumann introduced the product $\mathfrak{U}\mathfrak{V}$ of the varieties as being the variety of all groups that are extensions of a group in $\mathfrak{U}$ by a group in $\mathfrak{V}$. Still another way of producing a new variety out of $\mathfrak{U}$ and $\mathfrak{V}$ is taking their commutator product $[\mathfrak{U}, \mathfrak{V}]$ (see [13], Chapter 2, for properties and details). From the definition it follows immediately that $\mathfrak{U} \vee \mathfrak{V} \subseteq [\mathfrak{U}, \mathfrak{V}]$. Following C. R. Leedham-Green and S. McKay, we define a product of the varieties $\mathfrak{U}$ and $\mathfrak{V}$ which lies between $\mathfrak{U} \vee \mathfrak{V}$ and $[\mathfrak{U}, \mathfrak{V}]$.

(3.1) DEFINITION ([11], page 104). If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties, then the product $\mathfrak{U} * \mathfrak{V}$ is the variety of all groups G such that $U(G) \subseteq V^*(G)$.

It is not so difficult to show that this class of groups $\mathfrak{U} * \mathfrak{V}$ is in fact a variety and that the verbal subgroup of a group G with respect to that variety is $[U(G)V^*(G)]$ (see [11], Proposition 1.5).

Our notation differs from the one used by Leedham-Green and McKay; they write $\mathfrak{U}\mathfrak{V}^*$ rather than $\mathfrak{U} * \mathfrak{V}$. In general the product $*$ is not commutative (see [11], Example 2 on page 106) and in a moment we will see that it is not even associative. To give some examples, if $\mathfrak{V}$ is a variety, then $\mathfrak{E} * \mathfrak{V} = \mathfrak{V} = \mathfrak{V} * \mathfrak{E}$ and we have $[\mathfrak{V}, \mathfrak{E}] = \mathfrak{V} * \mathfrak{A}$. Further, for any non-negative integers m and n, $\mathfrak{N}_m * \mathfrak{N}_n = \mathfrak{N}_{m+n}$ (see [11], Example 1 on page 105).

With respect to the marginal subgroup of a group G corresponding to the variety $\mathfrak{U} * \mathfrak{V}$, we have the following.

(3.2) THEOREM. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be varieties and put $\mathfrak{W} = \mathfrak{U} * \mathfrak{V}$. Then for any group G the following hold.*

(a) $V^*(G) \subseteq W^*(G)$.

(b) $W^*(G)/V^*(G) \subseteq U^*(G/V^*(G)) \subseteq W^*(G/V^*(G))$.

We observe that the first inclusion of (3.2)(b) is equivalent to

$$[[W^*(G)U^*G]V^*G] = 1.$$

Moreover, if we put $U^*(G/V^*(G)) = M/V^*(G)$, then $M \trianglelefteq G$ and $M \in \mathfrak{W}$. Indeed, $[MU^*G] \subseteq V^*(G)$, so $[[MU^*G]V^*G] = 1$ by (2.3)(c). Hence by virtue of (2.3)(e) $[U(M)V^*M] = [[MU^*M]V^*M] \subseteq [[MU^*G]V^*G] = 1$, so that $U(M) \subseteq V^*(M)$, as desired.

For the proof of (3.2) we need a lemma.

(3.3) LEMMA. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be varieties and put $\mathfrak{W} = \mathfrak{U} * \mathfrak{V}$. Then the following are equivalent.*

(a) *For any group G: $W^*(G)/V^*(G) \subseteq U^*(G/V^*(G))$.*

(b) *For any group G and $N \trianglelefteq G$: $[[NU^*G]V^*G] \subseteq [NW^*G]$.*

Moreover the equality sign holds in (a) *if and only if the equality sign holds in* (b).

PROOF. (a)$\Rightarrow$(b): Let bar $\bar{\ }$ denote reduction modulo $[NW^*G]$. Hence $\overline{N} \subseteq W^*(\overline{G})$. Now $\overline{N}V^*(\overline{G})/V^*(\overline{G}) \subseteq W^*(\overline{G})/V^*(\overline{G}) \subseteq U^*(\overline{G}/V^*(\overline{G}))$ by assertion. It follows from (2.1) that $[\overline{N}V^*(\overline{G})U^*\overline{G}] \subseteq V^*(\overline{G})$. So $[\overline{N}U^*\overline{G}] \subseteq V^*(\overline{G})$, that is by (2.3)(c) $[[\overline{N}U^*\overline{G}]V^*\overline{G}] = \overline{1}$. But this is clearly equivalent to $[[NU^*G]V^*G] \subseteq [NW^*G]$.

(b)$\Rightarrow$(a): Take $N = W^*(G)$ and apply (2.3)(c). This concludes the proof of the first part of the theorem.

To prove the last assertion, assume first that for any group G it holds that $W^*(G)/V^*(G) = U^*(G/V^*(G))$. Now fix a group G and $N \trianglelefteq G$. By the first part of the proof $[[NU^*G]V^*G] \subseteq [NW^*G]$. To obtain the reverse inclusion let now bar $\bar{}$ denote reduction modulo $[[NU^*G]V^*G]$. Then apparently $[\overline{N}U^*\overline{G}] \subseteq V^*(\overline{G})$ and this implies that $\overline{N}V^*(\overline{G})/V^*(\overline{G}) \subseteq U^*(\overline{G}/V^*(\overline{G}))$. But $U^*(\overline{G}/V^*(\overline{G})) = W^*(\overline{G})/V^*(\overline{G})$ by assertion, whence $\overline{N} \subseteq W^*(\overline{G})$. Thus by (2.3)(c) $[\overline{N}W^*\overline{G}] = \overline{1}$, that is $[NW^*G] \subseteq [[NU^*G]V^*G]$, as required.

Finally assume that for any group G and $N \trianglelefteq G$ it holds that $[[NU^*G]V^*G] = [NW^*G]$. Pick a group G and put $U^*(G/V^*(G)) = M/V^*(G)$. Then $M \trianglelefteq G$ and $[MU^*G] \subseteq V^*(G)$. Thus $[[MU^*G]V^*G] = 1$ by (2.3)(c). By hypothesis $[[MU^*G]V^*G] = [MW^*G]$, so $M \subseteq W^*(G)$. Hence $U^*(G/V^*(G)) \subseteq W^*(G)/V^*(G)$ and the reverse inclusion is again assured by the first part of the lemma.

PROOF OF (3.2). It holds that $\mathfrak{V} \subseteq \mathfrak{W}$. Indeed, if $G \in \mathfrak{V}$, then by (2.3)(b) $G = V^*(G)$. Obviously $U(G) \subseteq V^*(G) = G$, thus $G \in \mathfrak{W}$. It follows that $V^*(G) \subseteq W^*(G)$ for an arbitrary group G, which proves (a).

The second inclusion in (3.2)(b) holds as soon as we have shown that $\mathfrak{U} \subseteq \mathfrak{W}$. Indeed, if $G \in \mathfrak{U}$, then $U(G) = 1$ by (2.3)(b). Hence $W(G) = [U(G)V^*G] = 1$, thus $G \in \mathfrak{W}$, again by virtue of (2.3)(b). We are left with proving the first inclusion of (3.2)(b). On invoking the Lemma (3.3) it suffices to prove that for any G and $N \trianglelefteq G$ we have $[[NU^*G]V^*G] \subseteq [NW^*G]$. So let $N \trianglelefteq G$. We use that if $u(x_1, \dots, x_r)$ and $v(x_1, \dots, x_s)$ are words in respectively $U(F_r)$ and $V(F_s)$, the laws which determine $\mathfrak{W}$ are given by

$$v(x_1, \dots, x_i u(x_{s+1}, \dots, x_{s+r}), x_{i+1}, \dots, x_s) v(x_1, \dots, x_i, \dots, x_s)^{-1},$$

where $1 \leq i \leq s$ (see [11], Proposition 1.5(ii)). A generating element of

$$[[NU^*G]V^*G]$$

is of the following form:

$$(*) \quad v(g_1, \dots, g_i u(g_{s+1}, \dots, g_{s+j}n, \dots, g_{s+r}) u(g_{s+1}, \dots, g_{s+r})^{-1}, \dots, g_s) \cdot v(g_1, \dots, g_s)^{-1},$$

where $g_1, \dots, g_{s+r} \in G$, $n \in N$, $1 \leq i \leq s$ and $1 \leq j \leq r$. Put $u = u(g_{s+1}, \dots, g_{s+r})$ and $u' = u(g_{s+1}, \dots, g_{s+j}n, \dots, g_{s+r})$. Then the element in $(*)$ takes the form:

$$\begin{aligned} &v(g_1, \dots, g_i u' u^{-1}, g_{i+1}, \dots, g_s) v(g_1, \dots, g_s)^{-1} \\ &\quad = v(g_1, \dots, g_i u^{-1} \cdot u u' u^{-1}, g_{i+1}, \dots, g_s) v(g_1, \dots, g_i u^{-1}, g_{i+1}, \dots, g_s)^{-1} \\ &\qquad \cdot v(g_1, \dots, g_i u^{-1}, g_{i+1}, \dots, g_s) v(g_1, \dots, g_s)^{-1} \end{aligned}$$

and this is clearly an element of $[NW^*G]$. We conclude that $[[NU^*G]V^*G] \subseteq [NW^*G]$, as required.

For later purposes we need the following facts.

(3.4) PROPOSITION. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be varieties and put $\mathfrak{W} = \mathfrak{U} * \mathfrak{V}$. Let $N \trianglelefteq G$. Then the following hold.*

(a) *If $N \subseteq W^*(G)$, then $[NU^*G] \subseteq V^*(G)$.*

(b) *If $N \cap V^*(G) = 1$, then $N \cap W^*(G) \subseteq U^*(G)$.*

PROOF. (a) If $N \subseteq W^*(G)$, then by (2.3)(c) $[NW^*G] = 1$. By utilizing (3.3) in connection with (3.2)(b) we get $[[NU^*G]V^*G] = 1$. Again (2.3)(c) gives $[NU^*G] \subseteq V^*(G)$.

(b) Here, $[(N \cap W^*(G))U^*G] \subseteq N \cap [W^*(G)U^*G] \subseteq N \cap V^*(G)$. Hence, if $N \cap V^*(G) = 1$, then (2.3)(c) gives the required result.

(3.5) PROPOSITION. *Let $\mathfrak{U}$, $\mathfrak{V}$ and $\mathfrak{W}$ be any varieties. Then*

$$\mathfrak{U} * (\mathfrak{V} * \mathfrak{W}) \subseteq (\mathfrak{U} * \mathfrak{V}) * \mathfrak{W}.$$

PROOF. Let $G \in \mathfrak{U} * (\mathfrak{V} * \mathfrak{W})$. Put $\mathfrak{T} = \mathfrak{V} * \mathfrak{W}$. Hence $U(G) \subseteq T^*(G)$. Now an application of (3.4)(a) yields $[U(G)V^*G] \subseteq W^*(G)$. Thus $S(G) \subseteq W^*(G)$, where $\mathfrak{S} = \mathfrak{U} * \mathfrak{V}$. We conclude that $G \in (\mathfrak{U} * \mathfrak{V}) * \mathfrak{W}$.

In general equality does not hold in (3.5). Equality does hold however for arbitrary $\mathfrak{U}$ with the stipulation that in (3.3) the equality sign holds everywhere for the varieties $\mathfrak{V}$ and $\mathfrak{W}$. That is, for any group G it holds that $T^*(G)/W^*(G) = V^*(G/W^*(G))$, where $\mathfrak{T} = \mathfrak{V} * \mathfrak{W}$. Indeed, if $G \in (\mathfrak{U} * \mathfrak{V}) * \mathfrak{W}$, then $[S(G)W^*G] = 1$, with $\mathfrak{S} = \mathfrak{U} * \mathfrak{V}$. By (3.3) we have that $[U(G)T^*G] = [[U(G)V^*G]W^*G] = [S(G)W^*G]$. Hence $G \in \mathfrak{U} * (\mathfrak{V} * \mathfrak{W})$, which proves the above claim. In a moment we will encounter an example where the inclusion of (3.5) is strict.

(3.6) PROPOSITION. *Let $\mathfrak{U} \subseteq \mathfrak{U}_1$ and $\mathfrak{V} \subseteq \mathfrak{V}_1$ be varieties. Then the following hold.*

(a) $\mathfrak{U} * \mathfrak{V} \subseteq \mathfrak{U}_1 * \mathfrak{V}_1$.

(b) $\mathfrak{U} * \mathfrak{A} \supseteq \mathfrak{A} * \mathfrak{U}$.

(c) *For any $m, n \geq 0$, $\mathfrak{U} * \mathfrak{N}_{m+n} = (\mathfrak{U} * \mathfrak{N}_m) * \mathfrak{N}_n$.*

PROOF. (a) If $G \in \mathfrak{U} * \mathfrak{V}$, then $U_1(G) \subseteq U(G) \subseteq V^*(G) \subseteq V_1(G)$, so $G \in \mathfrak{U}_1 * \mathfrak{V}_1$.

(b) Let $G \in \mathfrak{A} * \mathfrak{U}$, so $\gamma_2(G) \subseteq U^*(G)$. Then (2.3)(g) ensures the inclusion $U(G) \subseteq \varsigma(G)$. Thus $G \in \mathfrak{U} * \mathfrak{A}$.

(c) This follows immediately from the remarks made after (3.5) and the fact that for any group G and integers $m, n \geq 0$, $\varsigma_{m+n}(G)/\varsigma_n(G) = \varsigma_m(G/\varsigma_n(G))$ (see [15] 5.1.11(iv)).

(3.7) EXAMPLE. The example we describe has a threefold purpose. First it shows that the product $*$ as defined in (3.1) is not associative. Secondly, it provides an instance where in (3.3)(a) strict inclusion holds. Thirdly, it indicates that in (3.6)(b) the variety $\mathfrak{A}$ cannot be replaced by the variety $\mathfrak{N}_2$.

Consider the varieties $\mathfrak{U} = \mathfrak{V} = \mathfrak{A}$ and $\mathfrak{W} = \mathfrak{S}_2 = [\mathfrak{A}, \mathfrak{A}]$. We claim that, with this notation, strict inclusion in (3.5) holds. For suppose that $\mathfrak{A} * (\mathfrak{A} * \mathfrak{S}_2) = (\mathfrak{A} * \mathfrak{A}) * \mathfrak{S}_2$. Then by applying (3.6) several times we get $(\mathfrak{A} * \mathfrak{A}) * \mathfrak{S}_2 = \mathfrak{A} * (\mathfrak{A} * \mathfrak{S}_2) \subseteq (\mathfrak{A} * \mathfrak{S}_2) * \mathfrak{A} \subseteq (\mathfrak{S}_2 * \mathfrak{A}) * \mathfrak{A} = \mathfrak{S}_2 * (\mathfrak{A} * \mathfrak{A})$. Hence it would follow that $\mathfrak{N}_2 * \mathfrak{S}_2 \subseteq \mathfrak{S}_2 * \mathfrak{N}_2$, which is not the case by virtue of Example 2, page 106 in [11]. This also explains the third purpose, as referred to above.

Next, there exists a group G, such that $T^*(G)/W^*(G) \subsetneqq V^*(G/W^*(G))$, where $\mathfrak{T} = \mathfrak{V} * \mathfrak{W} = \mathfrak{A} * \mathfrak{S}_2$. Indeed, if this would not be the case, then the observations made after (3.5) would yield that $\mathfrak{A} * (\mathfrak{A} * \mathfrak{S}_2) = (\mathfrak{A} * \mathfrak{A}) * \mathfrak{S}_2$. However, we have just shown that this is false. Hence here we see that in (3.3) the inclusion signs can be strict. This justifies the following definition.

