ARITHMETICAL INVERSION FORMULAS

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1. Introduction. Let n and r be integers, r positive, and define the *core* $\gamma(r)$ of r to be the product of the distinct prime factors of r ($\gamma(1) = 1$). Let $f(n,r)$ be a complex-valued, arithmetical function of n and r. If for all n, $f(n,r) = f((n,r), r)$ then $f(n, r)$ is called an *even* function (mod r), and if $f(n,r) = f(\gamma(n, r), r)$ for all n, $\gamma(n, r) = \gamma((n, r))$, then $f(n, r)$ is said to be a *primitive* function (mod r). Clearly, both classes of functions are subclasses of the periodic functions (mod r), while the primitive functions form a subclass of the even functions (mod r).

In a series of three papers (**3**; **5**; **6**) the author developed parallel, though interrelated, trigonometric and arithmetical theories of the even and primitive functions (mod r). It was shown (**3**, Theorem 3) that $f(n, r)$ is even (mod r) if and only if it possesses a representation of the form

$$f(n, r) = \sum_{d \mid (n, r)} F\left(d, \frac{r}{d}\right), \tag{1.1}$$

and that $f(n, r)$ is primitive (mod r) if and only if it possesses a representation of the form (**5**, Theorem 8),

$$f(n, r) = \sum_{\substack{d \mid \gamma(r) \\ (d,n)=1}} G\left(d, \frac{r}{d}\right). \tag{1.2}$$

It is the purpose of the present paper to develop a purely arithmetical theory of these two classes of functions, built on the unifying idea of arithmetical inversion.

More precisely, the method of the paper is based on two arithmetical inversion principles, the first (Theorem 2.1) relating to the class of all even functions (mod r), while the second (Theorem 2.3) is limited to the primitive functions (mod r). We remark that the first of these two results becomes equivalent (Corollary 2.2) to the ordinary Möbius inversion formula in case $f(n, r)$ is restricted to the subclass of *completely even* functions (mod r), that is, functions satisfying $f(n, r) = f(n', r')$ for all n, n', and all positive r, r' such that $(n, r) = (n', r')$. An analogous result (Corollary 2.5) is proved for the *completely primitive* functions (mod r), that is, functions satisfying $f(n, r) = f(n', r')$ for all n, n' and all positive r, r' such that $\gamma(r)/\gamma(n, r) = \gamma(r')/\gamma(n', r')$.

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The characterizations (1.1) and (1.2) of the even and primitive functions (mod r) follow as immediate consequences (Theorems 2.2 and 2.4, respectively) of the above-mentioned inversion relations. Moreover, it also follows that the functions, $F(r_1, r_2)$ and $G(r_1, r_2)$, are *uniquely* determined, under appropriate restrictions on the integral variables r_1 and r_2.

Sections 3 and 4 are devoted to proofs of generalizations of three fundamental identities in the arithmetical theory of even functions. These identities are stated as follows. Let $\mu(r)$ denote the Möbius inversion function and $\phi(r)$ the Euler totient; then

$$\chi(n, r) \equiv \sum_{d\mid(n,r)} d\mu\left(\frac{r}{d}\right) = \frac{\phi(r)\mu(\delta)}{\phi(\delta)} \equiv \Phi(n, r), \tag{1.3}$$

where $\delta = r/(n, r)$;

$$\phi(r) \sum_{\substack{d\mid r\\(d,n)=1}} \frac{d}{\phi(d)}\, \mu\left(\frac{r}{d}\right) = \mu(r)\chi(n, r); \tag{1.4}$$

$$\sum_{\substack{d\mid r\\(d,n)=1}} \frac{\mu^2(d)}{\phi(d)} = \frac{r\phi((n, r))}{\phi(r)(n, r)}. \tag{1.5}$$

Formula (1.3) is Hölder's relation **(7)**, which asserts the equality between the Dedekind-von Sterneck function $\Phi(n, r)$ and Kluyver's function $\chi(n, r)$, or equivalently, the arithmetical form of Ramanujan's sum. The identity (1.4) is due to Brauer and Rademacher **(2**; **6**, § 5**)**, while (1.5) is due in the case $n = 1$ to Landau **(8**, p. 182**)**; for a proof of the extended form (1.5), we mention **(4**, Theorem 9**)**. In the sequel, these three relations will be referred to as the Hölder, Brauer-Rademacher, and Landau identities, respectively.

