# 1

# Examples

In this introductory chapter, we list a number of concrete examples of $\tau$-functions, noting the elements they have in common, but postponing a formal definition to subsequent chapters.

The first case is the simplest nonlinear periodic Hamiltonian system with one degree of freedom: the pendulum. In the Hamilton–Jacobi approach, Hamilton's characteristic function, evaluated on the energy level sets, is the logarithmic derivative of the Weierstrass $\sigma$-function. This is our first example of a $\tau$-function. The equations of motion are expressible as a bilinear equation for the $\sigma$-function, providing the first instance of an equation of Hirota type.

Turning to nonlinear integrable evolution equations that are PDE's in one spatial and one time dimension, such as the KdV equation, the simplest reduction is to travelling wave solutions with constant velocity. These again satisfy a Weierstrass-type equation, just like the pendulum. The separatrix, where the $\tau$-function is simply a hyperbolic cosine, corresponds to 1-soliton solutions. The $\tau$-functions corresponding to multisoliton solutions are expressed as the determinant of a matrix whose entries are linear exponential functions of the flow variables, which again satisfies a bilinear system of Hirota type. Multisoliton solutions of the more general integrable KP (Kadomtsev–Petviashvili) hierarchy are similarly given in terms of $\tau$-functions having determinantal exponential form, also satisfying the Hirota bilinear equations.

We next consider the basic building blocks from which all KP $\tau$-functions are constructed: the Schur functions, which are polynomials in the flow parameters, whose logarithmic derivatives provide rational solutions of the hierarchy. Other examples include the Toda lattice, an integrable multiparticle system on the line with exponential nearest-neighbour interactions and the Calogero–Moser system, another integrable multiparticle system on the line whose dynamics coincide with the pole dynamics of rational solutions of the KP hierarchy. The "ultimate" generalization of the pendulum then follows: the so-called *finite gap* or multi-quasi-periodic solutions of the KP hierarchy, where the $\tau$-function is

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simply expressible in term of multivariable Riemann $\theta$ functions associated to the period lattice of an algebraic curve of arbitrary genus.

Further specific examples of KP $\tau$-functions are provided by the partition functions for various types of random matrix models. These include cases where the matrix integrals do not necessarily converge, nor does the expansion in the basis of Schur functions. They may, however, be viewed as formal expansions that serve as generating functions for various combinatorial invariants such as: intersection indices on the moduli space of marked Riemann surfaces, or Hurwitz numbers, which enumerate branched covers of the Riemann sphere.

The characteristic features shared by all these examples are listed at the end of the chapter. A preliminary interpretation of these is given in Chapter 2, in terms of abelian group actions on a Grassmann manifold. This anticipates the Sato–Segal–Wilson approach to KP $\tau$-functions, whose detailed development begins in Chapters 3 and 4 and continues throughout the remainder of the book.

## 1.1 The pendulum and the KdV equation: elliptic function solutions

### *1.1.1 The pendulum*

Consider the motion of a simple pendulum, consisting of a point mass $m$ suspended on a massless rigid rod of length $L$, subject to the force of gravity (Fig. 1.1). The Lagrangian of the system, expressed in terms of the angle $\phi$ from the vertical, is the difference between kinetic and potential energies:

$$L(\phi, \dot{\phi}) = \frac{1}{2} m L^2 \dot{\phi}^2 - 2mgL \sin^2 \frac{\phi}{2}, \tag{1.1.1}$$

where $g$ is the acceleration due to gravity.