(3.8) DEFINITION. Let $\mathfrak{U}$ and $\mathfrak{V}$ be varieties and put $\mathfrak{W} = \mathfrak{U} * \mathfrak{V}$. For a group G let $\Delta_{\mathfrak{U},\mathfrak{V}}(G) = U^*(G/V^*(G))/(W^*(G)/V^*(G))$.

In other words, $\Delta_{\mathfrak{U},\mathfrak{V}}(G)$ measures to what extent the group $U^*(G/V^*(G))$ deviates from the group $W^*(G)/V^*(G)$, following (3.3). In Theorem (5.3) we will prove that $\Delta_{\mathfrak{U},\mathfrak{V}}(G)$ is a so-called family invariant for $\mathfrak{W}$-isologism.

We close this section with examining the relations between the varietal product (as defined by Hanna Neumann), the product $*$ and the commutator product of varieties.

(3.9) PROPOSITION. *Let $\mathfrak{U}$, $\mathfrak{V}$ and $\mathfrak{W}$ be varieties. Then the following hold.*
(a) $\mathfrak{U} \vee \mathfrak{V} \subseteq \mathfrak{U} * \mathfrak{V} \subseteq \mathfrak{V}\mathfrak{U}$.
(b) *If* $\mathfrak{U} \subseteq \mathfrak{W} * \mathfrak{A}$, *then* $\mathfrak{U} * \mathfrak{V} \subseteq [\mathfrak{W}, \mathfrak{V}]$. *In particular* $\mathfrak{U} * \mathfrak{V} \subseteq [\mathfrak{U}, \mathfrak{V}]$.

PROOF. (a) By (2.3)(e) we have $[U(G)V^*G] \subseteq U(G) \cap V(G)$ for any group G. Hence $\mathfrak{U} \vee \mathfrak{V} \subseteq \mathfrak{U} * \mathfrak{V}$. Now assume $G \in \mathfrak{U} * \mathfrak{V}$, so that $U(G) \subseteq V^*(G)$. According to (2.3)(a) we get $V(U(G)) = 1$. Thus $G \in \mathfrak{V}\mathfrak{U}$. We conclude that $\mathfrak{U} * \mathfrak{V} \subseteq \mathfrak{V}\mathfrak{U}$.

(b) Let $G \in \mathfrak{U} * \mathfrak{V}$, so that $U(G) \subseteq V^*(G)$. By hypothesis $U(G) \supseteq [W(G), G]$. Hence $[W(G), G] \subseteq V^*(G)$ and (2.3)(g) ensures $[V(G), W(G)] = 1$. Thus $G \in [\mathfrak{W}, \mathfrak{V}]$. This shows that $\mathfrak{U} * \mathfrak{V} \subseteq [\mathfrak{W}, \mathfrak{V}]$. Certainly $\mathfrak{U} \subseteq \mathfrak{U} * \mathfrak{A}$, so that the last assertion follows by setting $\mathfrak{W} = \mathfrak{U}$.

4. Isologisms

In a short paper, [7], Philip Hall introduced the notion of isologism, an equivalence relation on the class of all groups. This equivalence relation depends on some fixed variety $\mathfrak{V}$ and has the property that the groups in the variety $\mathfrak{V}$ form a single equivalence class. The notion of isoclinism, which also originated with Hall, is a special case of isologism by taking for $\mathfrak{V}$ the variety of all abelian groups. As Hall pointed out, the idea behind isologism is to build a general theory of classification of groups, more precisely, to "obtain a distinct system of classification corresponding to every fully invariant subgroup of the free group with a denumerable infinite number of generators".

(4.1) DEFINITION. Let $\mathfrak{V}$ be a variety and G and H be groups. A *$\mathfrak{V}$-isologism between G and H* is a pair of isomorphisms (α, β) with $\alpha \colon G/V^*(G) \xrightarrow{\sim} H/V^*(H)$ and $\beta \colon V(G) \xrightarrow{\sim} V(H)$, such that for all $s > 0$, all $v(x_1, \dots, x_s) \in V(F_s)$ and all $g_1, \dots, g_s \in G$, it holds that $\beta(v(g_1, \dots, g_s)) = v(h_1, \dots, h_s)$, whenever $h_i \in \alpha(g_i V^*(G))$ $(i = 1, \dots, s)$. We write $G \underset{\mathfrak{V}}{\sim} H$ and we will say that G and H are *$\mathfrak{V}$-isologic.*

Notice that if one puts $\mathfrak{V} = \mathfrak{N}_n$, the above definition is nothing else but the definition of n-isoclinism (see for example [8], Section 3). In that case we write $G \underset{n}{\sim} H$ in stead of $G \underset{\mathfrak{N}_n}{\sim} H$. For the rest of this paper the word "isoclinism" will mean "n-isoclinism", for some unspecified $n \geq 0$. (This, in contrast to [8], where "isoclinism" and "1-isoclinism" are synonymous.)

We collect some elementary properties of $\mathfrak{V}$-isologisms.

(4.2) LEMMA. *Let (α, β) be a $\mathfrak{V}$-isologism between G_1 and G_2. The following hold.*

(a) *If $V^*(G_1) \leq H_1 \leq G_1$ and $\alpha[H_1/V^*(G_1)] = H_2/V^*(G_2)$, then $H_1 \underset{\mathfrak{V}}{\sim} H_2$.*

(b) *If $N_1 \trianglelefteq G_1$ and $N_1 \subseteq V(G_1)$, then $G_1/N_1 \underset{\mathfrak{V}}{\sim} G_2/\beta[N_1]$.*

PROOF. Omitted. It runs along the same lines as in the n-isoclinic case, see [2], Lemma 1.2.

(4.3) PROPOSITION. *Let (α, β) be a $\mathfrak{V}$-isologism between G and H. Let $v \in V(G)$. Then the following hold.*

(a) $\alpha(vV^*(G)) = \beta(v)V^*(H)$.

(b) *If $g \in G$ and $h \in \alpha(gV^*(G))$, then $\beta(v^g) = \beta(v)^h$.*

PROOF. (a) This is clear from the definition of $\mathfrak{V}$-isologism.

(b) Let $w \in V(F_s)$, say $w = w(x_1, \dots, x_s)$. Let $g, g_1, \dots, g_s \in G$ and choose $h \in \alpha(gV^*(G))$, $h_i \in \alpha(g_i V^*(G))$ $(i = 1, \dots, s)$. Observe that $h_i^h \in \alpha(g_i^g V^*(G))$, because α is a homomorphism. We have $\beta((w(g_1, \dots, g_s))^g) = \beta(w(g_1^g, \dots, g_s^g)) = w(h_1^h, \dots, h_s^h) = (w(h_1, \dots, h_s))^h = \beta(w(g_1, \dots, g_s))^h$.

(4.4) LEMMA. *Let $H \leq G$, $N \trianglelefteq G$ and $\mathfrak{V}$ be a variety. Then the following hold.*

(a) *$H \underset{\mathfrak{V}}{\sim} HV^*(G)$. In particular, if $G = HV^*(G)$, then $G \underset{\mathfrak{V}}{\sim} H$. Conversely, if $G/V^*(G)$ satisfies the descending chain condition on subgroups and $G \underset{\mathfrak{V}}{\sim} H$, then $G = HV^*(G)$.*

(b) *$G/N \underset{\mathfrak{V}}{\sim} G/(N \cap V(G))$. In particular, if $N \cap V(G) = 1$, then $G \underset{\mathfrak{V}}{\sim} G/N$. Conversely, if $V(G)$ satisfies the ascending chain condition on normal subgroups and $G \underset{\mathfrak{V}}{\sim} G/N$, then $N \cap V(G) = 1$.*

PROOF. (a) We define a map α by putting $\alpha(hV^*(H)) = hV^*(HV^*(G))$ ($h \in H$). Since $V^*(HV^*(G)) = V^*(H)V^*(G)$, α is an isomorphism from $H/V^*(H)$ onto $HV^*(G)/V^*(HV^*(G))$. Since $V(H) = V(HV^*(G))$ and α induces the identity on $V(H)$, the pair $(\alpha, id_{V(H)})$ is a $\mathfrak{V}$-isologism between H and $HV^*(G)$.

Now suppose $H \leq G$ and $H \underset{\mathfrak{V}}{\sim} G$. By the above we may assume that $H \supseteq V^*(G)$. Now put $H = H_0$. There exists an isomorphism $\alpha_0 \colon G/V^*(G) \to H/V^*(H)$. Define $H_1 \leq H$ by $\alpha_0[H/V^*(G)] = H_1/V^*(H)$. So $H_1 \supseteq V^*(H)$ and by (4.2)(a) we have $H \underset{\mathfrak{V}}{\sim} H_1$, thus $G \underset{\mathfrak{V}}{\sim} H_1$. Observe that $G = H$ if and only if $H = H_1$. Apparently there exists an isomorphism $\alpha_1 \colon H/V^*(H) \to H_1/V^*(H_1)$. Define $H_2 \leq H_1$ by $\alpha_1[H_1/V^*(H)] = H_2/V^*(H_1)$. Hence $H_2 \supseteq V^*(H_1) \supseteq V^*(H)$ and by (4.2)(a) $H_1 \underset{\mathfrak{V}}{\sim} H_2$, so that $G \underset{\mathfrak{V}}{\sim} H_2$. Again, observe that $H = H_1$ if and only if $H_1 = H_2$. Continuing the above process, we get a sequence of subgroups of H, $H = H_0 \geq H_1 \geq H_2 \geq \cdots \geq V^*(H)$, with the property that $G \underset{\mathfrak{V}}{\sim} H_i$ for each $i \geq 0$. If however $G/V^*(G)$, and hence $H/V^*(H)$, satisfies the descending chain condition on subgroups, then it follows that for some $i \geq 0$ we have $H_i = H_{i+1}$. But this is equivalent to $G = H$, as desired.

(b) We denote $\overline{G} = G/N$ and $\tilde{G} = G/(N \cap V(G))$. Note that if $u, w \in V(G)$, then $\overline{u} = \overline{w} \Leftrightarrow \tilde{u} = \tilde{w}$. Hence, if $v(x_1, \ldots, x_s) \in V(F_s)$ and $g_1, \ldots, g_s \in G$, then for a $g \in G$ we have $(1 \leq i \leq s)$

$$\begin{aligned} v(\overline{g}_1, \ldots, \overline{g}_i\overline{g}, \ldots, \overline{g}_s) &= v(\overline{g}_1, \ldots, \overline{g}_i, \ldots, \overline{g}_s) \\ &\Leftrightarrow v(\tilde{g}_1, \ldots, \tilde{g}_i\tilde{g}, \ldots, \tilde{g}_s) = v(\tilde{g}_1, \ldots, \tilde{g}_i, \ldots, \tilde{g}_s). \end{aligned}$$

Hence $\overline{g} \in V^*(\overline{G})$ if and only if $\tilde{g} \in V^*(\tilde{G})$. If we define a map α by setting $\alpha(\overline{g}V^*(\overline{G})) = \tilde{g}V^*(\tilde{G})$, then α is an isomorphism from $\overline{G}/V^*(\overline{G})$ onto $\tilde{G}/V^*(\tilde{G})$. Let $u \in V(G)$ and put $\beta(\overline{u}) = \tilde{u}$. Then β is an isomorphism from $V(\overline{G})$ onto $V(\tilde{G})$ and clearly the pair (α, β) is a $\mathfrak{V}$-isologism between $\overline{G}$ and $\tilde{G}$.

Conversely, if $N \trianglelefteq G$ and $G \underset{\mathfrak{V}}{\sim} G/N$, then by the above we may assume that $N \subseteq V(G)$. Let $\beta_0 \colon V(G) \to V(G/N) = V(G)/N$ be an isomorphism. Write $N = N_0$. Define $N_1 \subseteq V(G)$ by $\beta_0[N_0] = N_1/N_0$. We are assured by (4.3)(b) that $N_1 \trianglelefteq G$. By virtue of (4.2)(b) $G \underset{\mathfrak{V}}{\sim} G/N_1$. Observe that $N = 1$ if and only if $N_0 = N_1$. Now there exists an isomorphism $\beta_1 : V(G) \to V(G/N_1)$. Define $N_2 \geq N_1$ by $\beta_1[N_1] = N_2/N_1$. Hence $N_2 \subseteq V(G)$ and $N_2 \trianglelefteq G$. Moreover $G \underset{\mathfrak{V}}{\sim} G/N_2$ and $N_0 = N_1$ if and only if $N_1 = N_2$. Continuing this construction we arrive at a sequence $N = N_0 \leq N_1 \leq N_2 \leq \cdots \leq V(G)$ of normal subgroups of G, with the property that $G \underset{\mathfrak{V}}{\sim} G/N_j$ for each $j \geq 0$. If $V(G)$ satisfies the ascending chain condition on normal subgroups, then we must have $N_j = N_{j+1}$ for some $j \geq 0$. It follows that $N = 1$, as required.

If (α, β) is a $\mathfrak{V}$-isologism between G and H, then it is not difficult to see that Proposition (4.3) implies that β induces an isomorphism from $V^*(G) \cap V(G)$ onto $V^*(H) \cap V(H)$. Also, for instance, $G/V^*(G)V(G) \simeq H/V^*(H)V(H)$. We are here dealing with so-called *family invariants*, groups which only depend on the $\mathfrak{V}$-isologism class. Other examples are contained in the following two propositions.

(4.5) PROPOSITION. *Let $\mathfrak{V}$ be a variety and let (α, β) be a $\mathfrak{V}$-isologism between G and H. Then the following hold.*

(a) *For all $n \geq 0$:* $\alpha[\gamma_{n+1}(G)V^*(G)/V^*(G)] = \gamma_{n+1}(H)V^*(H)/V^*(H)$.

(b) *For all $n \geq 0$:* $\beta[\varsigma_n(G) \cap V(G)] = \varsigma_n(H) \cap V(H)$.

PROOF. (a) Clear. In fact, $\gamma_{n+1}(G)$, respectively $\gamma_{n+1}(H)$, can be replaced by any verbal subgroup $W(G)$, respectively $W(H)$, where $\mathfrak{W}$ is any variety.

(b) This is proved by applying induction on n and utilizing (4.3)(b). The proof runs as in the analogous n-isoclinic case as given in [8], Theorem 3.12 (b).

(4.6) PROPOSITION. *Let $\mathfrak{V}$ be a variety and let (α, β) be a $\mathfrak{V}$-isologism between G and H. Let $M \trianglelefteq G$ and put $\alpha[MV^*(G)/V^*(G)] = N/V^*(H)$. Then $\beta[[MV^*G]] = [NV^*H]$.*

PROOF. Let $m \in M$, $g_1, \dots, g_s \in G$ and $v(x_1, \dots, x_s) \in V(F_s)$. Choose $h_i \in \alpha(g_iV^*(G))$ $(i = 1, \dots, s)$ and $n \in \alpha(mV^*(G))$. By definition of $\mathfrak{V}$-isologism we have that $\beta(v(g_1, \dots, g_im, \dots, g_s)) = v(h_1, \dots, h_in, \dots, h_s)$ and $\beta(v(g_1, \dots, g_i, \dots, g_s)) = v(h_1, \dots, h_i, \dots, h_s)$. Hence

$$\begin{aligned}\beta(v(g_1, \dots, g_im, \dots, g_s)&v(g_1, \dots, g_i, \dots, g_s)^{-1}) \\ &= v(h_1, \dots, h_in, \dots, h_s)v(h_1, \dots, h_i, \dots, h_s)^{-1}.\end{aligned}$$

We conclude that $\beta[[MV^*G]] \subseteq [NV^*G]$ and the reverse inclusion follows by applying the above arguments to β^{-1}.

