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MULTIDIMENSIONAL HARDY-TYPE INEQUALITIES VIA CONVEXITY

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Abstract

Let an almost everywhere positive function Φ be convex for $p > 1$ and $p < 0$, concave for $p \in (0, 1)$, and such that $Ax^p \leq \Phi(x) \leq Bx^p$ holds on $\mathbb{R}_+$ for some positive constants $A \leq B$. In this paper we derive a class of general integral multidimensional Hardy-type inequalities with power weights, whose left-hand sides involve $\Phi(\int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{t})\, d\mathbf{t})$ instead of $[\int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{t})\, d\mathbf{t}]^p$, while the corresponding right-hand sides remain as in the classical Hardy's inequality and have explicit constants in front of integrals. We also prove the related dual inequalities. The relations obtained are new even for the one-dimensional case and they unify and extend several inequalities of Hardy type known in the literature.

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1. Introduction

Let $p > 1$ and $f \in L^p(0, \infty)$ be a nonnegative function. If the function F is defined on $\mathbb{R}_+$ by

$$F(x) = \int_0^x f(t)\, dt,$$

then the inequality

$$\int_0^\infty \left[\frac{F(x)}{x}\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\, dx \tag{1.1}$$

holds and the constant $(p/(p-1))^p$ on its right-hand side is the best possible, that is, it cannot be replaced with any smaller constant. This highly important result, today referred to as the classical Hardy integral inequality, was obtained by Hardy, who

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first announced it in 1920 [5] and then proved it in 1925 [6] (see also [8, Chapter 9, Theorem 328]).

Three years later, in 1928, Hardy [7] (see also [8, Chapter 9, Theorem 330]) improved relation (1.1) by deriving its generalized form, namely that if $p > 1$, $m \neq 1$, and the function F is defined on $\mathbb{R}_+$ by

$$F(x) = \begin{cases} \displaystyle\int_0^x f(t)\,dt, & m > 1, \\ \displaystyle\int_x^\infty f(t)\,dt, & m < 1, \end{cases}$$

then the inequality

$$\int_0^\infty x^{-m} F^p(x)\,dx \le \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{p-m} f^p(x)\,dx \tag{1.2}$$

holds for all nonnegative functions f, such that $x^{1-m/p} f \in L^p(0,\infty)$, and the constant $(p/|m-1|)^p$ is the best possible. In the same paper (see also [8, Chapter 9, Theorem 347]), he also pointed out that if m and F fulfill the conditions of the above result, but $0 < p < 1$, then the sign of inequality in (1.2) is reversed, that is,

$$\int_0^\infty x^{-m} F^p(x)\,dx \ge \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{p-m} f^p(x)\,dx \tag{1.3}$$

holds.

On the other hand, the first unweighted Hardy-type inequality for $p < 0$ was considered by Knopp [12] in 1928, but in a discrete setting, for sequences of positive real numbers (see also [14] and the references therein). It is also interesting to note that general weighted integral Hardy-type inequalities for exponents $p, q < 0$ and $0 < p, q < 1$ were first studied much later, by Beesack and Heinig [1] and Heinig [9]. They obtained some necessary as well as some sufficient conditions for the reverse Hardy inequality

$$\left[\int_0^\infty (fv)^p(x)\,dx\right]^{1/p} \le C\left[\int_0^\infty \left(u(x)\int_0^x f(t)\,dt\right)^q dx\right]^{1/q}$$

and its dual version to hold with a positive real constant C, independent of f. For further information and remarks concerning the rich history, development, generalizations, and applications of Hardy's integral inequalities (1.1) and (1.2) see, for example, [8, 13–15, 18] and the references therein.

In this paper we consider Hardy's integral inequalities (1.2) and (1.3) in a multidimensional setting, with integrals taken over n-cells, that is, over axis-parallel rectangular blocks in $\mathbb{R}^n_+$. The first such result, a natural n-dimensional generalization of the classical Hardy integral inequality (1.1), was given by Pachpatte [19]. Using

Fubini's theorem and Jensen's inequality, he proved that the inequality

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n x_i\right)^{-p} F^p(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \\ \leq \left(\frac{p}{p-1}\right)^{np} \int_0^\infty \cdots \int_0^\infty f^p(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \tag{1.4}$$

holds for $p > 1$ and all nonnegative functions $f \in L^p(\mathbb{R}_+^n)$, where the function F is defined on $\mathbb{R}_+^n$ by

$$F(x_1, \ldots, x_n) = \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \ldots, t_n)\, dt_1 \ldots t_n,$$

and the constant $(p/(p-1))^{np}$ is the best possible. The corresponding results for the case $0 \neq p < 1$ were recently obtained by Oguntuase *et al.* [16] as consequences of much more general inequalities for convex and concave functions (see [17] for further details). Moreover, applying some different techniques (for example, mixed-means inequalities in [3]), Pachpatte [20] and Čižmešija and Pečarić [3] proved a multidimensional generalization of (1.2), but only for $p > 1$.

To present our idea, first note that the term $F^p(x)$, involved in the left-hand sides of the relations (1.1)–(1.4), is only a particular case of the expression $\Phi(F(x))$, where an almost everywhere positive function Φ is convex for $p > 1$ and $p < 0$, concave for $p \in (0, 1)$, and such that $Ax^p \leq \Phi(x) \leq Bx^p$ holds on $\mathbb{R}_+$ with some positive real constants $A \leq B$. Motivated by this simple observation, as a main result of this paper, we derive a general class of integral multidimensional Hardy-type inequalities with power weights, whose left-hand sides involve $\Phi(F(x))$ instead of $F^p(x)$, while the corresponding right-hand sides remain as usual, with explicit constants in front of integrals. The relations obtained are new even for the one-dimensional case ($n = 1$). As special cases, we obtain natural n-dimensional generalizations of (1.2) and (1.3) for all parameters $p \in \mathbb{R} \setminus \{0, 1\}$ and prove that the constant factors appearing on their right-hand sides are the best possible.

Our results unify and further extend several results of this type in the literature, including Pachpatte's results in [19, 20], as well as the fairly new results in [2–4, 10, 16, 17]. A technique used in our proofs is based on a convexity argument, essentially different from the classical one used by Hardy and others (see, for example, [6–9, 15]).