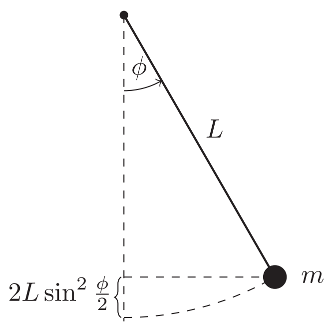


Fig. 1.1. The pendulum

The total energy, which is conserved, is the sum

$$E = \frac{1}{2} m L^2 \dot{\phi}^2 + 2mgL \sin^2 \frac{\phi}{2}. \tag{1.1.2}$$

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Introducing the coordinate

$$q =: \sin\frac{\phi}{2}, \tag{1.1.3}$$

we have

$$\dot{q} = \frac{1}{2}\cos\frac{\phi}{2}\dot{\phi} = \frac{1}{2}\sqrt{1-q^2}\,\dot{\phi}, \tag{1.1.4}$$

and the Lagrangian takes the form

$$L(q,\dot{q}) = 2mL^2\frac{\dot{q}^2}{1-q^2} - 2mgLq^2. \tag{1.1.5}$$

The momentum conjugate to $q$ is

$$p := \frac{\partial L}{\partial \dot{q}} = 4mL^2\frac{\dot{q}}{1-q^2}, \tag{1.1.6}$$

and the Legendre transformation gives the Hamiltonian as the sum of kinetic and potential energies:

$$H(q,p) = \frac{1}{8mL^2}(1-q^2)p^2 + 2mgLq^2. \tag{1.1.7}$$

Since the system is autonomous, the total energy is constant

$$\frac{1}{8mL^2}(1-q^2)p^2 + 2mgLq^2 \ = E \tag{1.1.8}$$

and a first integral of the equations of motion is given by its level curves. Substituting (1.1.6), this can be integrated directly, giving $q(t)$ implicitly in terms of an elliptic integral.

It is worthwhile, however, to also consider the problem using the Hamilton–Jacobi method. For this, we define a new momentum variable $P$, which is a constant of motion, by

$$2mgLP^2 := H(q,p), \tag{1.1.9}$$

and seek a generating function $S(q,P)$, Hamilton's principal function, for the transformation from $(q,p)$ to new canonical coordinates $(Q,P)$ in which the equations of motion are trivial. The transformation is defined by

$$p = \frac{\partial S}{\partial q}, \quad Q = \frac{\partial S}{\partial P}, \tag{1.1.10}$$

and the Hamilton–Jacobi equation is

$$H\left(q, \frac{\partial S}{\partial q}\right) = E \tag{1.1.11}$$

or, more explicitly,

$$\left(\frac{\partial S}{\partial q}\right)^2 = 16m^2gL^3\frac{P^2-q^2}{1-q^2}. \tag{1.1.12}$$

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The solution is given by an elliptic integral of the second kind:

$$S(q, P) = 4m\sqrt{gL^3} \int_{q_0}^{q} \sqrt{\frac{P^2 - x^2}{1 - x^2}} dx, \tag{1.1.13}$$

where the constant of integration is absorbed into the choice of initial point $q_0$. The coordinate canonically conjugate to $P$ is thus given by an elliptic integral of the first kind

$$Q = 4m\sqrt{gL^3} P \int_{q_0}^{q} \frac{dx}{\sqrt{(1 - x^2)(P^2 - x^2)}}, \tag{1.1.14}$$

defined on the curve

$$z^2 = (1 - x^2)(P^2 - x^2). \tag{1.1.15}$$

In the canonical coordinates $(Q, P)$, the equations of motion have the trivial form

$$\frac{dP}{dt} = -\frac{\partial H}{\partial Q} = 0, \tag{1.1.16}$$

$$\frac{dQ}{dt} = \frac{\partial H}{\partial P} = 4mgLP, \tag{1.1.17}$$

which, when integrated, give a linear flow in time

$$Q(t) = Q_0 + 4mgLPt, \quad P(t) = P_0. \tag{1.1.18}$$

Viewing Hamilton's characteristic function $S(q, P)$ as a function of time, evaluated on the energy level sets, we have

$$S(q(t), P) = 4mL\sqrt{gL} \int_{q(0)}^{q(t)} \sqrt{\frac{P^2 - x^2}{1 - x^2}} dx, \tag{1.1.19}$$