5. Constructions

Not all results known to be true for n-isoclinisms carry over to the general $\mathfrak{V}$-isologisms. We will encounter examples of this phenomenon later on. The next theorem however is fortunately an exception. As we will see, it will be very useful in proving certain results in Section 6.

(5.1) THEOREM. *Let $\mathfrak{V}$ be a variety and G_1 and G_2 be groups. Then $G_1 \underset{\mathfrak{V}}{\sim} G_2$ if and only if there exists a group G containing normal subgroups N_1 and N_2, such that $G_1 \simeq G/N_1$, $G_2 \simeq G/N_2$ and $G_1 \underset{\mathfrak{V}}{\sim} G \underset{\mathfrak{V}}{\sim} G_2$.*

PROOF. The "if" part is clear. So let us assume that $G_1 \underset{\mathfrak{V}}{\sim} G_2$, say (α, β) is a $\mathfrak{V}$-isologism between G_1 and G_2. Consider a subgroup G of $G_1 \times G_2$ given by

$$G = \{(g_1, g_2) \in G_1 \times G_2 \colon \alpha(g_1 V^*(G_1)) = g_2 V^*(G_2)\}.$$

Let $N_1 = \{(1, n_2) \colon n_2 \in V^*(G_2)\}$ and $N_2 = \{(n_1, 1) \colon n_1 \in V^*(G_1)\}$. Then N_1 and N_2 are normal subgroups of G. Moreover $G/N_i \simeq G_i$ $(i = 1, 2)$. By definition of $\mathfrak{V}$-isologism $V(G)$ is generated by elements of the form

$$(v(g_1, \dots, g_s), \beta(v(g_1, \dots, g_s)))(g_1, \dots, g_s \in G, v \in V(F_s)).$$

It follows that $N_i \cap V(G) = 1$ $(i = 1, 2)$. By virtue of (4.4)(b) we have $G \underset{\mathfrak{V}}{\sim} G/N_i \simeq G_i$ $(i = 1, 2)$.

We mention that the case $\mathfrak{V} = \mathfrak{A}$ was proved by P. M. Weichsel (see [21]), while the case $\mathfrak{V} = \mathfrak{N}_n$ is attributed to J. C. Bioch ([2], Theorem 1.4). In the latter case the reader is also referred to [8], Sections 4 and 5, for some applications. Here we exhibit the following corollary. It could also be proved directly, but is somewhat tricky to do so.

(5.2) COROLLARY. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be varieties. Then the following are equivalent.*

(a) $\mathfrak{U} \subset \mathfrak{V}$.

(b) *For any two groups G and H, $G \underset{\mathfrak{U}}{\sim} H$ implies $G \underset{\mathfrak{V}}{\sim} H$.*

PROOF. (a)⇒(b): Let G and H be groups with $G \underset{\mathfrak{U}}{\sim} H$. We may assume by (5.1) that $H \simeq G/N$ for some $N \trianglelefteq G$ with $N \cap U(G) = 1$. As $\mathfrak{U} \subseteq \mathfrak{V}$, we have $U(G) \supseteq V(G)$, whence $N \cap V(G) = 1$. By virtue of (4.4)(b) we get $G \underset{\mathfrak{V}}{\sim} G/N$, as desired.

(b)⇒(a): Let $G \in \mathfrak{U}$, so certainly $G \underset{\mathfrak{U}}{\sim} 1$, see (2.3)(b). By hypothesis this implies $G \underset{\mathfrak{V}}{\sim} 1$, so in particular $V(G) = 1$. Again (2.3)(b) shows that $G \in \mathfrak{V}$.

Note that (5.2) really says that the larger the variety, the cruder the equivalence relation of isologism becomes. Next we fulfill a promise made after (3.8) about $\Delta_{\mathfrak{U},\mathfrak{V}}(G)$. This group is a family invariant in the sense of Section 4.

(5.3) THEOREM. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be varieties and put $\mathfrak{W} = \mathfrak{U} * \mathfrak{V}$. Suppose $G \underset{\mathfrak{W}}{\sim} H$. Then $\Delta_{\mathfrak{U},\mathfrak{V}}(G) \simeq \Delta_{\mathfrak{U},\mathfrak{V}}(H)$.*

PROOF. Again, in view of (5.1), we may assume that $H = G/N$ for some $N \trianglelefteq G$ with $N \cap W(G) = 1$. By (2.3)(f) it follows that $N \subseteq W^*(G)$ and $W^*(G/N) = W^*(G)/N$. Now put $V^*(G/N) = M/N$, so that $M \trianglelefteq G$ and $NV^*(G) \subseteq M$. Hence

$$\begin{aligned}(*)\ \Delta_{\mathfrak{U},\mathfrak{V}}(G/N) &= U^*((G/N)/V^*(G/N))/(W^*(G/N)/V^*(G/N)) \\ &\simeq U^*((G/V^*(G))/(M/V^*(G)))/(W^*(G)/V^*(G))/(M/V^*(G)).\end{aligned}$$

We have that $[MV^*G] \subseteq N$. Thus

$$[(M \cap U(G))V^*G] \subseteq [MV^*G] \cap [U(G)V^*G] \subseteq N \cap W(G) = 1.$$

Then (2.3)(c) gives $M \cap U(G) \subseteq V^*(G)$, whence $M/V^*(G) \cap U(G/V^*(G)) = \bar{1}$. An application of (2.3)(f) again shows that

$$U^*((G/V^*(G))/(M/V^*(G))) = U^*(G/V^*(G))/(M/V^*(G)).$$

Substituting this last formula in $(*)$, we obtain that $\Delta_{\mathfrak{U},\mathfrak{V}}(G/N) \simeq \Delta_{\mathfrak{U},\mathfrak{V}}(G)$, as wanted.

As we have seen, given two $\mathfrak{V}$-isologic groups G_1 and G_2, one can construct a so-called common $\mathfrak{V}$-isologic *ancestor* G, that is, G_1 and G_2 occur as factor groups of G, whereas G, G_1 and G_2 are $\mathfrak{V}$-isologic to each other. This is the contents of (5.1). Now consider a dual question. Do two $\mathfrak{V}$-isologic groups G_1 and G_2 always have a common $\mathfrak{V}$-isologic descendant G, that is, can G_1 and G_2 be embedded in a group G, while G, G_1 and G_2 are $\mathfrak{V}$-isologic to each other? In case $\mathfrak{V} = \mathfrak{A}$ there is an affirmative answer to this question (see [8], Theorem 4.2), but in its full generality it remains unanswered. Here we supply a partial answer which shows for instance that for any abelian variety $\mathfrak{V}$ the above question can be solved.

(5.4) THEOREM. *Let $\mathfrak{V}$ be a variety and let (α, β) be a $\mathfrak{V}$-isologism between G_1 and G_2. Then there exists a group H with subgroups H_1 and H_2, such that $G_1/[G_1, V^*(G_1)] \simeq H_1$, $G_2/\beta[[G_1, V^*(G_1)] \cap V(G_1)] \simeq H_2$ and $H_1 \underset{\mathfrak{V}}{\sim} H \underset{\mathfrak{V}}{\sim} H_2$.*

PROOF. Consider the following subgroup of $G_1 \times G_2$ given by

$$X = \{(g_1, g_2) \in G_1 \times G_2 \colon \alpha(g_1 V^*(G_1)) = g_2 V^*(G_2)\}.$$

https://doi.org/10.1017/S1446788700030366 Published online by Cambridge University Press

Then $V(X) = \{(g_1, \beta(g_1)) : g_1 \in V(G_1)\}$, by the definition of $\mathfrak{V}$-isologism. Next, as in the proof of Theorem (5.1), put

$$N_1 = \{(1, n_2) : n_2 \in V^*(G_2)\} \text{ and } N_2 = \{(n_1, 1) : n_1 \in V^*(G_1)\}.$$

Thus, $N_i \trianglelefteq X$, $X/N_i \simeq G_i$ and $N_i \cap V(X) = 1$ $(i = 1, 2)$. Define

$$G = X/N_1 \times X/V(X). \tag{1}$$

Observe that $G \underset{\mathfrak{V}}{\sim} G_1 \underset{\mathfrak{V}}{\sim} X$. Since the intersection of N_1 and $V(X)$ is trivial, X can be embedded in G by an injective homomorphism $\iota : X \to G$ defined by

$$\iota(x) = (xN_1, xV(X)) \qquad (x \in X).$$

Now $\iota[N_2]$ is in general not a normal subgroup of G. Therefore let N be the normal closure of $\iota[N_2]$ in G, that is

$$N = \iota[N_2][G, \iota[N_2]].$$

One easily checks that

$$N = \iota[N_2] \cdot \langle((1,1)N_1, (c_1, 1)V(X)) : c_1 \in [G_1, V^*(G_1)]\rangle.$$

Next define two homomorphisms $f_1 : G_1 \to G/N$ and $f_2 : G_2 \to G/N$ as follows. First, let $g_1 \in G_1$ and choose $g_2 \in G_2$ such that $(g_1, g_2) \in X$. Then put

$$f_1(g_1) = ((g_1, g_2)N_1, (1,1)V(X)) \cdot N.$$

Secondly, let $h_2 \in G_2$ and choose $h_1 \in G_1$ such that $(h_1, h_2) \in X$. Then put

$$f_2(h_2) = ((h_1, h_2)N_1, (h_1, h_2)V(X)) \cdot N.$$

We claim that f_1 and f_2 are indeed well-defined homomorphisms and satisfy the following property

$$f_1[G_1]V^*(G/N) = G/N = f_2[G_2]V^*(G/N). \tag{2}$$

Now the definition of N_1 guarantees f_1 being well-defined, and the fact that $\iota[N_2] \subseteq N$ yields that f_2 is well-defined. Clearly f_1 and f_2 are homomorphisms. It follows from (1) and (2.3)(a)(f) that

$$V^*(G) = V^*(X)/N_1 \times X/V(X).$$

Now let (g_1, g_2) and (h_1, h_2) be elements of X. Then on the one hand we have

$$\begin{aligned}
&((g_1, g_2)N_1, (h_1, h_2)V(X)) \cdot N \\
&\quad = ((g_1, g_2)N_1, (1,1)V(X)) \cdot N \cdot ((1,1)N_1, (h_1, h_2)V(X)) \cdot N \\
&\quad \in f_1[G_1] \cdot V^*(G)N/N \subseteq f_1[G_1] \cdot V^*(G/N).
\end{aligned}$$

On the other hand it holds that

$$\begin{aligned}
&((g_1, g_2)N_1, (h_1, h_2)V(X)) \cdot N \\
&\quad = ((g_1, g_2)N_1, (g_1, g_2)V(X)) \cdot N \cdot ((1,1)N_1, (g_1^{-1}h_1, g_2^{-1}h_2)V(X)) \cdot N \\
&\quad \in f_2[G_2] \cdot V^*(G)N/N \subseteq f_2[G_2] \cdot V^*(G/N).
\end{aligned}$$

https://doi.org/10.1017/S1446788700030366 Published online by Cambridge University Press

We conclude that (2) holds, which proves the above claim.

Finally, we calculate the kernels $\ker(f_1)$ and $\ker(f_2)$.

(i) $\ker(f_1) = [G_1, V^*(G_1)]$.

PROOF OF (i). Suppose $g_1 \in \ker(f_1)$. Choose $g_2 \in G_2$ with $(g_1, g_2) \in X$. Hence $((g_1, g_2)N_1, (1,1)V(X)) \in N$, say for $n_1 \in V^*(G_1)$ and $c_1 \in [G_1, V^*(G_1)]$

$$((g_1, g_2)N_1, (1,1)V(X)) = ((n_1, 1)N_1, (n_1, 1)V(X)) \cdot ((1,1)N_1, (c_1, 1)V(X)).$$

Hence $g_1 = n_1$ and $(n_1 c_1, 1) \in V(X)$. Then $n_1 = c_1^{-1} \in [G_1, V^*(G_1)]$, so $g_1 \in [G_1, V^*(G_1)]$. Conversely, if $g_1 \in [G_1, V^*(G_1)]$, then $(g_1, 1) \in N_2$. Hence

$$\begin{aligned} &((g_1, 1)N_1, (1,1)V(X)) \\ &= ((g_1, 1)N_1, (g_1, 1)V(X)) \cdot ((1,1)N_1, (g_1^{-1}, 1)V(X)) \\ &\in \iota[N_2] \cdot \langle ((1,1)N_1, (c_1, 1)V(X)) : c_1 \in [G_1, V^*(G_1)] \rangle = N. \end{aligned}$$

So $f_1(g_1) = \overline{1}$. This proves (i).

(ii) $\ker(f_2) = \beta[V(G_1) \cap [G_1, V^*(G_1)]]$.

PROOF OF (ii). Suppose $g_2 \in \ker(f_2)$. Choose $g_1 \in G_1$ with $(g_1, g_2) \in X$. Hence $((g_1, g_2)N_1, (g_1, g_2)V(X)) \in N$, say for $n_1 \in V^*(G_1)$ and $c_1 \in [G_1, V^*(G_1)]$

$$((g_1, g_2)N_1, (g_1, g_2)V(X)) = ((n_1, 1)N_1, (n_1 c_1, 1)V(X)).$$

Hence $g_1 = n_1$ and $(g_1 c_1^{-1} n_1^{-1}, g_2) \in V(X)$. So $(g_1 c_1^{-1} n_1^{-1}) = g_2$ and $g_1 c_1^{-1} n_1^{-1} \in V(G_1)$. But as $c_1 \in [G_1, V^*(G_1)]$, we have that

$$g_1 c_1^{-1} n_1^{-1} = n_1 c_1^{-1} n_1^{-1} \in [G_1, V^*(G_1)].$$

So $g_2 \in \beta[V(G_1) \cap [G_1, V^*(G_1)]]$. Conversely, if $g_2 \in \beta[V(G_1) \cap [G_1, V^*(G_1)]]$, say $g_2 = \beta(g_1)$, where $g_1 \in V(G_1) \cap [G_1, V^*(G_1)]$, then $(g_1, g_2) \in X$ by virtue of (4.3)(a). In particular $(g_1, g_2) \in V(X)$. Now we can write

$$\begin{aligned} &((g_1, g_2)N_1, (g_1, g_2)V(X)) \\ &\quad = ((g_1, 1)N_1, (g_1, 1)V(X)) \cdot ((1, g_2)N_1, (1, g_2)V(X)) \\ &\quad = ((g_1, 1)N_1, (g_1, 1)V(X)) \cdot ((1,1)N_1, (g_1^{-1}, 1)V(X)) \\ &\quad \in \iota[N_2] \cdot \langle ((1,1)N_1, (c_1, 1)V(X)) : c_1 \in [G_1, V^*(G_1)] \rangle = N. \end{aligned}$$

Hence $f_2(g_2) = \overline{1}$. This proves (ii).

From (4.4)(a) and (2) it follows that

$$f_1[G_1] \underset{\mathfrak{V}}{\sim} G/N \underset{\mathfrak{V}}{\sim} f_2[G_2].$$

A simple application of the First Isomorphism Theorem shows us that $H = G/N$, $H_1 = f_1[G_1]$ and $H_2 = f_2[G_2]$ are the groups we are looking for.

(5.5) COROLLARY. *Let $\mathfrak{V}$ be a variety and $G_1 \underset{\mathfrak{V}}{\sim} G_2$. Assume that $V^*(G_1) \subseteq \varsigma(G_1)$. Then there exists a group G with subgroups H_1 and H_2 such that $G_1 \simeq H_1$, $G_2 \simeq H_2$ and $G_1 \underset{\mathfrak{V}}{\sim} G \underset{\mathfrak{V}}{\sim} G_2$.*

The above corollary certainly applies to abelian G_1 or abelian varieties $\mathfrak{V}$. Of course there are still other cases where we can apply (5.5). To give a small example: let $\mathfrak{V}$ be any variety that does not contain S_3 (any nilpotent variety will do). Then we must have $V^*(S_3) = 1$. For (2.3)(b)(g) show that $V^*(S_3) \neq S_3$ and $V^*(S_3) \neq A_3$. Hence $V^*(S_3) = \varsigma(S_3) = 1$. So here we can take G_1 to be S_3.

