CONJUGACY CLASSES IN SYLOW p-SUBGROUPS OF GL(n,q), IV†

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In this paper we give new information about the conjugacy vector of the group $\mathfrak{G}_n$, the Sylow p-subgroup of GL(n,q) consisting of the upper unitriangular matrices. The first two components of this vector are given in [**4**]. Here, we obtain the third component, that is, the number of conjugacy classes whose centralizer has q^{n+1} elements. Besides, we give the whole set of numbers which compose this vector:

$$\{|C_{\mathfrak{G}_n}(B)| \mid B \in \mathfrak{G}_n\} = \{q^u \mid n-1 \leq u \leq n(n-1)/2\}.$$

We keep the definitions and notations of [**2–4**]. We recall that an element $a_{i,j}$ of a matrix $A \in \mathfrak{G}_n$ is a *pivot* if it is the first nonzero element in its row, out of the main diagonal, that is $a_{i,k}=0$ for $k=i+1,\ldots,j-1$ and $a_{i,j}\neq 0$. In addition we introduce the following sets of indices;

$$\mathfrak{I}=\{(u,v)\mid 1\leq u<v\leq n\} \qquad \mathfrak{D}_i=\{(r,s)\in\mathfrak{I}\mid s-r=i\}, \qquad i=1,\ldots,n-1,$$

$$\mathfrak{F}_i=\{(i,s)\mid s=i+1,\ldots,n\}, \qquad i=1,\ldots,n-1,$$

$$\mathfrak{C}_j=\{(r,j)\mid r=1,\ldots,j-1\}, \qquad j=2,\ldots,n.$$

If $A\in\mathfrak{G}_n$ is a canonical matrix, we shall use the letters $\mathfrak{T}=\mathfrak{T}(A)$ and $\mathfrak{R}=\mathfrak{R}(A)$ to denote the sets of inert and ramification points of A, respectively.

THEOREM 1. *Let r and s be positive integers such that $1\leq r\leq n-2$, $1\leq s\leq n-r-1$. Then the matrices*:

$$A_{r,s}=I_n+\sum_{i=1}^{s-1}a_{i,i+r+1}E_{i,i+r+1}+\sum_{i=s+1}^{n-r}a_{i,i+r}E_{i,i+r},$$

$$a_{i,i+r+1}\neq 0 \quad \text{if} \quad 1\leq i\leq s-1, \qquad a_{i,i+r}\neq 0 \quad \text{if} \quad s+1\leq i\leq n-r,$$

are canonical in $\mathfrak{G}_n$ and each one has exactly $rn-r(r+1)/2+s-1$ ramification points, so that, $|C_{\mathfrak{G}_n}(A_{r,s})|=q^{rn-r(r+1)/2+s-1}$. Consequently, for any u with $n-1\leq u\leq n(n-1)/2$ there exists a canonical matrix A in $\mathfrak{G}_n$ such that $|C_{\mathfrak{G}_n}(A)|=q^u$ and for any v with $0\leq v\leq (n-1)(n-2)/2$ there exists a conjugacy class in $\mathfrak{G}_n$ with q^v elements.

Proof. We note, using Lemma 3.9 of [**3**], that the entries of the matrix $A_{r,s}$ which are not placed over pivots are ramification points. This also proves that the matrix is canonical, since all the non-zero values correspond to ramification points. Moreover, according to Lemma 3.7 of [**3**], the points which are over pivots are inert, so the character of all the entries of the matrix is determined and, consequently, the order of the centralizer of this matrix is the one given in the statement. The second part of this Theorem follows immediately, considering the matrices $A_{r,s}$ together with the identity matrix.

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THEOREM 2. *Let A be a canonical matrix of $\mathfrak{G}_n$. Then $|C_{\mathfrak{G}_n}(A)| = q^{n+1}$ if and only if A is one of the following matrices.*

(1)

$$A = I_n + \sum_{l\in[2,n-1]-\{i-1,i\}} a_{l,l+1}E_{l,l+1} + a_{1,i}E_{1,i} + a_{1,i+1}E_{1,i+1} + a_{i-1,i}E_{i-1,i} + a_{i-1,i+1}E_{i-1,i+1},$$

$$i = 3, \dots, n-1,$$

where

$$(a_{l,l+1} \mid l \in [2, n-1] - \{i-1, i\}) \in \mathbb{F}_q^{*n-4}$$

and

$$(a_{1,i}, a_{1,i+1}, a_{i-1,i}, a_{i-1,i+1}) \in X_i,$$

with

$$X_3 = (\mathbb{F}_q \times \{0\} \times \{0\} \times \mathbb{F}_q^*) \cup (\{0\} \times \mathbb{F}_q \times \mathbb{F}_q^* \times \mathbb{F}_q),$$