Changing the integration variable in eq. (1.1.13) from $x$ to $y := x^2$ gives

$$\int_{v_0}^{v(t)} \frac{dy}{\sqrt{y(y-1)(y-e)}} = 2\sqrt{\frac{g}{L}} t, \tag{1.1.20}$$

where

$$v(t) := q^2(t), \quad v_0 := v(0), \quad e := P^2 \tag{1.1.21}$$

and

$$S(q(t), P) = 2mL\sqrt{gL} \int_{v_0}^{v(t)} \sqrt{\frac{e - y}{y(1 - y)}} dy. \tag{1.1.22}$$

Introducing the rescaled, translated function

$$u(t) = v\left(\sqrt{\frac{L}{g}} t\right) - \frac{e + 1}{3}, \tag{1.1.23}$$

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the inverse of the elliptic integral in (1.1.20) becomes a first order differential equation in standard Weierstrass form:

$$(u')^2 = 4u^3 - g_2 u - g_3, \tag{1.1.24}$$

with coefficients

$$g_2 = \frac{4}{3}(e^2 - e + 1), \qquad g_3 = \frac{4}{27}(e+1)(e-2)(2e-1). \tag{1.1.25}$$

The general solution to (1.1.24) is given by the *Weierstrass* $\wp$*-function*

$$u(t) = \wp(t - t_0) \tag{1.1.26}$$

for these parameter values, and any initial value constant $t_0 \in \mathbf{C}$.

In general, $\wp$ is defined by

$$\wp(z) := \frac{1}{z^2} + \sum_{w \in \mathbf{L}\setminus\{0\}} \left[ \frac{1}{(z-w)^2} - \frac{1}{w^2} \right], \tag{1.1.27}$$

where the sum is over the integer lattice $\mathbf{L}$ in the complex plane

$$\mathbf{L} = \{2m\omega_1 + 2n\omega_2 \; : \; m, n \in \mathbf{Z}\} \tag{1.1.28}$$

generated by any non-collinear pair of elliptic periods $(2\omega_1, 2\omega_2 \in \mathbf{C}^+)$. This satisfies the Weierstrass equation [280]

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{1.1.29}$$

for modular constants $(g_2, g_3)$ determined from the lattice periods by the Eisenstein series

$$g_2 = 60 \sum_{w \in \mathbf{L}\setminus\{0\}} \frac{1}{w^4}, \qquad g_3 = 140 \sum_{w \in \mathbf{L}\setminus\{0\}} \frac{1}{w^6}. \tag{1.1.30}$$

It can also be expressed as a second logarithmic derivative:

$$\wp(z) = -\frac{d^2}{dz^2} \ln \sigma(z) \tag{1.1.31}$$

in terms of the Weierstrass $\sigma$-function

$$\sigma(z) := z \prod_{w \in \mathbf{L}\setminus\{0\}} \left(1 - \frac{z}{w}\right) \exp\left(\frac{z}{w} + \frac{z^2}{2w^2}\right). \tag{1.1.32}$$

Taking the first derivative of (1.1.29) gives

$$\wp''(t) = 6\wp^2(t) - \frac{g_2}{2}. \tag{1.1.33}$$

Substituting (1.1.31) in (1.1.33) gives the equation of motion in bilinear form in terms of $\sigma$

$$\sigma\sigma'''' - 4\sigma'\sigma''' + 3(\sigma'')^2 - \frac{g_2}{2}\sigma^2 = 0, \tag{1.1.34}$$