If we choose $\mathfrak{V} = \mathfrak{A}$ in (5.5), we obtain Theorem 4.2 of [8]. However the construction used there to derive the result is of a completely different nature than the one employed in the proof of (5.4). Nevertheless R. W. van der Waall pointed out (see [20]), that the group G we are concerned with here in (5.5) in case $\mathfrak{V} = \mathfrak{A}$ and the group G in Theorem 4.2 of [8], are in fact isomorphic. He also proved (5.4) in the special case $\mathfrak{V} = \mathfrak{N}_n$, inspired by an untransparent passage in the book of R. Beyl and J. Tappe (see [1], page 130). As we have seen things can be done for a general variety $\mathfrak{V}$.

6. Induced isologisms

Given any two varieties $\mathfrak{U}$ and $\mathfrak{V}$ we defined in Section 3 a new variety $\mathfrak{W} = \mathfrak{U} * \mathfrak{V}$. Here we investigate how the isologisms with respect to $\mathfrak{U}$, $\mathfrak{V}$ and $\mathfrak{W}$ are related. This will lead us to a generalization of Theorems 5.2 and 5.5 in [8].

(6.1) THEOREM. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be varieties and put $\mathfrak{W} = \mathfrak{U} * \mathfrak{V}$. Suppose $G \underset{\mathfrak{W}}{\sim} H$. Then the following hold.*

(a) $G/V^*(G) \underset{\mathfrak{U}}{\sim} H/V^*(H)$.

(b) $U(G) \underset{\mathfrak{V}}{\sim} U(H)$.

PROOF. By virtue of (5.1) it suffices in both (a) and (b) to assume that $H = G/N$, where $N \trianglelefteq G$ with $N \cap W(G) = 1$.

(a) Put $V^*(G/N) = M/N$, so that $M \trianglelefteq G$ and $NV^*(G) \subseteq M$. We claim that it is sufficient to show that $M \cap U(G) \subseteq V^*(G)$. Indeed, it implies $(M \cap U(G))V^*(G) = V^*(G)$, whence $M/V^*(G) \cap U(G/V^*(G)) = \bar{1}$, according to Dedekind's Rule and (2.3)(d). Therefore, by utilizing (4.4)(b),

$$\begin{aligned} G/V^*(G) &\underset{\mathfrak{U}}{\sim} (G/V^*(G))/(M/V^*(G)) \simeq G/M \simeq (G/N)/(M/N) \\ &\simeq (G/N)/V^*(G/N), \end{aligned}$$

https://doi.org/10.1017/S1446788700030366 Published online by Cambridge University Press

which is precisely what we want to prove. Certainly

$$[(M \cap U(G))V^*G] \subseteq [MV^*G] \cap [U(G)V^*G] \subseteq N \cap W(G) = 1.$$

So indeed by (2.3)(c) we get $M \cap U(G) \subseteq V^*(G)$.

(b) Here we have to show that

$$U(G) \underset{\mathfrak{V}}{\sim} U(G/N) = U(G)N/N \simeq U(G)/(N \cap U(G)).$$

In view of (4.4)(b) it therefore suffices to show that $N \cap U(G) \cap V(U(G)) = 1$. Now $N \cap U(G) \cap V(U(G)) = N \cap V(U(G))$ and it follows from (3.9)(a) that $W(G) \supseteq V(U(G))$. As $N \cap W(G) = 1$, we obtain $N \cap V(U(G)) = 1$, as desired.

(6.2) COROLLARY. *Let $n \geq 0$ and suppose $G \underset{n}{\sim} H$. Then for each $i \in \{0, \dots, n\}$ the following hold.*

(a) $G/\zeta_i(G) \underset{n-i}{\sim} H/\zeta_i(H)$.

(b) $\gamma_{i+1}(G) \underset{n-i}{\sim} \gamma_{i+1}(H)$.

PROOF. By employing (3.6)(c) we have $\mathfrak{N}_i * \mathfrak{N}_{n-i} = \mathfrak{N}_n = \mathfrak{N}_{n-i} * \mathfrak{N}_i$ for any i with $0 \leq i \leq n$. Hence the assertions follow immediately from (6.1).

For the moment let $\mathfrak{X}$ denote a class of groups which is invariant under 1-isoclinism (=$\mathfrak{A}$-isologism). Examples of $\mathfrak{X}$ are the class of abelian, nilpotent, supersolvable, monomial or solvable groups, respectively.

(6.3) COROLLARY. *Let $\mathfrak{V}$ be a variety and $\mathfrak{W}$ a subvariety of $\mathfrak{A} * \mathfrak{V}$. Suppose that $G \underset{\mathfrak{W}}{\sim} H$. Then the following hold.*

(a) *$G/V^*(G) \in \mathfrak{X}$ if and only if $H/V^*(H) \in \mathfrak{X}$.*

(b) *$V(G) \in \mathfrak{X}$ if and only if $V(H) \in \mathfrak{X}$.*

PROOF. We are assured by (3.6)(b) that $\mathfrak{W} \subseteq \mathfrak{A} * \mathfrak{V} \subseteq \mathfrak{V} * \mathfrak{A}$. Hence apply (5.2) and (6.1).

It should be remarked that (6.2) and (6.3) entail generalizations of a theorem of J. C. Bioch and R. W. van der Waall, which states that monomiality is invariant under 1-isoclinism (see [3], Theorem 4.6). A well-known example to which (6.3) applies is the following. Take $\mathfrak{W} = \mathfrak{N}_{2l}$ and $\mathfrak{V} = \mathfrak{S}_l$ (see [15], 5.1.12 for the inclusion $\mathfrak{N}_{2_l - 1} \subseteq \mathfrak{S}_l$ and apply (3.6)(a)(c)).

One may wonder whether the isologisms given in (6.1)(a)(b) are the best possible in the sense that examples can be found of varieties $\mathfrak{U}$ and $\mathfrak{V}$ and groups G and H with $G \underset{\mathfrak{U}*\mathfrak{V}}{\sim} H$, and hence $G/V^*(G) \underset{\mathfrak{U}}{\sim} H/V^*(H)$ and $U(G) \underset{\mathfrak{V}}{\sim} U(H)$, such that $G/V^*(G)$ and $H/V^*(H)$ are not $\mathfrak{U}_1$-isologic, respectively $U(G)$ and $U(H)$ are not $\mathfrak{V}_1$-isologic, for all subvarieties $\mathfrak{U}_1 \subsetneq \mathfrak{U}$, respectively $\mathfrak{V}_1 \subsetneq \mathfrak{V}$.

However, if one would like to handle such a definition of "best possible isologism", this requires a thorough knowledge of the subvarieties of the given varieties $\mathfrak{U}$ and $\mathfrak{V}$. But in general it is a difficult, if not impossible, matter to determine exactly all subvarieties of a variety. Therefore we will employ a slightly weaker notion of "best possible isologism" by admitting for the above $\mathfrak{U}_1$ and $\mathfrak{V}_1$ only subvarieties of a certain type. Moreover, we will use it only in connection with isoclinisms. An application of (3.9)(a), (5.2) and (6.1) shows that, if $n \geq 0$ and $G \underset{n}{\sim} H$, then for any variety $\mathfrak{V}$ it holds that $G/V^*(G) \underset{n}{\sim} H/V^*(H)$ and $V(G) \underset{n}{\sim} V(H)$. Therefore the following definition makes sense.

(6.4) DEFINITION. Let $\mathfrak{V}$ be a variety and $n \geq 0$. Put $k = \min\{m \in \mathbf{Z}_{\geq 0} : \forall G, H : G \underset{n}{\sim} H \Rightarrow G/V^*(G) \underset{m}{\sim} H/V^*(H)\}$. Thus $G \underset{n}{\sim} H$ implies $G/V^*(G) \underset{k}{\sim} H/V^*(H)$. This k-isoclinism is called the *best possible* for $\mathfrak{V}$ and n. An analogous terminology is used for $V(\cdot)$ instead of $\cdot/V^*(\cdot)$.

It was pointed out in [8], Remark 5.3 that in (6.2)(a) above the isoclinism is the best possible for $\mathfrak{N}_i$ and n indeed. On the contrary, the isoclinism in (6.2)(b) can be sharpened substantially. It was shown in [8], Theorem 5.5 that, if $G \underset{n}{\sim} H$, then $\gamma_{i+1}(G) \underset{\lceil (n-i)/(i_1) \rceil}{\sim} \gamma_{i+1}(H)$, whenever $0 \leq i \leq n$, and this isoclinism is now optimal in the sense of (6.4) (see [8], Remark 5.6). The special commutator structure of the word $[x_1, \ldots, x_{n+1}]$, which defines the variety $\mathfrak{N}_n$, underlies this sharpening. In particular the Three Subgroups Lemma ([15], 5.1.10) implies that for $k, l \geq 1$, $\gamma_k(\gamma_l(G)) \leq \gamma_{kl}(G)$. This property is the key to the proof of Theorem 5.5 of [8]. So there is the problem what conditions a variety must satisfy to allow a sharpening of some induced isologism like the case discussed above. In the case $\mathfrak{N}_n$, the simple commutator word of weight $n+1$ gives rise to the existence and properties of the lower and upper central series. These series can be generalized as follows.

(6.5) DEFINITION. Let $\mathfrak{V}$ be any variety. Put $\mathfrak{V}_0 = \mathfrak{E}$ and for $n \geq 0$ put $\mathfrak{V}_{n+1} = \mathfrak{V} * \mathfrak{V}_n$, so that in particular $\mathfrak{V}_1 = \mathfrak{V}$. The series of varieties $\mathfrak{V}_0 \subseteq \mathfrak{V}_1 \subseteq \mathfrak{V}_2 \subseteq \cdots$ is called the *$\mathfrak{V}$-marginal series*. The verbal and marginal subgroups of a group G with respect to the variety $\mathfrak{V}_n$ will be denoted by $V_n(G)$ and $V_n^*(G)$ respectively.

(6.6) LEMMA. *Let $\mathfrak{V}$ be a variety and $n \geq 0$. Then for any $i \in \{0, \ldots, n\}$ it holds that $\mathfrak{V}_n \subseteq \mathfrak{V}_i * \mathfrak{V}_{n-i}$.*

PROOF. Since $\mathfrak{V}_m * \mathfrak{V}_0 = \mathfrak{V}_m = \mathfrak{V}_0 * \mathfrak{V}_m$ for each $m \geq 0$, it follows by induction on n that for $n \geq 0$ and any $i \in \{0, \dots, n\}$, using (3.5) and (3.6)(a),

$$\begin{aligned}\mathfrak{V}_{n+1} &= \mathfrak{V} * \mathfrak{V}_n \subseteq \mathfrak{V} * (\mathfrak{V}_i * \mathfrak{V}_{n-i}) \subseteq (\mathfrak{V} * \mathfrak{V}_i) * \mathfrak{V}_{n-i} \\ &= \mathfrak{V}_{i+1} * \mathfrak{V}_{(n+1)-(i+1)}\end{aligned}$$

holds. Together with $\mathfrak{V}_{n+1} = \mathfrak{V}_0 * \mathfrak{V}_{n+1}$ this proves the assertion.

The next proposition justifies the name $\mathfrak{V}$-marginal of the series in (6.5).

(6.7) PROPOSITION. *Let $\mathfrak{V}$ be a variety. Then the following hold.*

(a) *For all $n \geq 0$: $V_n(G)/V_{n+1}(G) \subseteq V^*(G/V_{n+1}(G))$.*

(b) *For all $n \geq 0$: $V^*_{n+1}(G)/V^*_n(G) \subseteq V^*(G/V^*_n(G))$.*

PROOF. (a) By (2.3)(c) we have to show that $[V_n(G)V^*G] \subseteq V_{n+1}(G)$. But this follows from (6.6) as $\mathfrak{V}_n * \mathfrak{V} \subseteq \mathfrak{V}_{n+1}$.

(b) As $\mathfrak{V}_{n+1} = \mathfrak{V} * \mathfrak{V}_n$, the assertion follows immediately from (3.2)(b).

Observe that $\mathfrak{E} \subseteq \mathfrak{A} \subseteq \mathfrak{N}_2 \subseteq \mathfrak{N}_3 \subseteq \cdots$ is the $\mathfrak{A}$-marginal series which determines the lower and upper central series of a group.

Further we notice that there exists some connection between our $\mathfrak{V}$-marginal series and the lower and upper Φ-marginal series as defined by J. A. Hulse and J. C. Lennox in [9], page 140. In our notation the $\mathfrak{V}$-marginal series in (6.5) is defined by multiplication by $\mathfrak{V}$, with respect to $*$, *from the left*, whereas in [9] a similar construction is carried out by multiplying *from the right*. The difference is that we define a series of *varieties*, where Hulse and Lennox define a series of verbal and marginal *subgroups*. Now the verbal and marginal subgroups associated with our $\mathfrak{V}$-marginal series yield marginal series indeed according to (6.7). Conversely, it is not clear whether there exists a series of varieties, which achieves the same for the lower and upper Φ-series of Hulse and Lennox.

(6.8) PROPOSITION. *Let $\mathfrak{V}$ be a variety and $n \geq 0$. Assume that $G \underset{\mathfrak{V}_n}{\sim} H$. Then the following hold.*

(a) $G \underset{\mathfrak{V}_{n+1}}{\sim} H$.

(b) *For any $i \in \{0, \dots, n\}$: $G/V^*_i(G) \underset{\mathfrak{V}_{n-i}}{\sim} H/V^*_i(H)$.*

(c) *For any $i \in \{0, \dots, n\}$: $V_i(G) \underset{\mathfrak{V}_{n-i}}{\sim} V_i(H)$.*

PROOF. (a) Apply (5.2). (b) and (c) These follow immediately from (5.2), (6.6) and (6.1).

The above proposition should be compared with (6.2). Before seeking for conditions on $\mathfrak{V}$ to strenghten the isologism in (6.8)(c), we first prove a similar kind of result as in (6.8)(c), but of a surprising outcome. It depends on the special construction of the $\mathfrak{V}$-marginal series.

(6.9) LEMMA. *Let $\mathfrak{V}$ be a variety and $c, n \geq 0$. Then $\mathfrak{V}_{n(c+1)} \subseteq \mathfrak{N}_c \cdot \mathfrak{V}_n$.*

PROOF. We argue by induction on c, the case $c = 0$ being clear. Now

$$\begin{aligned} \mathfrak{N}_{c+1} \cdot \mathfrak{V}_n &= [\mathfrak{N}_c \cdot \mathfrak{V}_n, \mathfrak{V}_n] \\ &\supseteq [\mathfrak{V}_{n(c+1)}, \mathfrak{V}_n] && \text{(by the induction hypothesis)} \\ &\supseteq \mathfrak{V}_{n(c+1)} * \mathfrak{V}_n && \text{(by (3.9)(b))} \\ &\supseteq \mathfrak{V}_{n(c+2)} && \text{(by (6.6)).} \end{aligned}$$

(6.10) THEOREM. *Let $\mathfrak{V}$ be a variety and $n \geq 0$. Suppose that $G \underset{\mathfrak{V}_{n+1}}{\sim} H$. Then for each $i \in \{0, \ldots, n\}$,*

$$V_{i+1}(G) \underset{\lceil (n-i)/(i+1) \rceil}{\sim} V_{i+1}(H).$$

PROOF. In view of (5.1) it is sufficient to assume that $H = G/N$, with $N \trianglelefteq G$ and $N \cap V_{n+1}(G) = 1$. Put $j = \lceil (n-i)/(i+1) \rceil$ so that $(i+1)(j+1) \geq n+1$. We have to show that $V_{i+1}(G)$ and $V_{i+1}(G)/(N \cap V_{i+1}(G))$ are j-isoclinic. According to (4.4)(b) this is the case as soon as $\gamma_{j+1}(V_{i+1}(G)) \cap N = 1$. Since by (6.9) $\mathfrak{V}_{n+1} \subseteq \mathfrak{V}_{(j+1)(i+1)} \subseteq \mathfrak{N}_j \cdot \mathfrak{V}_{i+1}$, it follows that $V_{n+1}(G) \supseteq \gamma_{j+1}(V_{i+1}(G))$. As $N \cap V_{n+1}(G) = 1$, this finishes the proof.