In Theorem 3.1 we give a new proof of a generalization of the Landau identity, proved originally in **(5)**. The proof given in this paper is based on the theory of arithmetical inversion. As a consequence of the generalized Landau identity, we obtain in Theorem 3.2 a wide extension of the Brauer-Rademacher identity.

In Theorem 4.1 we give a new proof, based on arithmetical inversion, of a generalization of the Hölder relation, due to Anderson and Apostol **(1)**. The generalized Landau identity is also used in the proof of Theorem 4.1; moreover, a second proof of this identity is included in § 4, preceding the statement of the extended Hölder formula. The results of the paper are illustrated with a special case in § 5.

It is emphasized that the discussion of this paper is independent of the theory of even functions previously developed. We also mention that the results of the present paper remain valid when the field of values, assumed here to be complex, is replaced by an arbitrary field of characteristic 0.

2. Arithmetical inversion of even functions (mod r)**.** We now prove a general inversion principle for the even functions (mod r).

First we recall the characteristic property of $\mu(r)$,

$$\sum_{d \mid r} \mu(d) = 1 \quad \text{or} \quad 0 \tag{2.1}$$

according as $r = 1$ or $r > 1$.

THEOREM 2.1. *Let r_1, r_2 denote positive integral variables.*

(A) *If $F(r_1, r_2)$ is an arbitrary function of r_1, r_2 and $f(n, r)$ is an even function* (mod r) *defined by*

$$f(n, r) = \sum_{d \mid (n, r)} F\left(d, \frac{r}{d}\right) \tag{2.2}$$

then $F(r_1, r_2)$ has the form,

$$F(r_1, r_2) = \sum_{d \mid r_1} f\left(\frac{r_1}{d}, r\right) \mu(d), \qquad r = r_1 r_2. \tag{2.3}$$

(B) *Conversely, if $f(n, r)$ is an arbitrary even function* (mod r) *and $F(r_1, r_2)$ is defined by* (2.3), *then $f(n, r)$ has the form* (2.2).

Proof. (A) Assume first that $f(n, r)$ is defined by (2.2). Then, placing $r = r_1 r_2$ and using (2.1), it follows that

$$\sum_{d \mid r_1} f\left(\frac{r_1}{d}, r\right) \mu(d) = \sum_{d \mid r_1} \mu(d) \sum_{D \mid ((r_1/d), r)} F\left(D, \frac{r}{D}\right)$$
$$= \sum_{D \mid r_1} F\left(D, \frac{r}{D}\right) \sum_{d \mid (r_1/D)} \mu(d) = F(r_1, r_2).$$

Thus (A) is proved.

(B) Assuming $F(r_1, r_2)$ to be defined by (2.3), we have, again by (2,1),

$$\sum_{d \mid (n, r)} F\left(d, \frac{r}{d}\right) = \sum_{d \mid (n, r)} \sum_{D \mid d} f\left(\frac{d}{D}, r\right) \mu(D)$$
$$= \sum_{\substack{d \mid (n, r) \\ DE = d}} f(E, r)\, \mu(D) = \sum_{E \mid (n, r)} f(E, r) \sum_{D \mid ((n, r)/E)} \mu(D) = f((n, r), r),$$

so that by the definition of an even function (mod r), (B) is proved.

We are thus led immediately to a characterization of the class of even functions (mod r).

THEOREM 2.2. *A function $f(n, r)$ is even* (mod r) *if and only if it has a representation of the form* (2.2). *Moreover, the function $F(r_1, r_2)$ is uniquely determined by* (2.3) *for positive values of r_1 and r_2.*

Replacing $F(r_1, r_2)$ by $\mathrm{F}(r_1)$ and $f(n, r)$ by $g((n, r))$, we obtain from Theorem 2.1, with $r_2 = 1$, the following inversion formula for the completely even functions (mod r).