The paper is organized as follows. In Section 2 we introduce some necessary notation and state some already existing lemmas, germane to the proofs of our main results. General Hardy-type inequalities are presented, discussed and proved in Section 3. Finally, the concluding Section 4 is dedicated to some special cases of the general relations obtained, that is, to the previously mentioned direct n-dimensional generalizations of (1.2) and (1.3), and to some further remarks and examples.

CONVENTIONS. All functions in this paper are assumed to be measurable and expressions of the form $0 \cdot \infty$, $0/0$, a/∞ ($a \in \mathbb{R}$), and ∞/∞ are taken to be equal to zero.

As usual, by a weight function (or simply weight) we mean a nonnegative measurable function on a subset of $\mathbb{R}^n_+$, while an interval in $\mathbb{R}$ is any convex subset of $\mathbb{R}$.

2. Preliminaries

Before presenting our ideas and results, we need to introduce some notation. Throughout this paper we use bold letters to denote n-tuples of real numbers, for example $\mathbf{x} = (x_1, \ldots, x_n)$ or $\mathbf{y} = (y_1, \ldots, y_n)$. In particular, we set $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. Of course, it will be clear from the text whether (a, b) represents an open interval in $\mathbb{R}$ or a point in $\mathbb{R}^n$ for $n = 2$. Further, relations $<$, $\leq$, $>$, and $\geq$ are, as usual, defined componentwise. For example, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we write $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$, $i = 1, \ldots, n$. Moreover, $\mathbf{0} < \mathbf{b} \leq \boldsymbol{\infty}$ means $0 < b_i \leq \infty$, $i = 1, \ldots, n$. Finally, we introduce a notation for n-cells, that is, axis-parallel rectangular blocks in $\mathbb{R}^n$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a} < \mathbf{b}$, let

$$(\mathbf{a}, \mathbf{b}) = (a_1, b_1) \times \cdots \times (a_n, b_n) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} < \mathbf{x} < \mathbf{b}\},$$
$$(\mathbf{a}, \mathbf{b}] = (a_1, b_1] \times \cdots \times (a_n, b_n] = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a} < \mathbf{x} \leq \mathbf{b}\},$$

and analogously also for $[\mathbf{a}, \mathbf{b})$ and $[\mathbf{a}, \mathbf{b}]$. In particular, we have $\mathbb{R}^n_+ = (\mathbf{0}, \boldsymbol{\infty})$,

$$(\mathbf{0}, \boldsymbol{\infty}] = \{\mathbf{x} \mid \mathbf{0} < \mathbf{x} \leq \boldsymbol{\infty}\} \quad \text{and} \quad [\mathbf{0}, \boldsymbol{\infty}) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geq \mathbf{0}\}.$$

As announced in the Introduction, to prove our main results, we need two lemmas from the recent paper [17].

LEMMA 2.1. *Suppose that $\mathbf{b} \in (\mathbf{0}, \boldsymbol{\infty}]$, that $u : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$ is a weight function, such that $\mathbf{x} \mapsto u(\mathbf{x})/(x_1^2 \cdots x_n^2)$ is locally integrable in $(\mathbf{0}, \mathbf{b})$, and that the weight function v is defined by*

$$v(\mathbf{t}) = t_1 \cdots t_n \int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} \frac{u(\mathbf{x})}{x_1^2 \cdots x_n^2}\, d\mathbf{x}, \quad \mathbf{t} \in (\mathbf{0}, \mathbf{b}).$$

Let I be an interval in $\mathbb{R}$, $\Phi : I \to \mathbb{R}$, and $f : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$ be an integrable function, such that $f(\mathbf{x}) \in I$, for all $\mathbf{x} \in (\mathbf{0}, \mathbf{b})$.

(i) *If Φ is convex, then the following inequality holds:*

$$\int_0^{b_1} \cdots \int_0^{b_n} u(\mathbf{x}) \Phi\left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{t})\, d\mathbf{t}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n} \leq \int_0^{b_1} \cdots \int_0^{b_n} v(\mathbf{x}) \Phi(f(\mathbf{x}))\, \frac{d\mathbf{x}}{x_1 \cdots x_n}. \tag{2.1}$$

(ii) *If Φ is concave, the sign of the inequality in (2.1) is reversed, that is,*

$$\int_0^{b_1} \cdots \int_0^{b_n} u(\mathbf{x}) \Phi\left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{t})\, d\mathbf{t}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n} \geq \int_0^{b_1} \cdots \int_0^{b_n} v(\mathbf{x}) \Phi(f(\mathbf{x}))\, \frac{d\mathbf{x}}{x_1 \cdots x_n}. \tag{2.2}$$

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PROOF. Lemma 2.1 is an easy consequence of Jensen's inequality and Fubini's theorem (for details, see [17]). □

REMARK 2.2. In this paper, we shall particularly consider Lemma 2.1 for $u(\mathbf{x}) \equiv 1$. Then

$$v(\mathbf{x}) = t_1 \cdots t_n \int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} \frac{d\mathbf{x}}{x_1^2 \cdots x_n^2} = \prod_{i=1}^{n} \left(1 - \frac{x_i}{b_i}\right), \quad \mathbf{x} \in (\mathbf{0}, \mathbf{b}),$$

so (2.1) reads

$$\int_0^{b_1} \cdots \int_0^{b_n} \Phi\left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{t})\, d\mathbf{t}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n}$$
$$\le \int_0^{b_1} \cdots \int_0^{b_n} \Phi(f(\mathbf{x})) \prod_{i=1}^{n} \left(1 - \frac{x_i}{b_i}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n}, \tag{2.3}$$

while (2.2) reduces to (2.3) with the sign reversed.

The second lemma is dual to Lemma 2.1.

LEMMA 2.3. *For $\mathbf{b} \in [\mathbf{0}, \infty)$, let $u : (\mathbf{b}, \infty) \to \mathbb{R}$ be a locally integrable weight function in $(\mathbf{b}, \infty)$, and the weight function v be given by*

$$v(\mathbf{t}) = \frac{1}{t_1 \cdots t_n} \int_{b_1}^{t_1} \cdots \int_{b_n}^{t_n} u(\mathbf{x})\, d\mathbf{x}, \quad \mathbf{t} \in (\mathbf{b}, \infty).$$

Suppose that I is an interval in $\mathbb{R}$, $\Phi : I \to \mathbb{R}$, and that $f : (\mathbf{b}, \infty) \to \mathbb{R}$ is an integrable function, such that $f(\mathbf{x}) \in I$, for all $\mathbf{x} \in (\mathbf{b}, \infty)$.