By taking $\mathfrak{V} = \mathfrak{A}$ in (6.10) we get that if $G \underset{n}{\sim} H$, then it holds that $\gamma_{i+1}(G) \underset{\lceil (n/i) - 1 \rceil}{\sim} \gamma_{i+1}(H)$ for each $0 \leq i \leq n$. It was remarked before that this last isoclinism can be improved to an $\lceil (n-i)/(i+1) \rceil$-isoclinism. The structure of the commutator word defining $\mathfrak{A}$ underlies this.

(6.11) LEMMA. *Let $\mathfrak{V}$ be a variety. The following are equivalent.*
(a) *For all $n \geq 0$: $\mathfrak{V}_{n+1} = \mathfrak{V}_n * \mathfrak{V}$.*
(b) *For all $m, n \geq 0$: $\mathfrak{V}_{m+n} = \mathfrak{V}_m * \mathfrak{V}_n$.*

PROOF. (a)⇒(b): We use induction on n, the cases $n = 0, 1$ being clear. Now

$$\begin{aligned} \mathfrak{V}_{m+n+1} &\subseteq \mathfrak{V}_m * \mathfrak{V}_{n+1} && \text{(by (6.6))} \\ &= \mathfrak{V}_m * (\mathfrak{V}_n * \mathfrak{V}) && \text{(by assertion)} \\ &\subseteq (\mathfrak{V}_m * \mathfrak{V}_n) * \mathfrak{V} && \text{(by (3.5))} \\ &= \mathfrak{V}_{m+n} * \mathfrak{V} && \text{(by the induction hypothesis)} \\ &= \mathfrak{V}_{m+n+1}. \end{aligned}$$

Hence we must have equality everywhere.
(b)⇒(a): Trivial, taking $n = 1$.

https://doi.org/10.1017/S1446788700030366 Published online by Cambridge University Press

(6.12) THEOREM. *Let $\mathfrak{V}$ be a variety satisfying the following conditions.*

(i) *For all $n \geq 0$: $\mathfrak{V}_{n+1} = \mathfrak{V}_n * \mathfrak{V}$ and*

(ii) *For any group $G, n \geq 0$ and $N \trianglelefteq G$ with $N \subseteq V_n(G)$ it holds that $[NV^*V_n(G)] \subseteq [NV^*_{n+1}G]$.*

Then, if $G \underset{\mathfrak{V}_n}{\sim} H$, it holds that $V_i(G) \underset{\mathfrak{V}_{\lceil (n-i)/(i+1) \rceil}}{\sim} V_i(H)$ for each $i \in \{0, \ldots, n\}$.

PROOF. By virtue of (5.1) it suffices to assume that $H = G/N$ with $N \trianglelefteq G$ and $N \cap V_n(G) = 1$. Let $i \in \{0, \ldots, n\}$ be fixed and put $j = \lceil (n-i)/(i+1) \rceil$, so $j \geq 0$. Following the proof of (6.1)(b) we have to show that $N \cap V_j(V_i(G)) = 1$. We prove first that

$$V_j(V_i(G)) \subseteq V_{i+j+ij}(G).$$

For $j = 0$ this is trivial. Since $\mathfrak{V}_{j+1} = \mathfrak{V}_j * \mathfrak{V}$, induction on j gives

$$\begin{aligned} V_{j+1}(V_i(G)) &= [V_j(V_i(G))V^*V_i(G)] \\ &\subseteq [V_{i+j+ij}(G)V^*V_i(G)] \quad &&\text{(by the induction hypothesis)} \\ &\subseteq [V_{i+j+ij}(G)V^*_{i+1}G] \quad &&\text{(by (ii))} \\ &= V_{i+j+ij+i+1}(G) \quad &&\text{(by (6.11)(b))} \\ &= V_{i+j+1+i(j+1)}(G). \end{aligned}$$

Now $i + j + ij = i + (i+1)j \geq i + n - i = n$, so $V_{i+j+ij}(G) \subseteq V_n(G)$. Hence $N \cap V_j(V_i(G)) = 1$, as required.

The conditions in (6.12) are fulfilled if, for instance $\mathfrak{V} = \mathfrak{N}_c$. The condition (ii) of (6.12) may be looked upon as a kind of substitute for the Three Subgroups Lemma. For $\mathfrak{V} = \mathfrak{N}_c$ it reads: for any group G, $n \geq 0$ and $N \trianglelefteq G$ with $N \subseteq \gamma_{nc+1}(G)$ it holds that $[N, {}_c\gamma_{nc+1}(G)] \subseteq [N, {}_{c(n+1)}G]$. And this assertion is easily derived by utilizing the Three Subgroups Lemma. Still it is hard to avoid the impression that Theorem (6.12) is rather artificial. In fact $\mathfrak{V} = \mathfrak{N}_c$ is the only instance we know of that satisfies the conditions (i) and (ii) of (6.12). Continuing the discussion preceding (6.5), there is on the other hand a more natural way to generalize Theorems 5.2 and 5.5 in [8] to a wider class of varieties. Instead of fixing our attention to the simple commutators of some weight, we will now consider more complicated commutator words, to wit, outer commutators of some weight. Although we conjecture that the results below generalize to varieties defined by outer commutator words, we will restrict ourselves for the rest of this section to the variety $\mathfrak{N}_{c_1,\ldots,c_l}$ of polynilpotent groups of class row $(c_1, \ldots, c_l)$. Recall that the polynilpotent variety $\mathfrak{N}_{c_1,\ldots,c_l}$ equals the product $\mathfrak{N}_{c_l} \cdot \mathfrak{N}_{c_{l-1}} \cdot \ldots \cdot \mathfrak{N}_{c_2} \cdot \mathfrak{N}_{c_1}$ (the product in the sense of Hanna Neumann, see the beginning of Section 3). Observe that $\mathfrak{N}_{c_1,\ldots,c_l}$ generalizes both the nilpotent variety $\mathfrak{N}_c$ (take $l = 1$) and the solvable variety $\mathfrak{S}_l$ (take $c_1 = \cdots = c_l = 1$). Also

21.12 in [13] ensures us that if $\mathfrak{V} = \mathfrak{N}_{c_1,\ldots,c_l}$, then for a group G,

$$V(G) = \gamma_{c_l+1}(\gamma_{c_{l-1}+1}(\cdots(\gamma_{c_1+1}(G))\cdots)) =: \gamma_{c_1+1,c_2+1,\ldots,c_l+1}(G).$$

We will now derive a formula for $[NV^*G]$, where $N \trianglelefteq G$. First we need a lemma.

(6.13) LEMMA. *Let $\mathfrak{V} = \mathfrak{N}_{c_1,\ldots,c_l}$ and $N \trianglelefteq G$. Then the following are equivalent.*

(a) $N \subseteq V^*(G)$.

(b) $[N,{}_{c_1}G,{}_{c_2}\gamma_{c_1+1}(G),{}_{c_3}\gamma_{c_1+1,c_2+1}(G),\ldots,{}_{c_l}\gamma_{c_1+1,c_2+1,\ldots,c_{l-1}+1}(G)] = 1$.

PROOF. We employ induction on l, the case $l = 1$ being well-known. Now we can write $\mathfrak{V} = \mathfrak{N}_{c_l} \cdot \mathfrak{U}$, where $\mathfrak{U} = \mathfrak{N}_{c_1,\ldots,c_{l-1}}$. By a lemma of M. R. R. Moghaddam (see [12], Lemma 2.5(ii)), we have that

$$V^*(G)/\zeta_{c_l}(U(G)) = U^*(G/\zeta_{c_l}(U(G)).$$

We write bar $\bar{\ }$ for reduction modulo the normal subgroup $\zeta_{c_l}(U(G))$ of G. Then we have

$$\begin{aligned} N \subseteq V^*(G) &\Leftrightarrow \overline{N} \subseteq \overline{V^*(G)} \Leftrightarrow \overline{N} \subseteq U^*(\overline{G}) \qquad \text{(by the induction hypothesis)} \\ &\Leftrightarrow [\overline{N},{}_{c_1}\overline{G},{}_{c_2}\gamma_{c_1+1}(\overline{G}),\ldots,{}_{c_{l-1}}\gamma_{c_1+1,\ldots,c_{l-2}+1}(\overline{G})] = \overline{1} \\ &\Leftrightarrow [N,{}_{c_1}G,{}_{c_2}\gamma_{c_1+1}(G),\ldots,{}_{c_{l-1}}\gamma_{c_1+1,\ldots,c_{l-2}+1}(G)] \subseteq \zeta_{c_l}(U(G)) \\ &\Leftrightarrow [N,{}_{c_1}G,{}_{c_2}\gamma_{c_1+1}(G),\ldots,{}_{c_l}\gamma_{c_1+1,\ldots,c_{l-1}+1}(G)] = 1, \text{ as desired.} \end{aligned}$$

(6.14) COROLLARY. *Let $\mathfrak{V} = \mathfrak{N}_{c_1,\ldots,c_l}$ and $N \trianglelefteq G$. Then*

$$[NV^*G] = [N,{}_{c_1}G,{}_{c_2}\gamma_{c_1+1}(G),\ldots,{}_{c_l}\gamma_{c_1+1,\ldots,c_{l-1}+1}(G)].$$

PROOF. Put $T = [N,{}_{c_1}G,{}_{c_2}\gamma_{c_1+1}(G),\ldots,{}_{c_l}\gamma_{c_1+1,\ldots,c_{l-1}+1}(G)]$. As $N \trianglelefteq G$, it is clear that $T \subseteq N$. Also $T \trianglelefteq G$. From (6.13) it follows that $N/T \subseteq V^*(G/T)$. This yields $[NV^*G] \subseteq T$ (see the remarks made after Definition (2.1)). Conversely, by definition $N/[NV^*G] \subseteq V^*(G/[NV^*G])$. Hence by (6.13) again $T \subseteq [NV^*G]$. We conclude that $T = [NV^*G]$.

Note that for $\mathfrak{V} = \mathfrak{S}_l$ (6.14) gives for $N \trianglelefteq G$ that $[NV^*G] = [N, G, G', G^{(2)}, \ldots, G^{(l-1)}]$ (see also [9], Corollary 2.10).

We have now gathered enough information to prove the following theorem.

(6.15) THEOREM. *Let $n \geq 0$ and suppose that $G \underset{n}{\sim} H$. Let $\mathfrak{U} = \mathfrak{N}_{c_1,\ldots,c_l}$ and $\mathfrak{V} = \mathfrak{N}_{d_1,\ldots,d_m}$ and put $p + 1 = \prod_{i=1}^{l}(c_i + 1)$ and $q + 1 = \prod_{j=1}^{m}(d_j + 1)$. Then the following hold.*

(a) *If $p + q \geq n$, then $G/U^*(G) \underset{\mathfrak{V}}{\sim} H/U^*(H)$.*

(b) *If $p + pq + q \geq n$, then $U(G) \underset{\mathfrak{V}}{\sim} U(H)$.*

https://doi.org/10.1017/S1446788700030366 Published online by Cambridge University Press

PROOF. On invoking the Theorem (5.1) it is sufficient for both (a) and (b) to assume that $H = G/N$ with $N \trianglelefteq G$ and $N \cap \gamma_{n+1}(G) = 1$.

(a) Put $U^*(G/N) = M/N$. An argument similar to the one used in the proof of Theorem (6.1)(a) teaches us that all we have to show is that $M \cap V(G) \subseteq U^*(G)$. By (2.3)(c) this is equivalent to showing that $[(M \cap V(G))U^*G] = 1$. Now by definition $[MU^*G] \subseteq N$. Hence, in view of $N \cap \gamma_{n+1}(G) = 1$, we are left with proving $[V(G)U^*G] \subseteq \gamma_{n+1}(G)$. Now at this point we observe that by 5.1.11(ii) in [15]

$$V(G) = \gamma_{d_1+1,\dots,d_m+1}(G) \subseteq \gamma_{q+1}(G).$$

Hence, in view of (6.14),

$$\begin{aligned}
&[V(G)U^*G]\\
&\quad= [V(G), {}_{c_1}G, {}_{c_2}\gamma_{c_1+1}(G), {}_{c_3}\gamma_{c_1+1,c_2+1}(G), \dots, {}_{c_l}\gamma_{c_1+1,\dots,c_{l-1}+1}(G)]\\
&\quad\subseteq [\gamma_{q+1}(G), {}_{c_1}G, {}_{c_2}\gamma_{c_1+1}(G), {}_{c_3}\gamma_{c_1+1,c_2+1}(G), \dots, {}_{c_l}\gamma_{c_1+1,\dots,c_{l-1}+1}(G)]\\
&\quad= [\gamma_{q+1+c_1}(G), {}_{c_2}\gamma_{c_1+1}(G), {}_{c_3}\gamma_{c_1+1,c_2+1}(G), \dots, {}_{c_l}\gamma_{c_1+1,\dots,c_{l-1}+1}(G)]\\
&\quad\subseteq [\gamma_{q+1+c_1+c_2(c_1+1)}(G), {}_{c_3}\gamma_{c_1+1,c_2+1}(G), \dots, {}_{c_l}\gamma_{c_1+1,\dots,c_{l-1}+1}(G)]\\
&\quad\subseteq \cdots \subseteq \gamma_r(G),
\end{aligned}$$

where $r = q + 1 + c_1 + c_2(c_1+1) + c_3(c_1+1)(c_2+1) + \cdots + c_l \prod_{i=1}^{l-1}(c_i+1) = q + p + 1$. By hypothesis $r \geq n+1$, whence $\gamma_r(G) \subseteq \gamma_{n+1}(G)$. So indeed $[V(G)U^*G] \subseteq \gamma_{n+1}(G)$.

(b) Again, a similar reasoning as in the proof of Theorem (6.1)(b) gives us to show that $N \cap V(U(G)) = 1$. Indeed,

$$V(U(G)) = \gamma_{d_1+1,\dots,d_m+1}(\gamma_{c_1+1,\dots,c_l+1}(G)) \subseteq \gamma_{(q+1)(p+1)}(G),$$

and it follows from the hypothesis that $(p+1)(q+1) \geq n+1$. Hence $N \cap V(U(G)) \subseteq N \cap \gamma_{n+1}(G) = 1$.

(6.16) COROLLARY. *Let $n \geq 0$ and suppose that $G \underset{n}{\sim} H$. Let $\mathfrak{U} = \mathfrak{N}_{c_1,\dots,c_l}$ and put $p+1 = \prod_{i=1}^{l}(c_i+1)$. Assume that $p \leq n$. Then the following hold.*

(a) $G/U^*(G) \underset{n-p}{\sim} H/U^*(H)$.

(b) $U(G) \underset{\lceil (n-p)/(p+1) \rceil}{\sim} U(H)$.

Moreover these isoclinisms are the best possible for $\mathfrak{V}$ and n.