$$X_i = (\mathbb{F}_q \times \{0\} \times \{0\} \times \mathbb{F}_q^*) \cup (\{0\} \times \mathbb{F}_q \times \mathbb{F}_q^* \times \mathbb{F}_q^*), \quad \text{if } 4 \le i \le n-3,$$

$$X_{n-2} = (\mathbb{F}_q \times \{0\} \times \{0\} \times \mathbb{F}_q^*) \cup (\{0\} \times \mathbb{F}_q \times \mathbb{F}_q^* \times \mathbb{F}_q),$$

$$X_{n-1} = (\mathbb{F}_q \times \{0\} \times \{0\} \times \mathbb{F}_q) - \{(0,0,0,0)\}.$$

(2)

$$A = I_n + \sum_{l\in[1,n-2]-\{i-1,i\}} a_{l,l+1}E_{l,l+1} + a_{i-1,i}E_{i-1,i} + a_{i-1,i+1}E_{i-1,i+1} + a_{i,n}E_{i,n},$$

$$i = 2, \dots, n-2,$$

where

$$(a_{l,l+1} \mid l \in [1, n-2] - \{i-1, i\}) \in \mathbb{F}_q^{*n-4} \quad \textit{and} \quad (a_{i-1,i}, a_{i-1,i+1}, a_{i,n}) \in Y_i$$

with

$$Y_2 = \mathbb{F}_q^3 - \{(0,0,0)\},$$

$$Y_i = \mathbb{F}_q \times \mathbb{F}_q^* \times \mathbb{F}_q, \quad \text{if } 3 \le i \le n-4,$$

$$Y_{n-3} = (\{0\} \times \mathbb{F}_q^* \times \mathbb{F}_q) \cup (\mathbb{F}_q^* \times \mathbb{F}_q \times \mathbb{F}_q),$$

$$Y_{n-2} = \{0\} \times \mathbb{F}_q^* \times \mathbb{F}_q.$$

(3)

(i) *If* $n \ge 7$,

$$A = I_n + \sum_{l\in[1,n-1]-\{i-1,i\}} a_{l,l+1}E_{l,l+1} + a_{i-1,i}E_{i-1,i} + a_{i-2,i+1}E_{i-2,i+1}, \qquad i = 3, \dots, n-2,$$

where

$$(a_{l,l+1} \mid l \in [1, n-1] - \{i-1, i\}) \in \mathbb{F}_q^{*n-3} \quad \textit{and} \quad (a_{i-1,i}a_{i-2,i+1}) \in Z_i,$$

with

$$Z_3 = \mathbb{F}_q^2,$$

$$Z_i = \mathbb{F}_q \times \mathbb{F}_q^*, \qquad \text{if } 4 \le i \le n-4,$$

$$Z_{n-3} = \mathbb{F}_q^2 - \{(0,0)\},$$

$$Z_{n-2} = \{0\} \times \mathbb{F}_q.$$

(ii) *If* $n = 6$,

$$A = I_6 + a_{1,2}E_{1,2} + a_{4,5}E_{4,5} + a_{5,6}E_{5,6} + a_{2,3}E_{2,3} + a_{1,4}E_{1,4},$$

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with

$$(a_{1,2}, a_{4,5}, a_{5,6}) \in \mathbb{F}_q^{*3}, \qquad (a_{2,3}, a_{1,4}) \in \mathbb{F}_q^2,$$

or

$$A = I_6 + a_{1,2}E_{1,2} + a_{2,3}E_{2,3} + a_{5,6}E_{5,6} + a_{2,5}E_{2,5},$$

with

$$(a_{1,2}, a_{2,3}, a_{5,6}) \in \mathbb{F}_q^{*3}, \qquad a_{2,5} \in \mathbb{F}_q.$$

So the number of conjugacy classes of $\mathfrak{G}_n$ *of cardinality* $q^{(n^2-3n-2)/2}$ *equals*
(i) $(q-1)^{n-3}((3n-15)q^2-(n-15)q+1)$, *if* $n \geq 7$,
(ii) $(q-1)^{n-3}(4q^2+8q+2)$, *if* $n=6$.

The proof of this Theorem is based on the following Lemmas.

LEMMA 1. *Let A be a canonical matrix of* $\mathfrak{G}_n$ *and suppose that* $a_{u,u+1}=0$, $a_{u+1,u+2} \neq 0$ *for some* $2 \leq u \leq n-2$. *Then* $(u-1, u+1) \in \mathfrak{R}(A)$.