(i) *If Φ is convex, then*

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} u(\mathbf{x}) \Phi\left(x_1 \cdots x_n \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \frac{f(\mathbf{t})\, d\mathbf{t}}{t_1^2 \cdots t_n^2}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n}$$
$$\le \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} v(\mathbf{x}) \Phi(f(\mathbf{x})) \frac{d\mathbf{x}}{x_1 \cdots x_n}. \tag{2.4}$$

(ii) *If Φ is concave, then the sign of the inequality in (2.4) is reversed, that is,*

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} u(\mathbf{x}) \Phi\left(x_1 \cdots x_n \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \frac{f(\mathbf{t})\, d\mathbf{t}}{t_1^2 \cdots t_n^2}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n}$$
$$\ge \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} v(\mathbf{x}) \Phi(f(\mathbf{x})) \frac{d\mathbf{x}}{x_1 \cdots x_n}. \tag{2.5}$$

PROOF. Similarly to the proof of Lemma 2.1, it follows by applying Jensen's inequality and Fubini's theorem (see [17] for details). □

REMARK 2.4. As already mentioned in Remark 2.2, we shall take advantage of Lemma 2.3 in the case where $u(\mathbf{x}) \equiv 1$. In this setting,

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$$v(\mathbf{t}) = \frac{1}{t_1 \cdots t_n} \int_{b_1}^{t_1} \cdots \int_{b_n}^{t_n} d\mathbf{x} = \prod_{i=1}^{n} \left(1 - \frac{b_i}{x_i}\right), \qquad \mathbf{t} \in (\mathbf{b}, \infty).$$

Thus, (2.4) becomes

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \Phi\left(x_1 \cdots x_n \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \frac{f(\mathbf{t})\, d\mathbf{t}}{t_1^2 \cdots t_n^2}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n}$$
$$\leq \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \Phi(f(\mathbf{x})) \prod_{i=1}^{n} \left(1 - \frac{b_i}{x_i}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n}. \tag{2.6}$$

Similarly, (2.5) reduces to (2.6) with the sign $\geq$.

REMARK 2.5. It is important to observe that if the function Φ is both convex and concave, that is, affine ($\Phi(x) = \alpha x + \beta$), then inequalities (2.1), (2.3), (2.4), and (2.6) become equalities. One such particular case is $\Phi(x) = x$.

REMARK 2.6. Note that in the one-dimensional case ($n = 1$), Lemmas 2.1 and 2.3 reduce to the corresponding results from [4].

3. Main results

We are now ready to state and prove our main results in this paper. Applying Lemmas 2.1 and 2.3, we obtain a new class of general multidimensional strengthened Hardy-type inequalities with power weights, related to all possible choices of parameter $p \in \mathbb{R} \setminus \{0\}$ and to arbitrary almost everywhere positive convex (or concave) functions Φ, such that $Ax^p \leq \Phi(x) \leq Bx^p$ holds on $(0, \infty)$ with some positive real constants $A \leq B$. As particular cases of the relations obtained, we derive sharp natural n-dimensional generalizations of (1.1)–(1.3).

First, we consider the case with the parameter $p > 1$. The corresponding strengthened Hardy-type inequalities are given in the following theorem.

THEOREM 3.1. *Let* $1 < p < \infty$ *and* $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ *be such that* $m_i \neq 1$, $i = 1, \ldots, n$. *Let* $\Phi : [0, \infty) \to \mathbb{R}$ *be a convex, almost everywhere positive function, such that* $Ax^p \leq \Phi(x) \leq Bx^p$ *holds on* $[0, \infty)$ *for some constants* $0 < A \leq B < \infty$.

(i) *If* $\mathbf{b} \in (\mathbf{0}, \infty]$ *and* $\mathbf{m} > \mathbf{1}$, *then the inequality*

$$\int_0^{b_1} \cdots \int_0^{b_n} \Phi\left(\int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{t})\, d\mathbf{t}\right) \prod_{i=1}^{n} x_i^{-m_i}\, d\mathbf{x}$$
$$\leq \frac{B^2}{A} \left(\prod_{i=1}^{n} \frac{p}{m_i - 1}\right)^p \int_0^{b_1} \cdots \int_0^{b_n} f^p(\mathbf{x})$$
$$\times \prod_{i=1}^{n} x_i^{p-m_i} \left[1 - \left(\frac{x_i}{b_i}\right)^{(m_i-1)/p}\right] d\mathbf{x} \tag{3.1}$$

holds for all nonnegative integrable functions $f : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$.

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(ii) *If* $\mathbf{b} \in [\mathbf{0}, \infty)$ *and* $\mathbf{m} < \mathbf{1}$, *then the inequality*

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \Phi\bigg(\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(\mathbf{t})\, d\mathbf{t}\bigg) \prod_{i=1}^{n} x_i^{-m_i}\, d\mathbf{x}$$
$$\leq \frac{B^2}{A}\bigg(\prod_{i=1}^{n} \frac{p}{1-m_i}\bigg)^p \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} f^p(\mathbf{x}) \prod_{i=1}^{n} x_i^{p-m_i}\bigg[1 - \bigg(\frac{b_i}{x_i}\bigg)^{(1-m_i)/p}\bigg]\, d\mathbf{x} \tag{3.2}$$

holds for all nonnegative integrable functions $f : (\mathbf{b}, \infty) \to \mathbb{R}$.

Proof. The proof follows from Lemmas 2.1(i) and 2.3(i) by choosing the weight function $u(\mathbf{x}) \equiv 1$, that is, by using relations (2.3) and (2.6).