PROOF. This follows immediately from (6.15). To prove (a), let $m = 1$ and $d = q = n - p$, so that $p + q = n$. To prove (b), let $m = 1$ and $d = q = \lceil (n-p)/(p+1) \rceil$, so that $(q+1)(p+1) \geq n+1$, that is, $p+pq+q \geq n$. We are left with showing that the isoclinisms are the best possible in the sense of (6.4). Let G be the group of all $(n+1) \times (n+1)$ upper unitriangular matrices with coefficients in a finite field F. Then G is a Sylow subgroup of $\mathrm{GL}(n+1, \mathsf{F})$ and is of nilpotency class n. This group has the property that for $k, l \geq 1$ $[\gamma_k(G), \gamma_k(G)] = \gamma_{k+l}(G)$ (see [10], III Satz 16.3(b)). This implies that for any $r, s \geq 0$

$$[\gamma_{s+1}(G)U^*G] = \gamma_{s+p+1}(G) \tag{1}$$

and

$$\gamma_{r+1}(U(G)) = \gamma_{(p+1)(r+1)}(G). \tag{2}$$

Now put $H = G \times G$. Clearly $G \underset{n}{\sim} H \underset{n}{\sim} 1$. Hence, if we would have $G/U^*(G) \underset{s}{\sim} H/U^*(H)$, then consequently, as $H/U^*(H) = G/U^*(G) \times G/U^*(G)$, we would get $\gamma_{s+1}(G/U^*(G)) = 1$. Or, equivalently, $\gamma_{s+1}(G) \subseteq U^*(G)$, whence $[\gamma_{s+1}(G)U^*G] = 1$. By virtue of (1) we find $\gamma_{s+p+1}(G) = 1$, hence $s + p + 1 \geq n + 1$, thus $s \geq n - p$. Should we have $U(G) \underset{r}{\sim} U(H)$, then, as $U(H) = U(G) \times U(G)$, we would obtain $\gamma_{r+1}(U(G)) = 1$. According to (2) this yields $\gamma_{(p+1)(r+1)}(G) = 1$, hence $(p+1)(r+1) \geq n+1$, thus $r \geq (n-p)/(p+1)$. This completes the proof.

(6.17) COROLLARY. *Let $n \geq 0$ and suppose that $G \underset{n}{\sim} H$. Then for any i with $0 \leq i \leq^2 \log(n+1)$ the following hold.*

(a) $G/S_i^*(G) \underset{n+1-2^i}{\sim} H/S_i^*(H)$.

(b) $G^{(i)} \underset{\lceil (n+1)/2^i - 1 \rceil}{\sim} H^{(i)}$.

Moreover these isoclinisms are the best possible for $\mathfrak{S}_i$ and n.

PROOF. This is an application of (6.16) using the fact that $\mathfrak{S}_i = \mathfrak{N}_{1,\ldots,1}$ (i times the 1).

(6.18) REMARK. There is still another way to derive the results of (6.16). Let $n \geq 0$ and $c_1, \ldots, c_l \geq 1$. Put $p + 1 = \prod_{i=1}^{l}(c_i + 1)$ and assume that $p \leq n$. For any group G we have that $\gamma_{c_1+1,\ldots,c_l+1}(G) \subseteq \gamma_{p+1}(G)$. Hence it holds that $\mathfrak{N}_{c_1,\ldots,c_l} \supseteq \mathfrak{N}_p$. Multiplying this last inclusion on the left and the right by $\mathfrak{N}_{n-p}$ we obtain, on invoking (3.6)(a)(c),

$$\mathfrak{N}_{n-p} * \mathfrak{N}_{c_1,\ldots,c_l} \supseteq \mathfrak{N}_n \text{ and } \mathfrak{N}_{c_1,\ldots,c_l} * \mathfrak{N}_{n-p} \supseteq \mathfrak{N}_n.$$

These inclusions, together with (6.1) and (5.2), yield (6.16) again, albeit that (6.16)(b) does not appear in it sharpenest form. There are two reasons for not proving (6.16) in this manner. The first is that we wanted to derive a formula for $[NV^*G]$ in case $\mathfrak{V} = \mathfrak{N}_{c_1,\ldots,c_l}$ (see (6.14)). The second reason is that we could illustrate (6.1)(b) sometimes allows an amelioration.

7. Prescribing the marginal quotients

In [8], Theorem 7.7 it has been proved that if G is a group with $G/\zeta_n(G)$ finite, then, within the n-isoclinism class of G, there exists a group H, which is finite. Moreover, this group H can be obtained as a section of G. The main object of this section is to generalize such an assertion to varieties $\mathfrak{V}$ in the sense that given that the marginal quotient $G/V^*(G)$ belongs to a certain class of groups $\mathfrak{X}$, it is possible to indicate a group $H \in \mathfrak{X}$, inside the $\mathfrak{V}$-isologism class of G. For $\mathfrak{X}$ will be chosen not only the class of all finite groups, but also certain classes of infinite groups will be dealt with, for example the class of all polycyclic groups. Of course, if one wants to prove assertions like the ones described above, the variety $\mathfrak{V}$ and the class $\mathfrak{X}$ depend on each other. The question, to which variety $\mathfrak{V}$ belongs what class of groups $\mathfrak{X}$ and vice versa will not be settled in its full generality. However we prove theorems which to a large extent generalize the theorem in [8] mentioned just a moment ago.

In this section we were greatly inspired by the work of P. W. Stroud [18]. Let us first have a look at groups which do not possess a $\mathfrak{V}$-isologic section at all.

(7.1) DEFINITION. Let $\mathfrak{V}$ be a variety. A group G is called *subgroup irreducible with respect to V-isologism* if G contains no proper subgroup H satisfying $G = HV^*(G)$. A group G containing no non-trivial normal subgroup N satisfying $N \cap V(G) = 1$ is called *quotient irreducible with respect to $\mathfrak{V}$-isologism.*

(7.2) PROPOSITION. *Let $\mathfrak{V}$ be a variety and suppose $V^*(G) \subseteq V(G)$. Then G is both subgroup and quotient irreducible with respect to $\mathfrak{V}$-isologism.*

PROOF. Suppose $N \trianglelefteq G$ with $N \cap V(G) = 1$. Then (2.3)(f) gives $N \subseteq V^*(G)$. Hence $N \subseteq V(G)$, so $N = 1$. Assume now that $H \leq G$ with $G = HV^*(G)$. By (2.5)(b) we have $V(H) = V(G)$, so that $V^*(G) \subseteq V(H) \subseteq H$.

A simple application of Zorn's Lemma shows that, given a group and a variety $\mathfrak{V}$ one can always find a suitable quotient of G which is $\mathfrak{V}$-isologic to G and moreover is quotient irreducible with respect to $\mathfrak{V}$-isologism (see also [8], Theorem 7.6). This suggest the following problem: given a group G and a variety $\mathfrak{V}$, does there always exist a subgroup of G which is $\mathfrak{V}$-isologic to G and subgroup irreducible with respect to $\mathfrak{V}$-isologism? In general, we do not know whether the answer to this question is in the affirmative.

In the next two theorems partial characterizations are proved.

(7.3) THEOREM. *Let $\mathfrak{V}$ be a variety. If G is subgroup irreducible with respect to $\mathfrak{V}$-isologism, then $V^*(G) \subseteq \Phi(G)$. The converse holds if $V^*(G)/V(G) \cap V^*(G)$ is finitely generated.*

PROOF. Suppose first that G is subgroup irreducible with respect to $\mathfrak{V}$-isologism. We may assume that G has maximal subgroups, otherwise $G = \Phi(G)$ and the assertions become trivial. Let M be a maximal subgroup of G. Then clearly $MV^*(G) \subsetneq G$. Thus $V^*(G) \subseteq M$, by the maximality of M. We conclude that $V^*(G) \subseteq \Phi(G)$. Conversely, let $V^*(G)/V(G) \cap V^*(G)$ be finitely generated and $V^*(G) \subseteq \Phi(G)$. Let $H \leq G$ with $G = HV^*(G)$. Now there exist $g_1, \ldots, g_t \in V^*(G) (t \geq 1)$, such that $V^*(G) = \langle g_1, \ldots, g_t \rangle \cdot (V(G) \cap V^*(G))$. By (2.5)(c) we have $V(H) \cap V^*(H) = V(G) \cap V^*(G)$. As $g_1, \ldots, g_t \in \Phi(G)$, these elements are non-generators (see [15], 5.2.12), whence

$$\begin{aligned} G = HV^*(G) &= H \cdot \langle g_1, \ldots, g_t \rangle \cdot (V(G) \cap V^*(G)) \\ &= H \cdot (V(H) \cap V^*(H)) \cdot \langle g_1, \ldots, g_t \rangle = H \cdot \langle g_1, \ldots, g_t \rangle = H. \end{aligned}$$

So G is subgroup irreducible with respect to $\mathfrak{V}$-isologism.

(7.4) THEOREM. *Let $\mathfrak{V}$ be a variety. If G is quotient irreducible with respect to $\mathfrak{V}$-isologism, then* $\operatorname{soc}(G) \subseteq V(G)$ *and* $\varsigma(G)/\varsigma(G) \cap V(G)$ *is a torsion group. The converse holds if $\mathfrak{V}$ is nilpotent.*

PROOF. Assume that G has minimal normal subgroups, else $\operatorname{soc}(G) = 1$ and the first assertion in the statement of the theorem becomes trivially true. Let M be a minimal normal subgroup of G, where G is quotient irreducible with respect to $\mathfrak{V}$-isologism. Here $M \cap V(G) \neq 1$, whence $M \subseteq V(G)$ by the minimality of M. We conclude that $\operatorname{soc}(G) \subseteq V(G)$.

Next let $\bar{g} \in \varsigma(G)/\varsigma(G) \cap V(G)$ with $\bar{g} \neq \bar{1}$. Put $N = \langle g \rangle$. Then $N \subseteq \varsigma(G)$ and hence $N \trianglelefteq G$. Also $N \neq 1$, so that $N \cap V(G) \neq 1$. Thus for some positive integer k we must have $g^k \in V(G)$. It follows that $\bar{g}$ is a torsion element.

Conversely, suppose $\operatorname{soc}(G) \subseteq V(G)$, $\varsigma(G)/\varsigma(G) \cap V(G)$ is a torsion group and $\mathfrak{V}$ is nilpotent. Hence there is a natural number n, such that $V^*(G) \subseteq \varsigma_n(G)$. Now let $N \trianglelefteq G$ with $N \cap V(G) = 1$. We have to show that $N = 1$. Put $M = N \cap \varsigma(G)$. We have $M = N \cap \varsigma(G) \simeq (N \cap \varsigma(G))V(G)/V(G) \leq \varsigma(G)V(G)/V(G) \simeq \varsigma(G)/\varsigma(G) \cap V(G)$, so that M is a torsion group. If $M \neq 1$, choose $g \in M - \{1\}$ of minimal order. Then $\langle g \rangle$ is a minimal normal subgroup of G, whence $\langle g \rangle \subseteq V(G)$ by assumption. And this violates $N \cap V(G) = 1$. We conclude that $N \cap \varsigma(G) = 1$, so $N \cap \varsigma_i(G) = 1$, for all $i \geq 0$ (see for example [8], Theorem 2.3(b)). However, by virtue of (2.3)(f) we have $N \subseteq V^*(G)$. Hence $N \subseteq \varsigma_n(G)$ and we get $N = 1$, as required.

We now turn to the issue mentioned in the introduction of this section. For the rest of this section the following assumptions will be valid.

(7.5) HYPOTHESES. $\mathfrak{X}$ will denote a class of *finite* groups satisfying the following conditions.

(a) (Subgroups) $G \in \mathfrak{X}$ and $H \leq G \Rightarrow H \in \mathfrak{X}$.
(b) (Quotients) $G \in \mathfrak{X}$ and $N \trianglelefteq G \Rightarrow G/N \in \mathfrak{X}$.
(c) (Extensions) $N \trianglelefteq G$, $N \in \mathfrak{X}$ and $G/N \in \mathfrak{X} \Rightarrow G \in \mathfrak{X}$.
(d) (Cyclic groups) For every prime number p, $C_p \in \mathfrak{X}$.

Observe that (7.5)(c) implies that $\mathfrak{X}$ is closed with respect to taking finite direct products. Hence by virtue of (7.5)(d) $\mathfrak{X}$ contains all elementary abelian p-groups. In fact, it is easy to see that $\mathfrak{X}$ contains all finite solvable groups. Examples of $\mathfrak{X}$ are of course the class of all finite solvable groups, the class of all finite groups, or, between these two, the class of all finite π-separable groups.

(7.6) LEMMA. *Let G be a residually $\mathfrak{X}$-group and $H \leq G$ with H finite. Then there exists an $N \trianglelefteq G$ such that $N \cap H = 1$ and $G/N \in \mathfrak{X}$.*

PROOF. Let H be a finite subgroup of G. For every $h \in H - \{1\}$ there exists an $N_h \trianglelefteq G$ with $G/N_h \in \mathfrak{X}$ and $h \notin N_h$. Put $N = \bigcap_{h \in H-\{1\}} N_h$. Then $N \trianglelefteq G$ and $N \cap H = 1$. Further G/N can be embedded in the direct product $\prod_{h \in H-\{1\}}(G/N_h)$. By the remarks made after (7.5), and (7.5)(a) it follows that $G/N \in \mathfrak{X}$.

(7.7) LEMMA. *Let $N \trianglelefteq G$ such that $G/N \in \mathfrak{X}$. Then N is a residually $\mathfrak{X}$-group if and only if G is a residually $\mathfrak{X}$-group.*

PROOF. [$\Rightarrow$] Suppose N is a residually $\mathfrak{X}$-group. Let $g \in G - \{1\}$. Assume first that $g \in N$. Then there exists an $M \trianglelefteq N$ with $g \notin M$ and $N/M \in \mathfrak{X}$. Now let $\mathscr{T}$ be a transversal to M in G. Then $\mathscr{T}$ is a finite set and $K = \bigcap_{t \in \mathscr{T}} M^t = \mathrm{core}_G(M)$. Clearly $N^t/M^t = N/M^t \simeq N/M$ for any $t \in \mathscr{T}$. Hence $N/M^t \in \mathfrak{X}$ for all $t \in \mathscr{T}$. Now N/K can be embedded in the direct product $\prod_{t \in \mathscr{T}}(N/M^t)$. As $\mathfrak{X}$ is closed for taking finite direct products and subgroups, we get $N/K \in \mathfrak{X}$. Now $K \trianglelefteq G$, thus we have $G/N \simeq (G/K)/(N/K)$. It follows by (7.5)(c) that $G/K \in \mathfrak{X}$. In summary, if $g \in N$, then there exists a $K \trianglelefteq G$ with $G/K \in \mathfrak{X}$ and $g \notin K$. Together with $G/N \in \mathfrak{X}$ it follows that G is a residually $\mathfrak{X}$-group.

[$\Leftarrow$] Let $n \in N - \{1\}$. There exists an $M \trianglelefteq G$ with $G/M \in \mathfrak{X}$ and $n \notin M$. Put $K = M \cap N$. Then $K \trianglelefteq N$ and $n \notin K$. Moreover $N/K \simeq NM/M \leq G/M$. Hence by (7.5)(a) $N/K \in \mathfrak{X}$.