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COROLLARY 2.1. *If $F(r)$ is a function of a positive integral variable r, and $f(n, r)$ is a completely even function* (mod r) *defined by*

$$f(n, r) = g((n, r)) = \sum_{d \mid (n, r)} F(d), \tag{2.4}$$

then $F(r)$ has the form

$$F(r) = \sum_{d \mid r} f\left(\frac{r}{d}, r\right) \mu(d) \equiv \sum_{d \mid r} g\left(\frac{r}{d}\right) \mu(d). \tag{2.5}$$

Conversely, if $f(n, r) = g((n, r))$ is completely even (mod r), *and $F(r)$ is defined by* (2.5), *then $f(n, r)$ has the form* (2.4).

Replacing (n, r) in (2.4) by r, Corollary 2.1 becomes the ordinary Möbius inversion formula. In fact,

COROLLARY 2.2. *The inversion relation of Theorem* 2.1 *is equivalent to the Möbius inversion formula, provided the class of functions $f(n, r)$ is restricted to the completely even functions* (mod r).

We also have by Corollary 2.1, the following analogue of Theorem 2.2 (cf. **5**, Theorem 4).

COROLLARY 2.3. *A function $f(n, r)$ is completely even* (mod r) *if and only if it is representable in the form* (2.4). *The function $F(r)$ is uniquely determined by* (2.5) *for $r > 0$.*

The following lemmas are needed in the proof of the inversion theorem for the primitive functions (mod r).

Definition. An integer r is said to be *primitive* if r contains no square factors > 1.

LEMMA 2.1. *If $r = r_1 r_2$, $e | \gamma(r)$, and r_1 is primitive, then*

$$\sum_{\substack{d \mid r r_1 / \gamma(r) \\ (e, r/d) = 1}} \chi(r_2, d) = \begin{cases} r\mu(r_1)/\gamma(r) & (e = r_1) \\ 0 & (e \neq r_1). \end{cases}$$

LEMMA 2.2. *If $r = r_1 r_2$ then*

$$\sum_{\substack{de = r \\ (e, r_2) = 1}} \chi(r_1, d) = r_1 \mu(r_2). \tag{2.6}$$

LEMMA 2.3. *If r is primitive, $r_2 | r$, and $r_1 | r_2$, then*

$$\sum_{\substack{d\delta = r \\ (d, n) = 1 \\ \delta \mid r_1, d \mid r_2}} \mu(d) = \begin{cases} \mu\left(\dfrac{r}{(n, r)}\right) & \text{if } r_1 = (n, r), r_2 = r, \\ 0 & \text{otherwise.} \end{cases}$$

In view of the multiplicative property of $\mu(r)$ and $\chi(n, r)$ as functions of r, it is sufficient to verify the above lemmas in the case that r is the power of a prime. The details are omitted.

THEOREM 2.3. *Let r_1, r_2 represent positive integral variables, r_1 primitive.*

(A) *If $G(r_1, r_2)$ is an arbitrary function of r_1, r_2 and $f(n, r)$ is a primitive function* (mod r) *defined by*

$$f(n,r) = \sum_{\substack{d\mid\gamma(r)\\(d,n)=1}} G\left(d, \frac{r}{d}\right) \equiv \sum_{d\mid((\gamma(r))/(\gamma(n,r)))} G\left(d, \frac{r}{d}\right), \tag{2.7}$$

then $G(r_1, r_2)$ has the form,

$$G(r_1, r_2) = \frac{\gamma(r)\mu(r_1)}{r} \sum_{d\mid((rr_1)/(\gamma(r)))} f\left(\frac{r}{d}, r\right)\chi(r_2, d), \qquad r = r_1 r_2. \tag{2.8}$$

(B) *Conversely, if $f(n, r)$ is an arbitrary primitive function* (mod r) *and $G(r_1, r_2)$ is defined by* (2.8), *then $f(n, r)$ has the form* (2.7).