LEMMA 2. *Let A be a canonical matrix of* $\mathfrak{G}_n$ *suppose that* $a_{u,u+1}=0$, $a_{v,v+1}=0$, *and* $a_{w,w+1} \neq 0$, *for* $w=u+1, \ldots, v-1$. *Then*, $(u, v+1) \in \mathfrak{R}(A)$.

LEMMA 3. *Let A be a canonical matrix of* $\mathfrak{G}_n$ *and suppose that there exist two indices i and r such that* $2 \leq i \leq n-2$, $r \leq \min\{i-1, n-(i-1)\}$ *satisfying the following conditions*: (i) $a_{u,v}=0$, *if* $u<v$, $i-r \leq u \leq i-2$, $i-r+2 \leq v \leq i$, (ii) $a_{u,u+1} \neq 0$ *if* $i+1 \leq u \leq i+r$, (iii) $a_{u,i+1}=0$, *if* $i-r+1 \leq u \leq i$. *Then*, $(i-r, i+1) \in \mathfrak{R}$.

LEMMA 4. *Let A be a canonical matrix of* $\mathfrak{G}_n$ *and u and i two indices such that* $u<i \leq n-1$, *and* $a_{k,k+1} \neq 0$ *for* $k=i+1, \ldots, n$. *Then*:
(a) *If* $u \leq 2i-n$ *and* $a_{l,l+1} \neq 0$ *for* $u \leq l \leq u+n-i-1$, *then* $(u, i+1) \in \mathfrak{T}$.
(b) *If there exists* v, *with* $u<v<i$, *such that* $a_{l,l+1} \neq 0$ *for* $u \leq l<v$, $a_{v,i+1} \neq 0$ *and* $a_{l,i+1}=0$ *for* $v<l<i$, *then* $(k, i+1) \in \mathfrak{T}$ *for* $u \leq k \leq v-1$.

The proofs of these Lemmas are based on the fact that a point (i,j) is an inert or a ramification point depending on whether the linear form $L_{i,j}$ is linearly independent or dependent on the forms which precede it. As an example, we give in detail the proof of Lemma 3.

Proof of Lemma 3. The hypotheses imply that the entries $(u, u+1)$, $i+1 \leq u \leq i+r$ are pivots, so $a_{u,v}=0$, for $i+2 \leq v \leq i+r$, $u \leq v-2$, and we obtain the linear forms:

$$\begin{aligned} L_{i-r,i+1} &= a_{i-r,i-r+1}x_{i-r+1,i+1}, \\ L_{i-r+k,i+1+k} &= a_{i-r+k,i-r+1+k}x_{i-r+1+k,i+1+k} - a_{i+k,i+1+k}x_{i-r+k,i+k}, \qquad k=1, \ldots, r-1, \\ L_{i,i+1+r} &= -a_{i+r,i+1+r}x_{i,i+r}. \end{aligned}$$

Let Σ be the set of these $r+1$ linear forms in the unknowns $x_{i-r+k,i+k}$, $k=1, \ldots, r$, and Σ_1 the set obtained from Σ by eliminating the form $L_{i-r,i+1}$.

If we arrange the variables in the order

$$x_{i-1,i+r-1}, x_{i-2,i+r-2}, \ldots, x_{i-r+1,i+1}, x_{i,i+r},$$

the matrix defined by the coefficients of the forms in Σ_1 is

$$D = \begin{pmatrix} -a_{i+1,i+2} & a_{i-r+1,i-r+2} & 0 & \cdots & 0 & 0 \\ 0 & -a_{i+2,i+3} & a_{i-r+2,i-r+3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{i+r-1,i+r} & a_{i-1,i} \\ 0 & 0 & 0 & \cdots & 0 & -a_{i+r,i+r+1} \end{pmatrix}.$$

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All the elements in the main diagonal of this matrix are non-zero, so its rank is r and it follows that $L_{i-r,i+1}$ is a linear combination of the forms in Σ_1.

Proof of Theorem 2. We study the different possibilities for the configuration of the matrix A depending on the number of zeroes in the first diagonal.

Apart from the $n-1$ entries of the first diagonal, there must be exactly two additional ramification points, so the Lemma of [**4**] and Lemmas 1 and 2 yield the following conditions:

(a) There are at most three zeroes in $\mathfrak{D}_1$.

(b) If $\mathfrak{D}_1$ has exactly three zeroes, then one of them is one of the extremes of $\mathfrak{D}_1$ and the other two occupy consecutive positions.