(7.8) THEOREM. *Let $\mathfrak{V}$ be a finitely based and locally residually finite variety. If $G/V^*(G) \in \mathfrak{X}_\pi$, then $V(G) \in \mathfrak{X}_\pi$.*

PROOF. Assume $G/V^*(G) \in \mathfrak{X}_\pi$. In particular $G/V^*(G) \in \mathfrak{F}_\pi$ and then a theorem of P. W. Stroud ([18], Theorem 2) asserts that $V(G) \in \mathfrak{F}_\pi$. Next $V(G/V^*(G)) \simeq V(G)/V(G) \cap V^*(G)$ by (2.3)(d), so in view of (7.5)(a)

$V(G)/V(G) \cap V^*(G) \in \mathfrak{X}_\pi$. Hence we may assume that $V(G) \cap V^*(G) \neq 1$. We employ induction on the order of $V(G)$. If $V(G) = 1$, there is nothing to prove. So let $V(G) \neq 1$. There exists a characteristic subgroup $N \neq 1$ of the abelian group $V(G) \cap V^*(G)$, with N an elementary abelian p-group for some prime number p. Since $V(G) \in \mathfrak{F}_\pi$, $p \in \pi$. Hence by earlier remarks made after (7.5), $N \in \mathfrak{X}_\pi$. Now $(G/N)/V^*(G/N)$ is isomorphic to a quotient of $G/V^*(G)$, whence $(G/N)/V^*(G/N) \in \mathfrak{X}_\pi$. Moreover, $V(G/N) = V(G)/N$, so $|V(G/N)| < |V(G)|$. The induction hypothesis implies that $V(G/N) \in \mathfrak{X}_\pi$. We saw that $N \in \mathfrak{X}_\pi$, so by utilizing (7.5)(c) we get $V(G) \in \mathfrak{X}_\pi$, as desired.

(7.9) DEFINITION. Let $\mathfrak{V}$ be a variety. A class $\mathfrak{Y}$ of groups is called $\mathfrak{V}$-*closed* if for any group G with $G/V^*(G) \in \mathfrak{Y}$ it follows that $V(G) \in \mathfrak{Y}$.

To give an example, it is well known that $\mathfrak{F}$ is $\mathfrak{A}$-closed. More general, if $\mathfrak{V}$ is a finitely based variety such that $\mathfrak{F}_\pi$ is $\mathfrak{V}$-closed, then $\mathfrak{F}_\pi$ is $(\mathfrak{V} * \mathfrak{N}_c)$-closed for all $c \geq 0$. This follows with induction on c from (3.6)(c) and Theorem 1 of [18].

If a finite group possesses a normal subgroup, which after dividing out, produces a π-group, then this normal subgroup can be supplemented by a π-group. This assertion is attributed to H. Zassenhaus. Here a generalization is needed.

(7.10) LEMMA. *Let G be a finite group and $N \trianglelefteq G$ with $G/N \in \mathfrak{X}_\pi$. Then there exists a subgroup H of G with $G = HN$ and $H \in \mathfrak{X}_\pi$.*

PROOF. The proof runs along the same lines as the proof of Lemma 7 in [18]. The ingredients are induction on the order of G, the well-known Schur-Zassenhaus Theorem and the Frattini argument.

If G is a finite group and $\mathfrak{V}$ an arbitrary variety such that $G/V^*(G) \in \mathfrak{X}_\pi$, then there exists a subgroup H of G with $H \in \mathfrak{X}_\pi$ and $G = HV^*(G)$, in other words G is $\mathfrak{V}$-isologic to a $\mathfrak{X}_\pi$-group contained in G. This follows from a simple application of (7.10) and (4.4)(a). We now come to the main theorems of this section, which cover similar situations where G is infinite.

(7.11) THEOREM. *Let $\mathfrak{V}$ be a locally residually finite variety. Assume that $\mathfrak{F}$ is $\mathfrak{V}$-closed. Then the following properties are equivalent.*

(a) *G is $\mathfrak{V}$-isologic to a group in $\mathfrak{X}_\pi$.*

(b) *$G/V^*(G) \in \mathfrak{X}_\pi$.*

(c) *G is $\mathfrak{V}$-isologic to a section of itself lying in $\mathfrak{X}_\pi$.*

PROOF. The implications (c)$\Rightarrow$(b), (c)$\Rightarrow$(a) and (a)$\Rightarrow$(b) are obvious. So let us prove (b)$\Rightarrow$(c). Assume $G/V^*(G) \in \mathfrak{X}_\pi$. In particular $G/V^*(G)$ is finite and there exists a finitely generated subgroup H of G with $G = HV^*(G)$. By (4.4)(a) $G \underset{\mathfrak{V}}{\sim} H$. It follows that $H/V^*(H) \in \mathfrak{X}_\pi$. Hence $V^*(H)$, having finite

https://doi.org/10.1017/S1446788700030366 Published online by Cambridge University Press

index in H, is finitely generated. According to (2.3)(a)(b) $V^*(H) \in \mathfrak{V}$. Thus $V^*(H)$ is a residually finite group. Then we are assured by (7.7) that H is also a residually finite group. Now the class $\mathfrak{F}$ is $\mathfrak{V}$-closed. So $V(H)$ is a finite group, because $H/V^*(H)$ is. Hence (7.6) guarantees the existence of an $N \trianglelefteq H$ with $N \cap V(H) = 1$ and H/N finite. By virtue of (4.4)(b) we have $G \underset{\mathfrak{V}}{\sim} H \underset{\mathfrak{V}}{\sim} H/N$. Next, apply (7.10) to the finite group H/N and its normal subgroup $V^*(H)/N$ (notice that by (2.3)(f) $N \subseteq V^*(H)$). Apparently there exists a subgroup K of H, which contains N and such that $H = KV^*(H)$ and $K/N \in \mathfrak{X}_\pi$. Now by (2.5)(b) we have $V(H) = V(K)$, whence $N \cap V(H) = N \cap V(K) = 1$. Finally (4.4) yields $H/N \underset{\mathfrak{V}}{\sim} H \underset{\mathfrak{V}}{\sim} K \underset{\mathfrak{V}}{\sim} K/N$. So K/N is the desired section of G.

Note that on invoking Theorem (7.8), the assumption that $\mathfrak{F}$ has to be $\mathfrak{V}$-closed in the statement of (7.11) can be dropped, if $\mathfrak{V}$ is finitely based. We provide some examples to Theorem (7.11).

(7.12) EXAMPLES. (a) Let $\mathfrak{V}$ be any nilpotent variety. By a theorem of R. C. Lyndon (see [13], 34.14) $\mathfrak{V}$ is finitely based. Moreover it is well known that a finitely generated nilpotent group is residually finite. See also Theorem 7.7 and Corollary 7.9 in [8] for a special case.

(b) Let $\mathfrak{V}$ be any metabelian variety. By virtue of a theorem of D. E. Cohen (see [13], 36.11) $\mathfrak{V}$ is finitely based. It is known that a finitely generated metabelian group is residually finite (see [14], page 155 or the next example (c)).

(c) Let $\mathfrak{V}$ be any locally abelian-by-nilpotent variety. A celebrated theorem of Philip Hall (see for example [14], Theorem 9.51) states that a finitely generated abelian-by-nilpotent group is residually finite. Moreover $\mathfrak{F}$ is $\mathfrak{V}$-closed (see [11], Theorem 1.16). Consequently, we can restate (7.11) in this case as follows: If $\mathfrak{V}$ is a locally abelian-by-nilpotent variety, and G is a finite group of order m, then every $\mathfrak{V}$-isologism class with marginal quotient isomorphic to G has a finite representative or order dividing a power of m. This strenghtens Theorem 2.4 of [11].

(d) The following observation provides an abundance of examples.

THEOREM. *Let $\mathfrak{V}$ be a finitely based locally finite variety and $\mathfrak{U}$ a nilpotent variety. Then $\mathfrak{U}\mathfrak{V}$ is a finitely based locally residually finite variety. In particular $\mathfrak{F}$ is $\mathfrak{U}\mathfrak{V}$-closed.*

PROOF. $\mathfrak{U}\mathfrak{V}$ is again finitely based by a theorem of G. Higman (see [13], 34.24). So we have to show that $\mathfrak{U}\mathfrak{V}$ is locally residually finite. Therefore let $G \in \mathfrak{U}\mathfrak{V}$ be finitely generated. Let $N \trianglelefteq G$ with $N \in \mathfrak{U}$ and $G/N \in \mathfrak{V}$. Now G/N is finitely generated, whence by the locally finiteness of $\mathfrak{V}$, we have that G/N is finite. It follows that N is a finitely generated nilpotent group. In particular N

is residually finite. By (7.7) we obtain that G is residually finite. Finally (7.8) shows that $\mathfrak{F}$ is $\mathfrak{L}\mathfrak{V}$-closed.

We will now prove two fashions of Theorem (7.11) in which we allow the class $\mathfrak{X}$ to contain infinite groups. The question is what infinite groups should be considered. The smallest class of finite groups $\mathfrak{X}$ which satisfies Hypotheses (7.5) is the class of finite solvable groups. Therefore it is in line to consider infinite solvable groups, more precisely finitely generated solvable groups, like polycylic groups. Indeed, notice that the class of polycyclic groups satisfies the Hypotheses (7.5). We have the following.

(7.13) THEOREM. *Let $\mathfrak{V}$ be a finitely based locally finite variety. Let $\mathfrak{Y}$ be either the class of polycyclic, the class of finite-by-polycyclic or the class of polycyclic-by-finite groups. Then the following are equivalent.*

(a) *G is $\mathfrak{V}$-isologic to a group in $\mathfrak{Y}$.*

(b) *$G/V^*(G) \in \mathfrak{Y}$.*

(c) *G is $\mathfrak{V}$-isologic to a $\mathfrak{Y}$-group contained in G.*

(7.14) THEOREM. *Let $\mathfrak{V}$ be a nilpotent variety. Let $\mathfrak{Y}$ be either the class of polycyclic, the class of finite-by-polycyclic or the class of polycyclic-by-finite groups. Then the following are equivalent.*

(a) *G is $\mathfrak{V}$-isologic to a group in $\mathfrak{Y}$.*

(b) *$G/V^*(G) \in \mathfrak{Y}$.*

(c) *G is $\mathfrak{V}$-isologic to a $\mathfrak{Y}$-group contained in G.*

The proofs of the above two theorems depend on the following observations.

(7.15) LEMMA. *Let G be a finitely generated group. Let $\mathfrak{V}$ be a variety.*

(a) *If $\mathfrak{V}$ is nilpotent, then $G/V^*(G)$ is polycyclic implies G is polycyclic.*

(b) *If $\mathfrak{V}$ is either finitely based locally finite or nilpotent, then $G/V^*(G)$ is finite-by-polycyclic implies G is finite-by-polycyclic and $G/V^*(G)$ is polycyclic-by-finite implies G is polycyclic-by-finite.*

PROOF. (a) It is proved in [18], Theorem 3, part (b) that the class of polycyclic groups is $\mathfrak{V}$-closed if $\mathfrak{V}$ is nilpotent. However, a close inspection of the proof of that theorem reveals that in fact G is polycyclic, whenever G is finitely generated with $G/V^*(G)$ polycyclic.

(b) Let G be a finitely generated group. Suppose first that $\mathfrak{V}$ is a finitely based locally finite variety. Assume that $G/V^*(G)$ is polycyclic-by-finite, say $N \trianglelefteq G$, $N \supseteq V^*(G)$, $|G : N| < \infty$ and $N/V^*(G)$ polycyclic. Of course $V^*(N) \supseteq V^*(G)$, so $N/V^*(N)$ is polycyclic. As $|G : N| < \infty$, N is finitely generated. By Theorem 3 part (a) of [18], $V(N)$ is polycyclic (there we need the fact that $\mathfrak{V}$ is finitely based). But $N/V(N)$ is finitely generated and $N/V(N) \in \mathfrak{V}$. By the locally finiteness of $\mathfrak{V}$ we conclude that $|N/V(N)| < \infty$. Further $V(N)$ char $N \trianglelefteq G$, so $V(N) \trianglelefteq G$. And $|G : V(N)| = |G : N| \cdot |N : V(N)| < \infty$, so that G is polycyclic-by-finite. Next assume that $G/V^*(G)$ is finite-by-polycyclic, say $N \trianglelefteq G$, $N \supseteq V^*(G)$, G/N polycyclic and $|N : V^*(G)| < \infty$. Now any finite-by-polycyclic group is polycyclic-by-finite (see [18], Lemma 6). So by the former paragraph G is polycyclic-by-finite. In particular every subgroup of G is finitely generated. hence, as $V^*(G) \in \mathfrak{V}$, we have that $V^*(G)$ is finite, whence N is finite. So after all G is finite-by-polycyclic. This proves (b) in case $\mathfrak{V}$ is finitely based and locally finite.

From now on we take $\mathfrak{V}$ to be a nilpotent variety. Let $G/V^*(G)$ be polycyclic-by-finite, say $N \trianglelefteq G$, $N \supseteq V^*(G)$, $|G/N| < \infty$ and $N/V^*(G)$ polycyclic. Now $V^*(N) \supseteq V^*(G)$, so $N/V^*(N)$ is a polycyclic group. Moreover N is of finite index in G, so N is finitely generated. By virtue of (a) N is now polycyclic. We conclude that G is polycyclic-by-finite. Finally assume $G/V^*(G)$ to be finite-by-polycyclic, say $N \trianglelefteq G$, $N \supseteq V^*(G)$, G/N polycyclic and $|N\colon V^*(G)| < \infty$. As $V^*(N) \supseteq V^*(G)$, we have $|N/V^*(N)| < \infty$. Now $\mathfrak{V}$ is locally residually finite, so $\mathfrak{F}$ is $\mathfrak{V}$-closed by (7.8). Hence $|V(N)| < \infty$. Now N is finitely generated. In order to see this, observe that $G/V^*(G)$ is polycyclic-by-finite, so that by the previous paragraph G is polycyclic-by-finite. Therefore N is polycyclic-by-finite, so clearly N is finitely generated. It follows that $N/V(N)$ is a finitely generated nilpotent group. This implies that $N/V(N)$ is polycyclic (see [16], Chapter 1, Section B, Corollary 8). Also G/N is polycyclic and we get that $G/V(N)$ is polycyclic. We conclude that G is finite-by-polycyclic, as desired.

One may ask oneself whether it is true or not that, if $\mathfrak{V}$ is a finitely based locally finite variety and G is a finitely generated group with $G/V^*(G)$ polycyclic, G itself must be polycyclic. The following answers this question.

(7.16) THEOREM. *Let G be a finitely generated group. Let $\mathfrak{V}$ be a finitely based locally finite variety. Suppose that $G/V^*(G)$ is polycyclic. Then the following hold.*

(a) *$V^*(G)$ is finite. So in particular G is finite-by-polycyclic. In general G does not have to be polycyclic.*

(b) *G is $\mathfrak{V}$-isologic to a polycyclic subgroup.*

PROOF. (a) By hypothesis $G/V^*(G)$ is polycyclic, so in particular $G/V^*(G)$ is finite-by-polycyclic. Hence by virtue of (7.15)(b), G is finite-by-polycyclic. And this means that every subgroup of G is finitely generated. Thus $V^*(G)$ is finitely generated and moreover (2.3)(a)(b) ensure $V^*(G) \in \mathfrak{V}$. It follows that $V^*(G)$ is finite, as $\mathfrak{V}$ is locally finite.

Next we present an example showing that G is not polycyclic in general. Let G be a non-solvable finite group. Let $\mathfrak{V}$ be the variety generated by G. Of course $\mathfrak{V}$ is locally finite. A celebrated theorem of S. Oates and M. B. Powell (see [13], 52.12) states that $\mathfrak{V}$ is finitely based. Certainly $G \in \mathfrak{V}$, or equivalently, $G = V^*(G)$, according to (2.3)(b). Trivially $G/V^*(G) = \bar{1}$ is polycyclic. However G is not polycyclic (notice that for finite groups, being solvable is the same as being polycyclic).