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where $' := \frac{d}{dt}$. Equivalently, (1.1.34) may be expressed in a more symmetrical way [74] as

$$(\Delta^4 - g_2)\,(\sigma(t-t_0)\sigma(t'-t_0))\,|_{t=t'} = 0, \tag{1.1.35}$$

where

$$\Delta := \frac{d}{dt} - \frac{d}{dt'}. \tag{1.1.36}$$

Differentiating Hamilton's characteristic function (1.1.19) with respect to $t$ gives

$$\frac{\partial S(q(t),P)}{\partial t} = 4mgL^2\wp(t-t_0) + \mathcal{E} \tag{1.1.37}$$

where

$$\mathcal{E} := \frac{4}{3}(mgL^2 - E). \tag{1.1.38}$$

So, within an integration constant, we have

$$S(q(t),P) = -\frac{\partial(\ln\sigma)}{\partial t} + \mathcal{E}t. \tag{1.1.39}$$

**Remark 1.1.1.** *The logarithmic derivative formula (1.1.31) expressing the general solution $u(t)$ in terms of the Weierstrass $\sigma$-function will reappear in subsequent examples, as will the bilinear form (1.1.35) of the equation it satisfies. This is the first example of a $\tau$-function generating the solution of an integrable nonlinear equation. It is seen here as closely related to Hamilton's characteristic function $S(q(t),P)$; i.e., the complete solution of the Hamilton–Jacobi equation evaluated on the level sets of the conserved quantities.*

### *1.1.2 Travelling wave solutions of the KdV equation*

The Weierstrass $\wp$-function also appears in another context relating to integrable systems: travelling wave solutions of the nonlinear partial differential equation

$$4u_t = 6uu_x + u_{xxx}, \tag{1.1.40}$$

known as the *Korteweg–de Vries* (KdV) equation, which describes nondissipative shallow water waves in a narrow channel*. Choosing $u(x,t)$ to have the form of a travelling wave

$$u(x,t) = U(x+ct), \tag{1.1.41}$$

* The renewed study of the KdV equation, started in the mid 1960's, led to the discovery of solitons, the inverse scattering method [95–98] and the subsequent flood of interest in completely integrable systems with infinite degrees of freedom.

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where $U$ is a function of a single variable $z := x + ct$, and $c$ is the velocity, the KdV equation reduces to the ODE

$$4cU' = 6UU' + U'''. \tag{1.1.42}$$

Integration and multiplication by $U'$ gives

$$4cUU' = 3U^2U' + U''U' + \alpha U', \tag{1.1.43}$$

where $\alpha$ is an integration constant, which can again be integrated to give the first order equation

$$2cU^2 = U^3 + \frac{1}{2}\left(U'\right)^2 + \alpha U + \beta, \tag{1.1.44}$$

where $\beta$ is a second constant of integration. This can now be reduced to the Weierstrass standard form by the substitution

$$U(z) = -2\wp(z + z_0) + \frac{2c}{3}, \tag{1.1.45}$$

where $z_0 \in \mathbf{C}$ is an arbitrary constant and the modular forms $(g_2, g_3)$ determining $\wp(z)$ are

$$g_2 = \frac{4c^2}{3} - \frac{\alpha}{2} \quad \text{and} \quad g_3 = -\frac{8c^3}{27} + \frac{c\alpha}{6} + \frac{\beta}{4}. \tag{1.1.46}$$

The formula

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\left[e^{\frac{c}{6}x^2}\sigma(x + ct + z_0)\right] \tag{1.1.47}$$

thus gives the general travelling wave solution to the KdV equation, a simple example of an elliptic function solution to a nonlinear evolution equation.

The function

$$\begin{aligned}\tau(x,t) &:= e^{\frac{c}{6}x^2}\sigma(x + ct + z_0) && (1.1.48)\\ &= Ke^{\frac{c}{6}x^2}e^{\frac{\eta_1}{2\omega_1}(x+ct+z_0)^2}\theta\left(\frac{x + ct + z_0}{2\omega_1} + \frac{1}{2} + \frac{\omega_2}{2\omega_1}; \frac{\omega_2}{\omega_1}\right) && (1.1.49)\end{aligned}$$

where $\theta(z;\tau)$ is the Jacobi $\theta$ function

$$\theta(z;\tau) := \sum_{n\in\mathbf{Z}} e^{\pi i\tau n^2 + 2\pi izn}, \qquad \tau := \frac{\omega_2}{\omega_1}, \tag{1.1.50}$$

$K$ is a nonzero constant and

$$\eta_1 := \frac{\sigma'(\omega_1)}{\sigma(\omega_1)} \tag{1.1.51}$$

is another example of a $\tau$-function that, in this case, generates the elliptic function solution of the KdV equation representing generic travelling waves for this case.