First, we consider case (i), that is, $\mathbf{b} \in (\mathbf{0}, \infty]$ and $\mathbf{m} > \mathbf{1}$. The crucial step is to rewrite the inequality (2.3) with $\mathbf{a} \in (\mathbf{0}, \infty]$ and the function $g : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$, instead of $\mathbf{b}$ and f respectively, where $a_i = b_i^{(m_i-1)/p}$, $i = 1, \ldots, n$, and

$$g(\mathbf{x}) = f(x_1^{p/(m_1-1)}, \ldots, x_n^{p/(m_n-1)}) \prod_{i=1}^{n} x_i^{p/(m_i-1)-1}.$$

Hence, the left-hand side of (2.3) in this setting becomes

$$\begin{aligned}
&\int_0^{a_1} \cdots \int_0^{a_n} \Phi\bigg(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} g(\mathbf{t})\, d\mathbf{t}\bigg) \frac{d\mathbf{x}}{x_1 \cdots x_n} \\
&\quad\geq A \int_0^{a_1} \cdots \int_0^{a_n} \bigg(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} g(\mathbf{t})\, d\mathbf{t}\bigg)^p \frac{d\mathbf{x}}{x_1 \cdots x_n} \\
&\quad= A \int_0^{a_1} \cdots \int_0^{a_n} \bigg(\prod_{i=1}^{n} x_i\bigg)^{-p-1} \bigg(\int_0^{x_1} \cdots \int_0^{x_n} g(\mathbf{t})\, d\mathbf{t}\bigg)^p d\mathbf{x} \\
&\quad= A\bigg(\prod_{i=1}^{n} \frac{m_i-1}{p}\bigg)^p \int_0^{a_1} \cdots \int_0^{a_n} \bigg(\prod_{i=1}^{n} x_i\bigg)^{-p-1} \\
&\qquad\times\bigg(\int_0^{x_1^{p/(m_1-1)}} \cdots \int_0^{x_n^{p/(m_n-1)}} f(\mathbf{s})\, d\mathbf{s}\bigg)^p d\mathbf{x} \\
&\quad\geq \frac{A}{B}\bigg(\prod_{i=1}^{n} \frac{m_i-1}{p}\bigg)^p \int_0^{a_1} \cdots \int_0^{a_n} \bigg(\prod_{i=1}^{n} x_i\bigg)^{-p-1} \\
&\qquad\times\Phi\bigg(\int_0^{x_1^{p/(m_1-1)}} \cdots \int_0^{x_n^{p/(m_n-1)}} f(\mathbf{s})\, d\mathbf{s}\bigg) d\mathbf{x} \\
&\quad= \frac{A}{B}\bigg(\prod_{i=1}^{n} \frac{m_i-1}{p}\bigg)^{p+1} \int_0^{b_1} \cdots \int_0^{b_n} \Phi\bigg(\int_0^{y_1} \cdots \int_0^{y_n} f(\mathbf{s})\, d\mathbf{s}\bigg) \prod_{i=1}^{n} y_i^{-m_i}\, d\mathbf{y},
\end{aligned} \tag{3.3}$$

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where the second line in (3.3) is obtained by using the lower bound $\Phi(x) \geq Ax^p$ from the statement of Theorem 3.1, the fourth line by substituting $s_i = t_i^{p/(m_i-1)}$, $i = 1, \ldots, n$, in the inner integral, the fifth line by applying the condition $x^p \geq \Phi(x)/B$, while the last line follows by substituting $y_i = x_i^{p/(m_i-1)}$, $i = 1, \ldots, n$, in the outer integral. Note that $p/(m_i - 1) > 0$, $i = 1, \ldots, n$. Similarly, the right-hand side of (2.3), rewritten with the parameters mentioned, reads

$$\begin{aligned}
&\int_0^{a_1} \cdots \int_0^{a_n} \Phi(g(\mathbf{x})) \prod_{i=1}^n \left(1 - \frac{x_i}{a_i}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n} \\
&\quad \leq B \int_0^{a_1} \cdots \int_0^{a_n} f^p(x_1^{p/(m_1-1)}, \ldots, x_n^{p/(m_n-1)}) \\
&\quad \times \prod_{i=1}^n \left(1 - \frac{x_i}{a_i}\right) x_i^{p(p/(m_i-1)-1)-1} \, d\mathbf{x} \\
&\quad = B \left(\prod_{i=1}^n \frac{m_i - 1}{p}\right) \int_0^{b_1} \cdots \int_0^{b_n} f^p(\mathbf{y}) \prod_{i=1}^n y_i^{p-m_i} \left[1 - \left(\frac{y_i}{b_i}\right)^{(m_i-1)/p}\right] d\mathbf{y}.
\end{aligned} \tag{3.4}$$

Thus, (3.1) follows by combining (2.3), (3.3) and (3.4).

Case (ii), that is, $\mathbf{b} \in [\mathbf{0}, \infty)$ and $\mathbf{m} < \mathbf{1}$, holds analogously. Here we start from relation (2.6), considered with the parameter $\mathbf{a}$ and the function g, instead of $\mathbf{b}$ and f respectively, where $a_i = b_i^{(1-m_i)/p}$, $i = 1, \ldots, n$, and

$$g(\mathbf{x}) = f(x_1^{p/(1-m_1)}, \ldots, x_n^{p/(1-m_n)}) \prod_{i=1}^n x_i^{p/(1-m_i)+1}.$$

Considering that $p/(1 - m_i) > 0$, $i = 1, \ldots, n$, and applying the same estimates from the statement of Theorem 3.1 as in the previous case, together with the substitutions $s_i = t_i^{p/(1-m_i)}$ and $y_i = x_i^{p/(1-m_i)}$, $i = 1, \ldots, n$, the left-hand side of (2.6) transforms to

$$\begin{aligned}
&\int_{a_1}^\infty \cdots \int_{a_n}^\infty \Phi\left(x_1 \cdots x_n \int_{x_1}^\infty \cdots \int_{x_n}^\infty \frac{g(\mathbf{t})\, d\mathbf{t}}{t_1^2 \cdots t_n^2}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n} \\
&\quad \geq \frac{A}{B} \left(\prod_{i=1}^n \frac{1 - m_i}{p}\right)^{p+1} \int_{b_1}^\infty \cdots \int_{b_n}^\infty \Phi\left(\int_{y_1}^\infty \cdots \int_{y_n}^\infty f(\mathbf{s})\, d\mathbf{s}\right) \prod_{i=1}^n y_i^{-m_i} \, d\mathbf{y},
\end{aligned} \tag{3.5}$$

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while the right-hand side of (2.6) in this setting becomes

$$\int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \Phi(g(\mathbf{x})) \prod_{i=1}^{n} \left(1 - \frac{a_i}{x_i}\right) \frac{d\mathbf{x}}{x_1 \cdots x_n} \\ \leq B\left(\prod_{i=1}^{n} \frac{1-m_i}{p}\right) \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} f^p(\mathbf{y}) \prod_{i=1}^{n} y_i^{p-m_i} \left[1 - \left(\frac{b_i}{y_i}\right)^{(1-m_i)/p}\right] d\mathbf{y}. \tag{3.6}$$

Inequality (3.2) now follows from (2.6), (3.5) and (3.6). □

As a special case of Theorem 3.1 we get the following new multidimensional generalization of Hardy's inequality (1.2).