(b) Among all the counterexamples G to the assertion, choose one with $|V^*(G)|$ minimal (in view of (a) this is possible). So G is a finitely generated group with $G/V^*(G)$ polycyclic, but G is not $\mathfrak{V}$-isologic to any polycyclic subgroup. In particular G is not polycyclic, since always $G \underset{\mathfrak{V}}{\sim} G$. Now suppose there exists a proper subgroup H of G with $G = HV^*(G)$. Owing to (2.5)(a) $V^*(H) = H \cap V^*(G)$. Hence $V^*(G) = V^*(H)$ would imply $H \supseteq V^*(G)$, whence $G = H$, a contradiction. Thus $|V^*(H)| < |V^*(G)|$. Further $G/V^*(G) \simeq H/V^*(H)$, so $H/V^*(H)$ is polycyclic. Also H is finitely generated, since H is a subgroup of the finite-by-polycyclic group G. Apparently H is not a counterexample, so there exists a polycyclic subgroup K of H, such that $H \underset{\mathfrak{V}}{\sim} K$. However (4.4)(a) guarantees that $G \underset{\mathfrak{V}}{\sim} H$. It follows that $G \underset{\mathfrak{V}}{\sim} K$, violating the choice of G. We conclude that G is subgroup irreducible with respect to $\mathfrak{V}$-isologism. By employing (7.3) we have $V^*(G) \subseteq \Phi(G)$. This implies that $G/\Phi(G)$ is polycyclic. Now G is polycyclic-by-finite by Lemma 6 of [18]. It is known that the Frattini subgroup of a polycyclic-by-finite group is nilpotent (see [16], Chapter 1, Section C, Theorem 3). Hence $\Phi(G)$ is a finitely generated nilpotent group and therefore polycyclic (see [16], Chapter 1, Section B, Corollary 8). We now have that both $G/\Phi(G)$ and $\Phi(G)$ are polycyclic, whence G itself is polycyclic, a contradiction.

PROOFS OF (7.13) AND (7.14). In both cases the implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(a) are obvious. So let us prove the implication (b)$\Rightarrow$(c). Let G be an arbitrary group. Let the variety $\mathfrak{V}$ and the class $\mathfrak{Y}$ be either as in the statement of (7.13) or as in the statement of (7.14). Suppose that $G/V^*(G) \in \mathfrak{Y}$. Since the class $\mathfrak{Y}$ consists in all cases of finitely generated groups, we can find a finitely generated subgroup of H of G with the property that $G = HV^*(G)$. By virtue of (4.4)(a) we have $G \underset{\mathfrak{V}}{\sim} H$. Now the theorems follow immediately from (7.15) and (7.16)(b).

We end this section with the following six consequences of the above Theorems (7.13) and (7.14) (see also [18], Lemma 5, Lemma 6 and Theorem 3).

(7.17) COROLLARY. *Let $\mathfrak{V}$ be either a finitely based locally finite variety or nilpotent variety. Let $\mathfrak{Y}$ be either the class of polycyclic, the class of finite-by-polycyclic or the class of polycyclic-by-finite groups. Then $\mathfrak{Y}$ is $\mathfrak{V}$-closed.*

PROOF. Clear from (7.13) and (7.14).

8. Stemgroups

It was Philip Hall who discovered towards the end of the thirties that for any group G there exists a group S with the property that $G \underset{1}{\sim} S$ and $\gamma_2(S) \supseteq \varsigma(S)$. He coined the name *stemgroup* for these group S. The stemgroups S are finite as long as G is finitely generated and $G/\varsigma(G)$ is finite. The question arises whether such stemgroups also can be constructed in case of a general variety $\mathfrak{V}$. More precisely: given a variety $\mathfrak{V}$ and a group G, does there exist a group S, such that $G \underset{\mathfrak{V}}{\sim} S$ and $V^*(S) \subseteq V(S)$? In general, the answer to this question is in the negative, as the following examples show.

(8.1) EXAMPLES. (a) (see also [1], Chapter IV, 7.25) Let $\mathfrak{V} = \mathfrak{A}_{p^2}$, where p is a prime number. For a group G it then holds that

$$V(G) = \gamma_2(G)G^{p^2} \quad \text{and} \quad V^*(G) = \{g \in \varsigma(G) \colon g^{p^2} = 1\}.$$

Hence, if we take $G = C_{p^3}$, then $V(G) \simeq C_p$ and $V^*(G) \simeq C_{p^2}$. Suppose now that there exists a group S with $G \underset{\mathfrak{V}}{\sim} S$ and $V^*(S) \subseteq V(S)$. Apparently $|S/V^*(S)| = |G/V^*(G)| = p$ and $|V^*(S)|$ divides $|V(S)| = |V(G)| = p$. Hence $|S|$ divides p^2 thus we can conclude that $S \in \mathfrak{V}$. Now (2.3)(b) yields $|V(S)| = 1$, which is clearly absurd.

(b) Let $\mathfrak{V} = \mathfrak{N}_n$, where $n \geq 2$. Let G be a group of nilpotency class $n+1$ (for instance take G to be the dihedral group of order 2^{n+2}, or see the proof of Corollary (6.16)). Hence $G/\varsigma_n(G)$ is a non-trivial abelian group. Suppose now that there exists a group S with $G \underset{n}{\sim} S$ and $\varsigma_n(S) \subseteq \gamma_{n+1}(S)$. Hence $S/\varsigma_n(S)$ is non-trivial and abelian, so $S = \varsigma_{n+1}(S)$. Moreover we have $\gamma_2(S) \subseteq \varsigma_n(S) \subseteq \gamma_{n+1}(S) \subseteq \gamma_2(S)$, whence $\gamma_2(S) = \gamma_{n+1}(S)$. This yields that $\gamma_2(S) = \gamma_{n+2}(S)$. However $\gamma_{n+2}(S) = 1$. We conclude that S is abelian, a contradiction to $S/\varsigma_n(S)$ being non-trivial.

It is not a surprise that the variety $\mathfrak{A}_{p^2}$ features in the above Example (8.1)(a). Namely the proof of the result of Hall mentioned above depends on the following two requirements a variety $\mathfrak{V}$ should come up to:

(8.2) Subgroups of $\mathfrak{V}$-free groups are $\mathfrak{V}$-splitting.

(8.3) Subgroups of $\mathfrak{V}$-marginal subgroups are normal.

However we have the following (see also [11] at the top of page 115).

(8.4) PROPOSITION. *Let $\mathfrak{V}$ be a variety. Then $\mathfrak{V}$ satisfies* (8.2) *and* (8.3) *if and only if $\mathfrak{V} = \mathfrak{E}$, $\mathfrak{A}_m$ or $\mathfrak{A}$ (with m a squarefree positive integer).*

PROOF. [$\Leftarrow$] This is clear as subgroups of free abelian groups are again free abelian, and because any quotient of a finite cyclic group C is isomorphic to a subgroup of C.

[$\Rightarrow$] Let $G \in \mathfrak{V}$, so that in view of (2.3)(b) $G = V^*(G)$. According to (8.3) every subgroup of G is normal in G. Hence G is a Dedekind group. It follows (see [15] 5.3.7) that G is either abelian or $G \simeq Q \times T$, where Q is the quaternion group of order 8 and T is an abelian torsion group. In the latter case we would have $Q \in \mathfrak{V}$, as $\mathfrak{V}$ is closed with respect to taking subgroups. This would imply that $Q \times Q \in \mathfrak{V}$, as $\mathfrak{V}$ is closed with respect to taking subcartesian products. However, $Q \times Q$ possesses subgroups which are not normal, for instance the diagonal will do. We conclude that G is abelian after all. Hence $\mathfrak{V}$ is an abelian variety. Consequently $\mathfrak{V}$ is one of the varieties $\mathfrak{E}$, $\mathfrak{A}_m$ or $\mathfrak{A}$, where m is a positive integer. Next assume that m is not squarefree, say p is a prime number with $p^2|m$. Put $\mathfrak{V} = \mathfrak{A}_m$. Then $\mathbf{Z}/V(\mathbf{Z}) \simeq \mathbf{Z}/m\mathbf{Z}$, whence $\mathbf{Z}/m\mathbf{Z}$ is $\mathfrak{V}$-free (of rank 1). In view of (8.2) $\mathbf{Z}/p\mathbf{Z}$ should be a $\mathfrak{V}$-splitting group. However the exact sequence $0 \to (m/p)\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$ does not split as $p = \gcd(m/p, p)$. This is a contradiction.

(8.5) COROLLARY. *Let m be a squarefree positive integer. For every group G there exists a group S, such that*

(a) $G \underset{\mathfrak{A}_m}{\sim} S$.

(b) $\{g \in \varsigma(S) \colon g^m = 1\} \subseteq \gamma_2(S)S^m$.

The above considerations show that for a variety $\mathfrak{V}$ it is not always possible to find a group inside a $\mathfrak{V}$-isologism class having its marginal subgroup contained in its verbal subgroup. On the other hand it was proved in [8], Section 6, that for a given group G and $n \geq 0$ there exists a group S satisfying $G \underset{n}{\sim} S$ and $\varsigma(S) \subseteq \gamma_{n+1}(S)$. Groups with the latter property are called n-stemgroups and for $n = 1$ they are exactly the stemgroups in the sense of Hall. Indeed a lot of properties shared by Hall's stemgroups can be generalized to n-stemgroups. To mention one, if G is finitely generated and $G/\varsigma_n(G)$ is finite, then a finite n-stemgroup S can be constructed within the n-isoclinism class of G (see [8], Theorem 6.3). Moreover the n-stemgroups are of importance in cohomology theory of groups, notably with respect to induced central extensions (see [17], pages 129 and 130, and also [19]). This leads us to the following definition.

(8.6) DEFINITION. Let $\mathfrak{V}$ be a variety. A group S is called a *$\mathfrak{V}$-stemgroup* if it satisfies $\varsigma(S) \subseteq V(S)$.

The proof of the existence of $\mathfrak{V}$-stemgroups in case $\mathfrak{V} = \mathfrak{N}_n$ rests among others on the fact that for an (absolutely) free group F, the group $V(F)/[F, V(F)]$

https://doi.org/10.1017/S1446788700030366 Published online by Cambridge University Press

is free again. This seems to be a very special situation and it is not known which varieties $\mathfrak{V}$ satisfy this property. Fairly recently C. K. Gupta showed that $\mathfrak{V} = \mathfrak{S}_2$ can be left out of account, for it holds that $\varsigma(F/[F,F'']) = F''/[F,F'']$ and this group is not even torsion-free (see [4]). A proof of the existence of $\mathfrak{V}$-stemgroups within a $\mathfrak{V}$-isologism class along the lines the proof of the existence of n-stemgroups within an n-isoclinism class seems to be hopeless. Nevertheless, in case of polynilpotent varieties we are in favorable circumstances.

(8.7) THEOREM. *Let $\mathfrak{V} = \mathfrak{N}_{c_1,\dots,c_l}$. For any finitely generated group G there exists a finitely generated group S such that $G \underset{\mathfrak{V}}{\sim} S$ and $\varsigma(S) \subseteq V(S)$.*

PROOF. For $l = 1$, see [8], Theorem 6.3. Assume in the sequel that $l > 1$. If G is cyclic, $S = 1$ will do, as $G \in \mathfrak{V}$ in this case. Hence we can choose a free presentation of G, say $G \simeq F/R$ with F free of finite rank at least two. Set $T = \gamma_{c_1+1,\dots,c_{l-1}+1}(F)$, so $V(F) = \gamma_{c_l+1}(T)$. Now put $S = F/R \cap V(F)$. Regarding (4.4)(b) $S \underset{\mathfrak{V}}{\sim} G$. Certainly S is finitely generated. Now $\varsigma(S)V(S)/V(S) \subseteq \varsigma(S/V(S))$ and $S/V(S) \simeq F/V(F) = F/\gamma_{c_l+1}(T)$. At this point observe that F/T is infinite, since $c_1 \geq 1$, so that $T \subseteq F'$. By virtue of a theorem of N. D. Gupta and C. K. Gupta and the fact that $c_l \geq 1$, it follows that $\varsigma(F/\gamma_{c_l+1}(T)) = \bar{1}$ (see [5]). We draw the conclusion that $\varsigma(S) \subseteq V(S)$, as wanted.

As already mentioned before, the n-stemgroups S ($= \mathfrak{N}_n$-stemgroups) in (8.7) can be chosen to be finite if one assumes for example that $\gamma_{n+1}(G)$ is finite. One of that facts that underlies this is $\mathfrak{F}$ being $\mathfrak{N}_n$-closed for all $n \geq 1$. Now Theorem B in [18] asserts that $\mathfrak{F}$ is $\mathfrak{N}_{c_1,\dots,c_l}$-closed. The question remains whether, by imposing some extra conditions on G in (8.7) like $\gamma_{c_1+1,\dots,c_l+1}(G)$ being finite, a finite S can be constructed.

References

[1] F. R. Beyl and J. Tappe, *Group extensions, representations and the Schur multiplicator*, (Lecture Notes in Math., vol. 958, Springer, Berlin-Heidelberg-New York, 1982).

[2] J. C. Bioch, 'On n-isoclinic groups', *Indag. Math.* **38** (1976), 400-407.

[3] J. C. Bioch and R. W. van der Waall, 'Monomiality and isoclinism of groups', *J. Reine Angew. Math.* **298** (1978), 74–88.

[4] C. K. Gupta, 'The free centre-by-metabelian groups', *J. Austral. Math. Soc.* **16** (1973), 294–299.

[5] C. K. Gupta and N. D. Gupta, 'Generalized Magnus embeddings and some applications', *Math. Z.* **160** (1978), 75–87.

[6] Ph. Hall, 'The classification of prime-power groups', *J. Reine Angew. Math.* **182** (1940), 130–141.

[7] Ph. Hall, 'Verbal and marginal subgroups', *J. Reine Angew. Math.* **182** (1940), 156–157.

[8] N. S. Hekster, 'On the structure of n-isoclinism classes of groups', *J. Pure Appl. Algebra* **40** (1986), 63–85.
[9] J. A. Hulse and J. C. Lennox, 'Marginal series in groups', *Proc. Roy. Soc. Edinburgh Sect.* A **76** (1976), 139–154.
[10] B. Huppert, *Endliche Gruppen* I (Springer, Berlin-Heidelberg-New York, 1979).
[11] C. R. Leedham-Green and S. McKay, 'Baer-invariants, isologism, varietal laws and homology', *Acta Math.* **137** (1976), 99–150.
[12] M. R. R. Moghaddam, 'On the Schur-Baer property', *J. Austral. Math. Soc. Ser.* A **31** (1981), 343–361.
[13] H. Neumann, *Varieties of groups*, (Ergebnisse der Math., Neue Folge 37, Springer, Berlin, 1967).
[14] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, (Ergebnisse der Math., Neue Folge 62, 63, Springer, Berlin, 1972).
[15] D. J. S. Robinson, *A course in the theory of groups*, (Graduate Texts in Math. 80, Springer, Berlin-Heidelberg-New York, 1982).
[16] D. Segal, *Polycyclic groups* (Cambridge Tracts in Math. 82, Cambridge Univ. Press, 1983).
[17] U. Stammbach, *Homology in group theory*, (Lecture Notes in Math. 359, Springer, Berlin-Heidelberg-New York, 1973).
[18] P. W. Stroud, 'On a property of verbal and marginal subgroups', *Proc. Cambridge Philos. Soc.* **61** (1965), 41–48.
[19] L. R. Vermani, 'A note on induced central extensions', *Bull. Austral. Math. Soc.* **20** (1979), 411–420.
[20] R. W. van der Waall, 'On n-isoclinic embedding of groups', to appear in *J. Pure Appl. Algebra*.
[21] P. M. Weichsel, 'On isoclinism', *J. London Math. Soc.* **38** (1963), 63–65.
[22] P. M. Weichsel, 'On critical p-groups', *Proc. London Math. Soc.* (3) **14** (1964), 83–100.

Mathematisch Instituut
Universiteit van Amsterdam
Roetersstraat 15
1018 WB Amsterdam
The Netherlands