Proof. (A) Assume that $f(n, r)$ is defined by (2.7), and let $T(r_1, r_2)$ denote the right member of (2.8). Then

$$T(r_1, r_2) = \frac{\gamma(r)\mu(r_1)}{r} \sum_{d\mid((rr_1)/(\gamma(r)))} \left(\sum_{\substack{e\mid\gamma(r)\\(e,(r/d))=1}} G\left(e, \frac{r}{e}\right)\right) \chi(r_2, d)$$

$$= \frac{\gamma(r)\mu(r_1)}{r} \sum_{e\mid\gamma(r)} G\left(e, \frac{r}{e}\right) \sum_{\substack{d\mid((rr_1)/(\gamma(r)))\\(e,(r/d))=1}} \chi(r_2, d).$$

Application of Lemma 2.1 yields $T(r_1, r_2) = G(r_1, r_2)$, which proves (A).

(B) Assume $G(r_1, r_2)$ to be given in the form (2.8) and denote the right member of (2.7) by $S(n, r)$.

$$S(n,r) = \frac{\gamma(r)}{r} \sum_{\substack{d\mid\gamma(r)\\(d,n)=1}} \mu(d) \sum_{e\mid((dr)/(\gamma(r)))} f\left(\frac{r}{e}, r\right)\chi\left(\frac{r}{d}, e\right)$$

$$= \frac{\gamma(r)}{r} \sum_{e\mid r} f\left(\frac{r}{e}, r\right) \sum_{\substack{d\mid\gamma(r)\\(d,n)=1\\((\gamma(r))/d)\mid(r/e)}} \mu(d) \sum_{D\mid((r/d),e)} D\mu\left(\frac{e}{D}\right)$$

$$= \frac{\gamma(r)}{r} \sum_{e\mid r} f\left(\frac{r}{e}, r\right) \sum_{D\mid e} D\mu\left(\frac{e}{D}\right) \sum_{\substack{d\delta=\gamma(r)\\(d,n)=1\\d\mid(r/D),\,\delta\mid(r/e)}} \mu(d).$$

By Lemma 2.3, the innermost sum of the last expression is 0 unless $\gamma(r/e) = \gamma(n, r)$, $\gamma(r/D) = \gamma(r)$, and under these conditions it has the value $\mu(\gamma(r)/\gamma(n, r))$. Moreover, since $f(n, r)$ is primitive (mod r), we must have then $f(r/e, r) = f(\gamma(r/e), r) = f(\gamma(n, r), r) = f(n, r)$, and it therefore follows, with $m = \gamma(r)/\gamma(n, r)$, that

$$S(n,r) = \frac{\gamma(r)\mu(m)f(n,r)}{r} \sum_{\substack{e\mid r\\\gamma(r/e)=\gamma(n,r)}} \sum_{\substack{D\mid e\\\gamma(r/D)=\gamma(r)}} D\mu\left(\frac{e}{D}\right).$$

Note that the conditions $e|r$, $\gamma(r/e) = \gamma(n, r)$ are equivalent to the conditions, $e|(r/\gamma(n, r))$, $(r/e, \gamma(r)) = \gamma(n, r)$. Similarly, $\gamma(r/D) = \gamma(r)$ and $D|(r/\gamma(r))$

are equivalent conditions for a divisor D of r. Therefore by definition of $\chi(n, r)$, one obtains

$$S(n, r) = \frac{\gamma(r)\mu(m)f(n, r)}{r} \sum_{\substack{e\delta = r/\gamma(n, r) \\ (\delta, m)=1}} \chi\left(\frac{r}{\gamma(r)}, e\right).$$

Thus by Lemma 2.2,

$$S(n, r) = \frac{\gamma(r)\mu(m)f(n, r)}{r} \cdot \frac{r\mu(m)}{\gamma(r)} = f(n, r).$$

This completes the proof.