COROLLARY 3.2. *Let $1 < p < \infty$ and $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ be such that $m_i \neq 1$, $i = 1, \ldots, n$. Let $\Phi : [0, \infty) \to \mathbb{R}$ be a convex, almost everywhere positive function, such that $Ax^p \leq \Phi(x) \leq Bx^p$ holds on $[0, \infty)$ for some constants $0 < A \leq B < \infty$. Then the inequality*

$$\int_{0}^{\infty} \cdots \int_{0}^{\infty} \Phi(F(\mathbf{x})) \prod_{i=1}^{n} x_i^{-m_i}\, d\mathbf{x} \\ \leq \frac{B^2}{A}\left(\prod_{i=1}^{n} \frac{p}{|m_i - 1|}\right)^p \int_{0}^{\infty} \cdots \int_{0}^{\infty} f^p(\mathbf{x}) \prod_{i=1}^{n} x_i^{p-m_i}\, d\mathbf{x} \tag{3.7}$$

holds for all nonnegative functions $f : \mathbb{R}_+^n \to \mathbb{R}$, such that

$$\prod_{i=1}^{n} x_i^{1-m_i/p} f \in L^p(\mathbb{R}_+^n),$$

where the function F is on $\mathbb{R}_+^n$ defined by

$$F(\mathbf{x}) = \begin{cases} \displaystyle\int_{0}^{x_1} \cdots \int_{0}^{x_n} f(\mathbf{t})\, d\mathbf{t}, & \mathbf{m} > \mathbf{1}, \\ \displaystyle\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(\mathbf{t})\, d\mathbf{t}, & \mathbf{m} < \mathbf{1}. \end{cases} \tag{3.8}$$

PROOF. Corollary 3.2 follows directly from Theorem 3.1 by setting $\mathbf{b} = \boldsymbol{\infty}$ in (3.1) if $\mathbf{m} > \mathbf{1}$, and $\mathbf{b} = \mathbf{0}$ in (3.2) if $\mathbf{m} < \mathbf{1}$. □

REMARK 3.3. Obviously, the function $\Phi(x) = x^p$ fulfills the conditions from Theorem 3.1 and Corollary 3.2, with $A = B = 1$. In that case, inequalities (3.1), (3.2), and (3.7) reduce to the results from [3, 16], that is, to a natural multivariable generalization of the one-dimensional Hardy integral inequality (1.2) and its sharp strengthened versions given in [2, 4]. Those results will be addressed in the concluding section of this paper.

REMARK 3.4. Note that for $\mathbf{m} = m\mathbf{1}$, that is, for $m_1 = \cdots = m_n = m$, where $1 \neq m \in \mathbb{R}$, relations (3.1) and (3.2) respectively read

$$\int_0^{b_1} \cdots \int_0^{b_n} \Phi(F(\mathbf{x}))\left(\prod_{i=1}^n x_i\right)^{-m} d\mathbf{x} \leq \frac{B^2}{A}\left(\frac{p}{m-1}\right)^{np} \int_0^{b_1} \cdots \int_0^{b_n} f^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m} \left[1 - \left(\frac{x_i}{b_i}\right)^{(m-1)/p}\right] d\mathbf{x}, \quad (3.9)$$

for $m > 1$, and

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \Phi(F(\mathbf{x}))\left(\prod_{i=1}^n x_i\right)^{-m} d\mathbf{x} \leq \frac{B^2}{A}\left(\frac{p}{1-m}\right)^{np} \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} f^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m} \left[1 - \left(\frac{b_i}{x_i}\right)^{(1-m)/p}\right] d\mathbf{x}, \quad (3.10)$$

for $m < 1$, where F is defined by (3.8). In particular, for $m = p$, relation (3.9) becomes

$$\int_0^{b_1} \cdots \int_0^{b_n} \Phi(F(\mathbf{x}))\left(\prod_{i=1}^n x_i\right)^{-p} d\mathbf{x} \leq \frac{B^2}{A}\left(\frac{p}{p-1}\right)^{np} \int_0^{b_1} \cdots \int_0^{b_n} f^p(\mathbf{x}) \prod_{i=1}^n \left[1 - \left(\frac{x_i}{b_i}\right)^{(p-1)/p}\right] d\mathbf{x},$$

while for $m = 0$ the inequality (3.10) reduces to

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \Phi(F(\mathbf{x}))\, d\mathbf{x} \leq \frac{B^2}{A} p^{np} \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} f^p(\mathbf{x}) \prod_{i=1}^n x_i^p \left[1 - \left(\frac{b_i}{x_i}\right)^{1/p}\right] d\mathbf{x}.$$

Moreover, for $\mathbf{m} = p\mathbf{1}$, the generalized Hardy inequality (3.7) can be written in the form

$$\int_0^{\infty} \cdots \int_0^{\infty} \Phi(F(\mathbf{x}))\left(\prod_{i=1}^n x_i\right)^{-p} d\mathbf{x} \leq \frac{B^2}{A}\left(\frac{p}{p-1}\right)^{np} \|f\|_{L^p(\mathbb{R}_+^n)},$$

while for $\mathbf{m} = \mathbf{0}$ we obtain

$$\int_0^{\infty} \cdots \int_0^{\infty} \Phi(F(\mathbf{x}))\, d\mathbf{x} \leq \frac{B^2}{A} p^{np} \int_0^{\infty} \cdots \int_0^{\infty} \left(f(\mathbf{x}) \prod_{i=1}^n x_i\right)^p d\mathbf{x}.$$

Finally, we emphasize that all the results obtained in this section are new even for $n = 1$.

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We continue our analysis by exploring the case where $p < 0$. Since the function $x \mapsto x^p$ is positive, convex, and decreasing on $\mathbb{R}_+$, we obtain the following theorem, corresponding to Theorem 3.1.