(c) If $\mathfrak{D}_1$ has exactly two zeroes and they are not consecutive, then each one occupies one of the extremes of $\mathfrak{D}_1$.

On the other hand, if the number of additional ramification points is less than two, we apply Theorem 1.3 of [**4**] to conclude the following conditions:

(d) If there are exactly two zeroes in $\mathfrak{D}_1$ and they are consecutive, then they are not in the extremes of $\mathfrak{D}_1$.

(e) If $\mathfrak{D}_1$ has exactly one zero which is in the entry $(i, i+1)$, then $2 \le i \le n-2$ and $a_{i-1,i+1} = 0$.

Indeed, if $a_{1,2} = 0$ or $a_{n-1,n} = 0$, it would follow from Theorem 1.2 of [**4**] that the centralizer of the matrix has cardinality q^{n-1}, which is impossible.

The equality $a_{i-1,i+1} = 0$ follows from the Theorem 1.3 of [**4**], since, otherwise the order of the centralizer of A would be q^n.

(f) There is at least one zero in $\mathfrak{D}_1$.

So the proof is finally reduced to the following three mutually exclusive cases:

(C_1) $a_{12} = 0$ and there are one or two more zeroes in $\mathfrak{D}_1$. In the latter case, these zeroes are contiguous (the case $a_{12} = a_{23} = a_{34} = 0$ is included).

(C_2) $a_{n-1,n} = 0$ and there are one or two more zeroes in $\mathfrak{D}_1$. In the latter case, these zeroes are contiguous (the case $a_{n-3,n-2} = a_{n-2,n-1} = a_{n-1,n} = 0$ is included).

(C_3) The first diagonal has two contiguous zeroes or a single zero, which are not in the extremes of $\mathfrak{D}_1$ and the rest of the entries are non-zero.

(C_1) In this case there exists an integer i with $3 \le i \le n-1$, such that

$$a_{1,2} = 0, \qquad a_{i,i+1} = 0, \qquad a_{l,l+1} \ne 0 \quad \text{for} \quad l \in [2, n-1] - \{i-1, i\}.$$

We note that $a_{i-1,i}$ may have any possible value since there can be one or two contiguous zeroes in $\mathfrak{D}_1$. In these conditions, the elements $a_{l,l+1} \ne 0$, $l \in [2, n-1] - \{i-1, i\}$ are pivots and so:

$$\mathfrak{C}_{l+1} - \mathfrak{D}_1 \subseteq \mathfrak{T}, \qquad l \in [2, n-1] - \{i-1, i\}.$$

From this and the fact that $a_{i,i+1} = 0$ it follows that all elements in the ith row are zero (except for the entry in this row which lies in the main diagonal) and so, according to Lemma 1.1 of [**4**], the entries of the ith column which are preceded by pivots are inert points, that is:

$$\mathfrak{C}_i - \{(1, i), (i-1, i)\} \subseteq \mathfrak{T}.$$

Besides,

$$(i-1, i+1) \in \mathfrak{R}.$$

This follows from Lemma 1 if $i < n-1$, since $a_{i,i+1} = 0$, $a_{i+1,i+2} \neq 0$, and from Lemma 3.9 of [4] if $i = n-1$, since, in this case $a_{n-2,n-1} = 0$ by (c).

We set $i_0 = i$ or $i+1$ according to $a_{i-1,i} = 0$ or $a_{i-1,i} \neq 0$ respectively. Then we have from Lemma 2 that $(1, i_0) \in \mathfrak{R}$. Thus, we have proved that:

$$\mathfrak{D}_1 \cup \{(1, i_0), (i-1, i+1)\} \subseteq \mathfrak{R} \subseteq \mathfrak{D}_1 \cup \{(1, i), (1, i+1), (i-1, i+1)\} \cup \mathfrak{C}_{i+1}. \quad (1)$$

So, the equality $|\mathfrak{R}| = n+1$ holds if and only if the first inclusion in (1) is an equality. In any case, a necessary condition is that $\mathfrak{C}_{i+1} - \mathfrak{D}_1 - \mathfrak{D}_2 - \mathfrak{F}_1 \subseteq \mathfrak{T}$.