As a consequence of Theorem 2.3, we have the following characterization of the class of primitive functions (mod r).

THEOREM 2.4. *A function $f(n, r)$ is primitive* (mod r) *if and only if it has a representation of the form* (2.7). *Moreover, the function $G(r_1, r_2)$ is uniquely determined, provided r_1 and r_2 are positive and r_1 is primitive.*

Corresponding to Corollaries 2.1, 2.2, and 2.3 in the case of the completely even functions (mod r), we deduce from Theorem 2.3 the following analogous properties of the completely primitive functions (mod r).

COROLLARY 2.4. *If $f(n, r) = k(m)$ is a completely primitive function* (mod r), *$m = \gamma(r)/\gamma(n, r)$, then*

$$f(n, r) = \sum_{\substack{d \mid \gamma(r) \\ (d, n)=1}} G(d) \rightleftarrows G(r_1) = \sum_{d \mid r_1} f\left(\frac{r_1}{d}, r_1\right) \mu\left(\frac{r_1}{d}\right), \tag{2.9}$$

where $G(r_1)$ is defined for primitive integers r_1.

Remark. The equivalence in (2.9) is to be interpreted in the same precise sense as Theorem 2.3.

Proof. Place $r = r_1$, $r_2 = 1$ in (2.7) and note that $\chi(1, d) = \mu(d)$.

Formula (2.9) may be reformulated as

$$k(m) = \sum_{d \mid m} G(d) \rightleftarrows G(r_1) = \sum_{d \mid r_1} k(d) \mu\left(\frac{r_1}{d}\right), \tag{2.10}$$

where r_1 is primitive and m is defined as in Corollary 2.4. Hence one obtains

COROLLARY 2.5. *If r is primitive, then the inversion relation of Theorem* 2.3 *is equivalent to the Möbius inversion formula, provided $f(n, r)$ is restricted to the completely primitive functions* (mod r).

COROLLARY 2.6 (cf. **5**, Theorem 10). *A function $f(n, r)$ is completely primitive* (mod r) *if and only if it is representable in the form* (2.7) *with $G(r_1, r_2) = G(r_1)$. The function $G(r_1)$ is uniquely determined for positive, primitive r_1.*

3. The generalized Landau and Brauer-Rademacher identities. We first introduce some notation. Let $g(r)$ and $h(r)$ be functions of r and define

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$$f(n, r) = \sum_{d \mid (n,r)} h(d) g\left(\frac{r}{d}\right) \mu\left(\frac{r}{d}\right), \quad F(r) = f(0, r). \tag{3.1}$$

Definition. A function $f(r)$ is said to be *completely multiplicative* if $f(1) = 1$, $f(r_1 r_2) = f(r_1) f(r_2)$ for all r_1, r_2.

We now recall two simple lemmas proved in (**5**, § 4).

LEMMA 3.1. *If $h(r)$ is completely multiplicative, then*

$$F(r) = h\left(\frac{r}{\gamma(r)}\right) F(\gamma(r)). \tag{3.2}$$

LEMMA 3.2. *If $g(r)$ is multiplicative, $h(r)$ is completely multiplicative, and for all primes p, $h(p) \neq 0$, $h(p) \neq g(p)$, then $F(r) \neq 0$ for all r.*

We now prove a theorem which generalizes the Landau identity, (1.5).

THEOREM 3.1 (**5**, Theorem 9). *If $g(r)$ and $h(r)$ satisfy the conditions of Lemma 3.2, then*

$$\sum_{\substack{d \mid r \\ (d,n)=1}} \left(\frac{g(d)}{F(d)}\right) \mu^2(d) = \frac{h(r) F((n, r))}{F(r) h((n, r))}. \tag{3.3}$$

Remark. Since $\mu(r)$, $g(r)$, and $h(r)$ are multiplicative, it follows that $F(r)$ is also multiplicative.