THEOREM 3.5. *Suppose that $-\infty < p < 0$ and that $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ is such that $m_i \neq 1$, $i = 1, \ldots, n$. Let $\Phi : (0, \infty) \to \mathbb{R}$ be a positive convex function, such that $Ax^p \leq \Phi(x) \leq Bx^p$ holds on $(0, \infty)$ for some constants $0 < A \leq B < \infty$. If $\mathbf{b} \in (\mathbf{0}, \infty]$ and $\mathbf{m} < \mathbf{1}$, then inequality (3.1) holds for all positive integrable functions f on $(\mathbf{0}, \mathbf{b})$. If $\mathbf{b} \in [\mathbf{0}, \infty)$ and $\mathbf{m} > \mathbf{1}$, inequality (3.2) holds for all positive integrable functions f on $(\mathbf{b}, \infty)$.*

PROOF. The proof follows the same lines as that of Theorem 3.1, so it will be omitted. If $\mathbf{m} < \mathbf{1}$ here, then $p/(m_i - 1) > 0$ for all $i = 1, \ldots, n$, while if $\mathbf{m} > \mathbf{1}$, then $p/(1 - m_i) > 0$ holds for all $i = 1, \ldots, n$. □

In particular, as limit cases of the parameter $\mathbf{b}$, namely for $\mathbf{b} = \mathbf{0}$ and $\mathbf{b} = \infty$, we get an analogue of Corollary 3.2, that is, a new n-dimensional Hardy-type inequality related to a convex function Φ and a real parameter $p < 0$.

COROLLARY 3.6. *Suppose that $-\infty < p < 0$ and that $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ is such that $m_i \neq 1$, $i = 1, \ldots, n$. If $\Phi : (0, \infty) \to \mathbb{R}$ is a positive convex function, such that $Ax^p \leq \Phi(x) \leq Bx^p$, $x \in (0, \infty)$, for some constants $0 < A \leq B < \infty$, then inequality (3.7) holds for all positive functions f, such that*

$$\prod_{i=1}^{n} x_i^{1-m_i/p} f \in L^p(\mathbb{R}^n_+),$$

where the function F is on $\mathbb{R}^n_+$ given by

$$F(\mathbf{x}) = \begin{cases} \displaystyle\int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{t})\, d\mathbf{t}, & \mathbf{m} < \mathbf{1}, \\ \displaystyle\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(\mathbf{t})\, d\mathbf{t}, & \mathbf{m} > \mathbf{1}. \end{cases} \tag{3.11}$$

REMARK 3.7. It is evident that the rest of analysis done for $p > 1$ can be repeated for $p < 0$ as well. Therefore, all relations from Remark 3.4 hold also in this setting if the condition $\mathbf{m} > \mathbf{1}$ is replaced with $\mathbf{m} < \mathbf{1}$, and vice versa. Moreover, as in the previous case, the function $\Phi(x) = x^p$ is convex and fulfills the conditions from the statement of Theorem 3.5 and Corollary 3.6 with $A = B = 1$, so these results can be regarded as n-dimensional generalizations of the classical one-dimensional Hardy integral inequality (1.2). For $\mathbf{m} = m\mathbf{1}$, they also cover the recent multidimensional results from [16].

The next case to be considered is $0 < p < 1$. Since the function $x \mapsto x^p$ is now concave, the sign of inequality in the corresponding relations will be reversed.

THEOREM 3.8. *Let* $0 < p < 1$ *and* $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ *be such that* $m_i \neq 1$, $i = 1, \ldots, n$. *Suppose that* $\Phi : [0, \infty) \to \mathbb{R}$ *is a concave, almost everywhere positive function, such that* $Bx^p \leq \Phi(x) \leq Ax^p$, $x \in [0, \infty)$, *for some constants* $0 < B \leq A < \infty$. *For a function* f *let* F *be defined by* (3.8).

(i) *If* $\mathbf{b} \in (\mathbf{0}, \boldsymbol{\infty}]$ *and* $\mathbf{m} > \mathbf{1}$, *then*

$$\int_0^{b_1} \cdots \int_0^{b_n} \Phi(F(\mathbf{x})) \prod_{i=1}^n x_i^{-m_i}\, d\mathbf{x} \geq \frac{B^2}{A} \left(\prod_{i=1}^n \frac{p}{m_i - 1} \right)^p \int_0^{b_1} \cdots \int_0^{b_n} f^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{x_i}{b_i} \right)^{(m_i-1)/p} \right] d\mathbf{x} \tag{3.12}$$

holds for all nonnegative integrable functions $f : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$.

(ii) *If* $\mathbf{b} \in [\mathbf{0}, \boldsymbol{\infty})$ *and* $\mathbf{m} < \mathbf{1}$, *then*

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \Phi(F(\mathbf{x})) \prod_{i=1}^n x_i^{-m_i}\, d\mathbf{x} \geq \frac{B^2}{A} \left(\prod_{i=1}^n \frac{p}{1 - m_i} \right)^p \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} f^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{b_i}{x_i} \right)^{(1-m_i)/p} \right] d\mathbf{x} \tag{3.13}$$

holds for all nonnegative integrable functions $f : (\mathbf{b}, \boldsymbol{\infty}) \to \mathbb{R}$.

PROOF. To prove (3.12), we start from Lemma 2.1(ii), rewritten for the same function g and parameter $\mathbf{a}$ as in the proof of (3.1). Relation (3.12) follows by the same sequence of transformations and estimates as in (3.3) and (3.4), but with the inequalities reversed. Similarly, Lemma 2.3(ii) and relations (3.5) and (3.6), with the inequalities both reversed, yield (3.13). □

For $b = \mathbf{0}$ and $\mathbf{b} = \boldsymbol{\infty}$, Theorem 3.8 can be written in the following form.

COROLLARY 3.9. *Let* $0 < p < 1$ *and* $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ *be such that* $m_i \neq 1$, $i = 1, \ldots, n$. *If* $\Phi : [0, \infty) \to \mathbb{R}$ *is a concave, almost everywhere positive function, such that* $Bx^p \leq \Phi(x) \leq Ax^p$ *holds on* $[0, \infty)$ *for some constants* $0 < B \leq A < \infty$, *then*

$$\int_0^{\infty} \cdots \int_0^{\infty} \Phi(F(\mathbf{x})) \prod_{i=1}^n x_i^{-m_i}\, d\mathbf{x} \geq \frac{B^2}{A} \left(\prod_{i=1}^n \frac{p}{|m_i - 1|} \right)^p \int_0^{\infty} \cdots \int_0^{\infty} f^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i} d\mathbf{x}$$

holds for all nonnegative functions f *on* $\mathbb{R}_+^n$, *such that* $\prod_{i=1}^n x_i^{1-m_i/p} f \in L^p(\mathbb{R}_+^n)$, *where the function* F *is on* $\mathbb{R}_+^n$ *defined by* (3.8).