We consider the following four cases according to the value of i:

$i = 3$. In this case we have $a_{1,2} = 0 = a_{3,4}$. If $a_{2,3} = 0$, then $a_{2,4} \neq 0$, since, otherwise arguing as in b1) we would have the contradiction $(1, 4) \in \mathfrak{R}$. So $(2, 4)$ is a pivot and $(1, 4)$ is an inert point. Now it follows from (1) that $\mathfrak{R} = \mathfrak{D}_1 \cup \{(1, 3), (2, 4)\}$ and thus the canonical matrices corresponding to the subset $\mathbb{F}_q \times \{0\} \times \{0\} \times \mathbb{F}_q^*$ of X_3 have centralizers of order q^{n+1}. If $a_{2,3} \neq 0$, then $(1, 3)$ is an inert point because it is over a pivot and so $\mathfrak{R} = \mathfrak{D}_1 \cup \{(1, 4), (2, 4)\}$, for any value of $a_{2,4}$. The canonical matrices obtained in this way are considered in the subset $\{0\} \times \mathbb{F}_q \times \mathbb{F}_q^* \times \mathbb{F}_q$ of X_3.

$4 \leq i \leq n-3$. As we have pointed out before, $(i-1, i+1) \in \mathfrak{R}$. We note that $a_{i-1,i+1} \neq 0$. Indeed, if $a_{i-1,i+1} = 0$, then Lemma 3 (with $i = i$, $r = 2$) implies that $(i-2, i+1) \in \mathfrak{R}$ and $|C_G(A)| \geq q^{n+2}$, which is impossible. So $a_{i-1,i+1} \neq 0$. If $a_{i-1,i} = 0$, then $a_{i-1,i+1} \neq 0$ is a pivot and $\mathfrak{C}_i - \mathfrak{D}_1 - \mathfrak{D}_2 \subseteq \mathfrak{T}$ so that:

$$\mathfrak{R} = \mathfrak{D}_1 \cup \{(1, i), (i-1, i+1)\}, \qquad |C_G(A)| = q^{n+1}.$$

This justifies the subset $\mathbb{F}_q \times \{0\} \times \{0\} \times \mathbb{F}_q^*$ which is contained in X_i. If $a_{i-1,i} \neq 0$, then $(i-1, i)$ is a pivot and Lemma 4(b) (with $u = 2, v = i-1$) yields $(k, i+1) \in \mathfrak{T}$ for any $2 \leq k \leq i-2$, so

$$\mathfrak{R} = \mathfrak{D}_1 \cup \{(1, i+1), (i-1, i+1)\}, \qquad |C_G(A)| = q^{n+1},$$

and this completes the set X_i.

$i = n-2$. Suppose that $a_{n-3,n-2} = 0$. Then $a_{n-3,n-1} \neq 0$, since otherwise the linear form $L_{n-4,n-1} = a_{n-4,n-3} x_{n-3,n-1}$ would be linearly dependent on the form $L_{n-3,n} = -a_{n-1,n} x_{n-3,n-1}$ (note that $a_{n-1,n} \neq 0$) and consequently, $(n-4, n-1) \in \mathfrak{R}$ and $|C_G(A)| \geq q^{n+2}$, which is impossible. So we have:

$$\mathfrak{R} = \mathfrak{D}_1 \cup \{(1, n-2), (n-3, n-1)\}, \qquad |C_G(A)| = q^{n+1}$$

and we obtain the first subset of X_{n-2}. Suppose now that $a_{n-3,n-2} \neq 0$. Then it follows from Lemma 4(a) (with $2 \leq u \leq 2(n-2) - n = n-4$) that $(u, i+1) \in \mathfrak{T}$ and for any value of $a_{n-3,n-1}$, we have that:

$$\mathfrak{R} = \mathfrak{D}_1 \cup \{(1, n-1), (n-3, n-1)\}, \qquad |C_G(A)| = q^{n+1},$$

which completes the set X_{n-2}.

$i = n-1$. The case $a_{n-2,n-1} \neq 0$ is ruled out by (c). We have from the Lemma given in [4] that the points (u, n), $2 \leq u \leq n-3$ are inert.

If $a_{1,n-1} = 0 = a_{n-2,n}$, then $(1, n) \in \mathfrak{R}$ and $|C_G(A)| \geq q^{n+2}$ and we obtain a contradiction. If $a_{1,n-1} \neq 0$, then $(1, n) \in \mathfrak{T}$ (since it is preceded by the pivot of its row). If $a_{n-2,n} \neq 0$, $(1, n) \in \mathfrak{T}$ because it is over the pivot of its column. So we have

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$(a_{1,n-1}, a_{n-2,n}) \in \mathbb{F}_q^2 - \{(0,0)\}$, and

$$\mathfrak{R} = \mathfrak{D}_1 \cup \{(1, n-1), (n-2, n)\}, \qquad |C_G(A)| = q^{n+1}.$$

This defines the set X_{n-1}.

The discussion of the cases (C_2) and (C_3) is analogous.

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