Proof. Denote the right member of (3.3) by $J(n, r)$; in view of the non-vanishing of $F(r)$ and $h(r)$, $J(n, r)$ is properly defined. We verify by Lemma 3.1, and the multiplicative property of $h(r)$ and $F(r)$, that

$$J(n, r) = \frac{h(m)}{F(m)} \left(m = \frac{\gamma(r)}{\gamma(n, r)}\right).$$

Hence $J(n, r)$ is completely primitive (mod r) and we may apply Corollary 2.4. In particular, we have

$$J(n, r) = \sum_{\substack{d \mid \gamma(r) \\ (d,n)=1}} G(d), \tag{3.4}$$

where, assuming r_1 *primitive*,

$$G(r_1) = \sum_{d \mid r_1} J\left(\frac{r_1}{d}, r_1\right) \mu\left(\frac{r_1}{d}\right) = \sum_{d \mid r_1} \frac{h(d)}{F(d)} \mu\left(\frac{r_1}{d}\right). \tag{3.5}$$

Hence, by the multiplicativity of $\mu(r)$ and $F(r)$, and by Lemma 3.2,

$$\begin{aligned} G(r_1) &= \frac{\mu(r_1)}{F(r_1)} \sum_{d \mid r_1} h(d) \mu(d) F\left(\frac{r_1}{d}\right) \\ &= \frac{\mu(r_1)}{F(r_1)} \sum_{d \mid r_1} h(d) \mu(d) \sum_{D\delta = (r_1/d)} h(D) g(\delta) \mu(\delta). \end{aligned}$$

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The complete multiplicativity of $h(r)$ gives, with $Dd = E$,

$$G(r_1) = \frac{\mu(r_1)}{F(r_1)} \sum_{E \mid r_1} h(E) g\left(\frac{r_1}{E}\right) \mu\left(\frac{r_1}{E}\right) \sum_{d \mid E} \mu(d).$$

Hence by (2.1),

$$G(r_1) = \frac{\mu^2(r_1) g(r_1)}{F(r_1)}. \tag{3.6}$$

By (3.4) and (3.6) the theorem is proved.

Corollary 3.1 ($n = 1$). *Under the conditions of the Theorem,*

$$\sum_{d \mid r} \left(\frac{g(d)}{F(d)}\right) \mu^2(d) = \frac{h(r)}{F(r)} \equiv J(1, r). \tag{3.7}$$

Next we prove a generalization of the Brauer-Rademacher identity (1.4).

Theorem 3.2. *Under the conditions of Lemma* 3.2,

$$F(r) \sum_{\substack{d \mid r \\ (d,n)=1}} \frac{h(d)}{F(d)} \mu\left(\frac{r}{d}\right) = \mu(r) f(n, r). \tag{3.8}$$

Proof. Denote the left member of (3.8) by $Q(n, r)$. Let r_1 and r_2 be the uniquely determined positive integers such that $r = r_1 r_2$, $\gamma(r_2) = \gamma(n, r)$, $(r_1, r_2) = 1$. Then on the basis of Corollary 3.1 and the multiplicative property of $\mu(r)$,

$$\begin{aligned} Q(n, r) &= F(r)\mu(r_2) \sum_{d \mid r_1} \frac{h(d)}{F(d)} \mu\left(\frac{r_1}{d}\right) \\ &= F(r)\mu(r_2) \sum_{d \mid r_1} \mu\left(\frac{r_1}{d}\right) \sum_{D \mid d} \frac{g(D)\mu^2(D)}{F(D)}. \end{aligned}$$

With $d = DE$, one obtains then

$$Q(n, r) = F(r)\mu(r_2) \sum_{D \mid r_1} \frac{g(D)\mu^2(D)}{F(D)} \sum_{E \mid (r_1/D)} \mu\left(\frac{r_1/D}{E}\right),$$

so that by (2.1) and the multiplicative property of $\mu(r)$ and $F(r)$,

$$Q(n, r) = F(r_2)\mu(r)\mu(r_1)g(r_1). \tag{3.9}$$

By definition of $F(r)$ and the multiplicativity of $\mu(r)$, $g(r)$, it follows that

$$Q(n, r) = \mu(r)\mu(r_1)g(r_1) \sum_{d \mid r_2} h(d) g\left(\frac{r_2}{d}\right) \mu\left(\frac{r_2}{d}\right) = \mu(r) \sum_{d \mid r_2} h(d) g\left(\frac{r}{d}\right) \mu\left(\frac{r}{d}\right).$$