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REMARK 3.10. Note that for $0 < p < 1$ and under the same conditions on $\mathbf{m}$, all relations obtained in Remark 3.4 hold with the inequality reversed. Since the function $\Phi(x) = x^p$ is concave in this setting, and trivially fulfills the conditions from the statement of Theorem 3.8 and Corollary 3.9, the inequalities obtained may be seen as new multidimensional generalizations of (1.3). In particular, the relations related to $\mathbf{m} = m\mathbf{1}$ generalize recent results from [16].

We conclude this section with the parameter $p = 1$. It is not hard to see that Theorem 3.1 and Corollary 3.2 hold also for such p, as well as Theorem 3.8 and Corollary 3.9, together with their special cases (3.9) and (3.10) from Remark 3.4 if Φ is convex, and the corresponding relations with inequality reversed, mentioned in Remark 3.10, if Φ is concave. In particular, if $p = 1$ and $\Phi(x) = x$ (that is, $A = B = 1$ and Φ is both convex and concave), all the Hardy-type inequalities obtained become equalities. Therefore, the case $p = 1$ is not of particular interest.

4. Concluding remarks

This final section is dedicated to natural multidimensional generalizations of Hardy's inequalities (1.2) and (1.3), that is, to the case where $\Phi(x) = x^p$, $p \in \mathbb{R} \setminus \{0, 1\}$. First, we state n-dimensional strengthened Hardy inequalities. For $1 \neq p > 0$, we get the following consequences of Theorems 3.1 and 3.8.

COROLLARY 4.1. *Let $1 < p < \infty$, $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ be such that $m_i \neq 1$, $i = 1, \ldots, n$, and for a real-valued, locally integrable function f on a subset of $\mathbb{R}^n_+$, let F be defined by (3.8). If $\mathbf{b} \in (\mathbf{0}, \boldsymbol{\infty}]$ and $\mathbf{m} > \mathbf{1}$, then*

$$\int_0^{b_1} \cdots \int_0^{b_n} F^p(\mathbf{x}) \prod_{i=1}^n x_i^{-m_i}\, d\mathbf{x} \\ \leq \left(\prod_{i=1}^n \frac{p}{m_i - 1}\right)^p \int_0^{b_1} \cdots \int_0^{b_n} f^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{x_i}{b_i}\right)^{(m_i-1)/p}\right] d\mathbf{x} \quad (4.1)$$

holds for all nonnegative integrable functions f on $(\mathbf{0}, \mathbf{b})$. If $\mathbf{b} \in [\mathbf{0}, \boldsymbol{\infty})$ and $\mathbf{m} < \mathbf{1}$, then

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} F^p(\mathbf{x}) \prod_{i=1}^n x_i^{-m_i}\, d\mathbf{x} \\ \leq \left(\prod_{i=1}^n \frac{p}{1 - m_i}\right)^p \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} f^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{b_i}{x_i}\right)^{(1-m_i)/p}\right] d\mathbf{x} \quad (4.2)$$

holds for all nonnegative integrable functions f on $(\mathbf{b}, \boldsymbol{\infty})$. The constant factor $(\prod_{i=1}^n p/|m_i - 1|)^p$ is the best possible for both inequalities (4.1) and (4.2). Moreover, if $p \in (0, 1)$, then (4.1) and (4.2) hold with the inequalities reversed.

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PROOF. The proof follows directly from Theorems 3.1 and 3.8, considering $\Phi(x) = x^p$. In particular, for $p > 1$ here we provide a new proof of (4.1) and (4.2) which is, in our opinion, simpler that the proof by mixed-means techniques from [3]. Moreover, the reversed inequalities in the case $p \in (0, 1)$ are new.

We now prove that the constant $C = (\prod_{i=1}^n p/(m_i - 1))^p$ is the best possible for (4.1). If this is not true, then there exists a smaller constant $0 < D < C$, such that

$$\int_0^{b_1} \cdots \int_0^{b_n} F^p(\mathbf{x}) \prod_{i=1}^n x_i^{-m_i}\, d\mathbf{x} \leq D \int_0^{b_1} \cdots \int_0^{b_n} f^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{x_i}{b_i}\right)^{(m_i-1)/p}\right] d\mathbf{x} \tag{4.3}$$

holds for all nonnegative functions f on $(\mathbf{0}, \mathbf{b})$. Since $0 < (\prod_{i=1}^n p/(m_i - 1 + \xi))^p \nearrow C$ as $\xi \searrow 0$, there exists a small $\varepsilon > 0$, such that $0 < D < (\prod_{i=1}^n p/(m_i - 1 + \varepsilon))^p < C$. Hence, for the function $f_\varepsilon : (\mathbf{0}, \mathbf{b}) \to \mathbb{R}$, defined by $f_\varepsilon(\mathbf{x}) = \prod_{i=1}^n x_i^{m_i - 1 + \varepsilon/p - 1}$,

$$\begin{aligned}
\int_0^{b_1} \cdots \int_0^{b_n} f_\varepsilon^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{x_i}{b_i}\right)^{(m_i-1)/p}\right] d\mathbf{x}
&\leq \int_0^{b_1} \cdots \int_0^{b_n} f_\varepsilon^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i}\, d\mathbf{x} \\
&= \int_0^{b_1} \cdots \int_0^{b_n} \prod_{i=1}^n x_i^{\varepsilon-1} d\mathbf{x} = \frac{(\prod_{i=1}^n b_i)^\varepsilon}{\varepsilon^n},
\end{aligned}$$

so

$$\begin{aligned}
\int_0^{b_1} \cdots \int_0^{b_n} &\left(\int_0^{x_1} \cdots \int_0^{x_n} f_\varepsilon(\mathbf{t})\, d\mathbf{t}\right)^p \prod_{i=1}^n x_i^{-m_i}\, d\mathbf{x} \\
&= \left(\prod_{i=1}^n \frac{p}{m_i - 1 + \varepsilon}\right)^p \frac{(\prod_{i=1}^n b_i)^\varepsilon}{\varepsilon^n} > D \cdot \frac{(\prod_{i=1}^n b_i)^\varepsilon}{\varepsilon^n} \\
&\geq D \int_0^{b_1} \cdots \int_0^{b_n} f_\varepsilon^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i} \left[1 - \left(\frac{x_i}{b_i}\right)^{(m_i-1)/p}\right] d\mathbf{x}.
\end{aligned}$$

This contradicts (4.3), so the constant C is the best possible for (4.1). The proof that $(\prod_{i=1}^n p/(1 - m_i))^p$ is the best possible constant for (4.2) follows the same lines, considering $f_\varepsilon(\mathbf{x}) = \prod_{i=1}^n x_i^{(m_i - 1 - \varepsilon)/p - 1}$, for a small enough $\varepsilon > 0$. The same goes also for the reversed inequalities in the case $0 < p < 1$. Note that the analysis of the best possible constants presented here is different and more elegant than that in [3]. □

The remaining case, $p < 0$, is given in the next corollary.