In view of the presence of the factor $\mu(r)$ and the fact that $\gamma(r_2) = \gamma(n, r)$, one obtains then

$$Q(n, r) = \mu(r) \sum_{d \mid (n, r)} h(d) g\left(\frac{r}{d}\right) \mu\left(\frac{r}{d}\right) = \mu(r) f(n, r).$$

The theorem is proved.

4. The generalized Hölder identity and a second proof of the generalized Landau identity. In the proof of the generalized Landau identity (3.3), we used as starting point the right member $J(n, r)$. This was the natural approach, relative to the application of (3.3) in proving Theorem 3.2, because it was $J(n, r)$, with $m = 1$, that arose in the proof of that theorem. We shall also use (3.3) in the proof of the generalized Hölder theorem below. However, in this proof, it is the left member of (3.3) which arises; therefore, it is proper to give another proof of the generalized Landau identity, proceeding from the left side of (3.3).

Second proof of the generalized Landau identity, Theorem 3.1. Denote the left member of (3.3) by $S(n, r)$. We obtain then by the multiplicative property of $F(r)$ and $g(r)$, with $m = \gamma(r)/\gamma(e)$, $e = (n, r)$,

$$S(n, r) = \sum_{d|m} \frac{g(d)}{F(d)} = \frac{1}{F(m)} \sum_{d|m} g(d) F\left(\frac{m}{d}\right)$$
$$= \frac{1}{F(m)} \sum_{d|m} g(d) \sum_{D|(m/d)} h(D) g\left(\frac{m/d}{D}\right) \mu\left(\frac{m/d}{D}\right)$$
$$= \frac{1}{F(m)} \sum_{D|m} h(D) g\left(\frac{m}{D}\right) \sum_{d|(m/D)} \mu\left(\frac{m/D}{d}\right).$$

Hence by (2.1) and multiplicativity,

$$S(n, r) = \frac{h(m)}{F(m)} = \frac{h(\gamma(r))F(\gamma(e))}{F(\gamma(r))h(\gamma(e))}. \tag{4.1}$$

Multiplying both numerator and denominator of the last expression in (4.1) by $h(r/\gamma(r))h(e/\gamma(e))$, one obtains, by the complete multiplicativity of $h(r)$ and by 3.2,

$$S(n, r) = \frac{h(r)F(e)}{F(r)h(e)} \qquad (e = (n, r)),$$

which is (3.3). The proof is complete.

We shall need the following lemma in the proof of the generalized Hölder identity.

LEMMA 4.1. *Under the conditions of Lemma* 3.2, *if a and b are positive integers, then*

$$F(ab) = \frac{F(a)F(b)h((a, b))}{F((a, b))}. \tag{4.2}$$

Proof. In view of the multiplicative property of the functions concerned, it suffices to verify (4.2) in case $a = p^t$, $b = p^s$, p prime, $t \geqq s > 0$. Since $h(r)$ is completely multiplicative, it follows that for $q > 0$, $F(p^q) = h^{q-1}(p)$ $(h(p) - g(p))$. Hence by Lemmas 3.1 and 3.2, one deduces, for the above values of a and b,

$$\frac{F(a)F(b)h((a,b))}{F((a,b))} = \frac{F(p^t)F(p^s)h(p^s)}{F(p^s)} = F(p^t)h^s(p)$$
$$= h^{s+t-1}(p)F(p) = F(p^{s+t}) = F(ab).$$

By multiplicativity, the lemma follows for arbitrary values of a and b.

We now prove the following generalizations of Hölder's identity (1.3).