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COROLLARY 4.2. *Let $-\infty < p < 0$, $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ be such that $m_i \neq 1$, $i = 1, \ldots, n$, and for a real-valued, locally integrable function f on a subset of $\mathbb{R}^n_+$, let F be defined by (3.11). If $\mathbf{b} \in (\mathbf{0}, \infty]$ and $\mathbf{m} < \mathbf{1}$, then inequality (4.1) holds for all positive integrable functions f on $(\mathbf{0}, \mathbf{b})$. On the other hand, if $\mathbf{b} \in [\mathbf{0}, \infty)$ and $\mathbf{m} > \mathbf{1}$, then inequality (4.2) holds for all positive integrable functions f on $(\mathbf{b}, \infty)$. Moreover, the constant $(\prod_{i=1}^n p/|m_i - 1|)^p$ is the best possible for (4.1) and (4.2).*

PROOF. Corollary 4.2 is a direct consequence of Theorem 3.5 for the concave function $\Phi(x) = x^p$. To prove that the constant $(\prod_{i=1}^n p/|m_i - 1|)^p$ is the best possible, we use the same test functions as in the proof of Corollary 4.1. □

REMARK 4.3. Note that, for $\mathbf{m} = m\mathbf{1}$, Corollaries 4.1 and 4.2 reduce to the recent results in [16]. Moreover, for $n = 1$ we obtain the results in [4].

REMARK 4.4. Observe that, by setting $\mathbf{b} = \infty$ in (4.1) and $\mathbf{b} = \mathbf{0}$ in (4.2), we get

$$\int_0^\infty \cdots \int_0^\infty F^p(\mathbf{x}) \prod_{i=1}^n x_i^{-m_i}\, d\mathbf{x} \leq \left(\prod_{i=1}^n \frac{p}{|m_i - 1|}\right)^p \int_0^\infty \cdots \int_0^\infty f^p(\mathbf{x}) \prod_{i=1}^n x_i^{p-m_i}\, d\mathbf{x}, \tag{4.4}$$

thus obtaining a natural multidimensional Hardy inequality with power weights and with the best possible constant. As above, the function F is defined either by (3.8) if $1 \neq p > 0$, or by (3.11) if $p < 0$. Of course, if $p \in (0, 1)$ the inequality in (4.4) is reversed. The result obtained is a special case of Corollaries 3.2, 3.6, and 3.9 from the previous section. On the other hand, for $n = 1$ (4.4) reduces to (1.2), while the corresponding reversed inequality becomes (1.3). In particular, for $m = p$ we obtain (1.1). Hence, (4.4) can also be considered as a generalization of the classical Hardy integral inequality.

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References

[1] P. R. Beesack and H. P. Heinig, 'Hardy's inequalities with indices less than 1', *Proc. Amer. Math. Soc.* **83**(3) (1981), 532–536.
[2] A. Čižmešija and J. Pečarić, 'Some new generalizations of inequalities of Hardy and Levin–Cochran–Lee', *Bull. Austral. Math. Soc.* **63**(1) (2001), 105–113.
[3] A. Čižmešija and J. Pečarić, 'Multivariable mixed means and inequalities of Hardy and Levin–Cochran–Lee type', *Math. Inequal. Appl.* **5**(3) (2002), 397–415.
[4] A. Čižmešija, J. Pečarić and L.-E. Persson, 'On strengthened Hardy and Pólya–Knopp's inequalities', *J. Approx. Theory* **125** (2003), 74–84.

[5] G. H. Hardy, 'Note on a theorem of Hilbert', *Math. Z.* **6** (1920), 314–317.
[6] G. H. Hardy, 'Notes on some points in the integral calculus LX: An inequality between integrals', *Messenger of Math.* **54** (1925), 150–156.
[7] G. H. Hardy, 'Notes on some points in the integral calculus LXIV', *Messenger of Math.* **57** (1928), 12–16.
[8] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities* (Cambridge University Press, Cambridge, 1959).
[9] H. P. Heinig, 'Variations of Hardy's inequality', *Real Anal. Exchange* 5 (1979–80), 61–81.
[10] S. Kaijser, L. Nikolova, L.-E. Persson and A. Wedestig, 'Hardy type inequalities via convexity', *Math. Inequal. Appl.* **8**(3) (2005), 403–417.
[11] S. Kaijser, L.-E. Persson and A. Öberg, 'On Carleman and Knopp's inequalities', *J. Approx. Theory* **117** (2002), 140–151.
[12] K. Knopp, 'Über Reihen mit positiven Gliedern', *J. London Math. Soc.* **3** (1928), 205–211.
[13] A. Kufner, L. Maligranda and L.-E. Persson, 'The prehistory of the Hardy inequality', *Amer. Math. Monthly* **113** (2006), 715–732.
[14] A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy inequality—about its history and some related results* (Vydavatelsky Servis Publishing House, Pilsen, 2007).
[15] A. Kufner and L.-E. Persson, *Weighted inequalities of Hardy type* (World Scientific, Singapore, 2003).
[16] J. A. Oguntuase, C. A. Okpoti, L.-E. Persson and F. K. A. Allotey, 'Multidimensional Hardy type inequalities for $p < 0$ and $0 < p < 1$', *J. Math. Inequal.* **1** (2007), 1–11.
[17] J. A. Oguntuase, C. A. Okpoti, L.-E. Persson and F. K. A. Allotey, 'Weighted multidimensional Hardy and Pólya–Knopp's type inequalities', Research report **15**, Department of Mathematics, Luleå University of Technology, Sweden, 2006, 15 pages.
[18] B. Opic and A. Kufner, *Hardy-type inequalities*, Pitman Series, 219 (Longman Scientific & Technical, Harlow, 1990).
[19] B. G. Pachpatte, 'On multivariate Hardy type inequalities', *An. Stiint. Al. I. Cuza Iasi Sect. I a Mat.* **38**(3) (1992), 355–361.
[20] B. G. Pachpatte, 'On some generalizations of Hardy's integral inequality for functions of several variables', *Demonstratio Math.* **27**(1) (1994), 43–51.

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