THEOREM 4.1 (**1**, Theorem 2; **5**, Theorem 2). *If $g(r)$ and $h(r)$ satisfy the conditions of Lemma* 3.2, *then*

$$f(n,r) = \frac{F(r)g(\delta)\mu(\delta)}{F(\delta)} \qquad (\delta = r/(n,r)), \tag{4.3}$$

where $f(n,r)$ is defined by (3.1).

Proof. Denote the right member of (4.3) by $T(n,r)$. Evidently $T(n,r)$ is even (mod r). Hence by Theorem 2.1, $T(n,r)$ has the representation,

$$T(n,r) = \sum_{d\mid(n,r)} H\left(d, \frac{r}{d}\right), \tag{4.4}$$

where, with $r = r_1 r_2$,

$$H(r_1, r_2) = \sum_{d\mid r_1} T\left(\frac{r_1}{d}, r\right)\mu(d)$$
$$= F(r)\sum_{d\mid r_1} \frac{g(r_2 d)\mu(r_2 d)\mu(d)}{F(r_2 d)}.$$

But by definition of $\mu(r)$ and multiplicativity, it follows that

$$H(r_1, r_2) = \frac{F(r)g(r_2)\mu(r_2)}{F(r_2)} \sum_{\substack{d\mid r_1\\(d,r_2)=1}} \left(\frac{g(d)}{F(d)}\right)\mu^2(d).$$

Applying Theorem 3.1 one obtains

$$H(r_1, r_2) = \frac{F(r)g(r_2)\mu(r_2)}{F(r_2)}\cdot\frac{h(r_1)F((r_1,r_2))}{F(r_1)h((r_1,r_2))},$$

so that by Lemma 4.1,

$$H(r_1, r_2) = h(r_1)g(r_2)\mu(r_2). \tag{4.5}$$

The theorem follows from (4.4) and (4.5) and the definition of $f(n,r)$.

Combination of (3.8) and (4.3) yields the following result.

COROLLARY 4.1. *Under the conditions of the Theorem,*

$$\sum_{\substack{d\mid r\\(d,n)=1}} \frac{h(d)}{F(d)}\mu\left(\frac{r}{d}\right) = \frac{\mu(r)g(\delta)\mu(\delta)}{F(\delta)} \qquad \left(\delta = \frac{r}{(n,r)}\right). \tag{4.6}$$

5. A special case. In this section we illustrate the results of §§ 3 and 4 with a particular example. Let $J(r) = \phi_2(r)$ denote the Jordan totient of rank 2. We recall the following identity proved in (**5**, Corollary 24; $t = 2$, $n = 1$):

$$\sum_{d|r} \frac{\mu(d)\phi(d)}{J(d)} = \frac{r\phi(r)}{J(r)}. \tag{5.1}$$

Placing $h(r) = 1$ and $g(r) = \phi(r)/J(r)$ in (3.3), (3.8), (4.3), and (4.6), respectively, one obtains on the basis of (5.1) the following relations.

$$\sum_{\substack{d|r \\ (d,n)=1}} \frac{\mu^2(d)}{d} = \frac{J(r)}{r\phi(r)} \left(\frac{(n,r)\phi((n,r))}{J((n,r))}\right); \tag{5.2}$$

$$\frac{r\phi(r)}{J(r)} \sum_{\substack{de=r \\ (d,n)=1}} \frac{\mu(e)J(d)}{d\phi(d)} = \mu(r) \sum_{\substack{d|(n,r) \\ de=r}} \frac{\mu(e)\phi(e)}{J(e)}; \tag{5.3}$$

$$\sum_{\substack{d|(n,r) \\ de=r}} \frac{\mu(e)\phi(e)}{J(e)} = \frac{\phi(r)(n,r)}{J(r)} \mu\left(\frac{r}{(n,r)}\right); \tag{5.4}$$

$$\sum_{\substack{de=r \\ (d,n)=1}} \frac{\mu(e)J(d)}{d\phi(d)} = \frac{\mu(r)(n,r)}{r} \mu\left(\frac{r}{(n,r)}\right). \tag{5.5}$$

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