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### *1.1.3 Degeneration to the trigonometric/hyperbolic case: the separatrix*

In terms of the pendulum, the elliptic integral (1.1.20) degenerates to a trigonometric one at the critical energy

$$E_{crit} = 2mgL, \tag{1.1.52}$$

and therefore the solution, either for the pendulum or the travelling wave of the KdV equation, can be written in terms of elementary trigonometric/hyperbolic functions. The corresponding solution of the pendulum problem is known as the *separatrix*, i.e., the special level curve $E = E_{crit}$ of the energy on the phase space of the pendulum in the coordinates $(\phi, \dot{\phi})$. (See Fig. 1.2, where the separatrix $E = E_{crit}$ is indicated.)

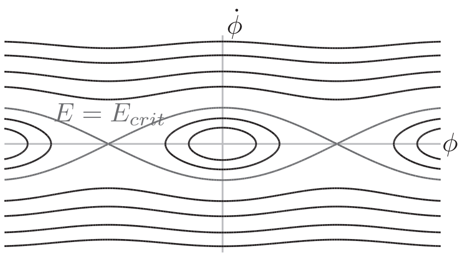


Fig. 1.2. Level curves of the energy $E$ of a pendulum in the $(\phi, \dot{\phi})$ plane

Similarly, if both integration constants $\alpha$ and $\beta$ in (1.1.44) are chosen to be zero, the discriminant for the Weierstrass equation vanishes:

$$\Delta = g_2^3 - 27g_3^2 = 0, \tag{1.1.53}$$

and the general solution to (1.1.44) can be obtained by using elementary hyperbolic functions:

$$U(z) = 2c\,\mathrm{sech}^2(\sqrt{c}(z + z_0)). \tag{1.1.54}$$

Since $U(z)$ can be written as a second logarithmic derivative

$$U(z) = 2\frac{d^2}{dz^2}\ln\cosh(\sqrt{c}(z + z_0)), \tag{1.1.55}$$

the corresponding solution to the KdV equation can be represented as

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\left[\cosh(\sqrt{c}(x + ct + z_0))\right], \tag{1.1.56}$$

which is known as the *one-soliton solution* to the KdV equation, associated to the simple exponential type of $\tau$-function

$$\tau(x,t) = \cosh(\sqrt{c}(x + ct + z_0)) = \frac{1}{2}\left(e^{\sqrt{c}(x+ct+z_0)} + e^{-\sqrt{c}(x+ct+z_0)}\right). \tag{1.1.57}$$

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## 1.2 Multisoliton solutions of KdV and KP

For a given positive integer $N$, choose $2N$ complex numbers

$$\{\alpha_k\}_{k=1,\dots,N} \text{ and } \{\gamma_k\}_{k=1,\dots,N} \tag{1.2.1}$$

with all $\alpha_k$'s pairwise distinct and all $\gamma_k$'s nonzero. Define $N$ functions

$$y_k(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i \alpha_k^i} + \gamma_k e^{\sum_{i=1}^{\infty} t_i (-\alpha_k)^i}, \quad k = 1, \dots, N, \tag{1.2.2}$$

where $\mathbf{t}$ is an infinite sequence of variables

$$\mathbf{t} = (t_1, t_2, \dots), \tag{1.2.3}$$

referred to as the higher KdV flow variables or *times*. Note that

$$y_k^{(l)}(\mathbf{t}) := \frac{\partial^l}{\partial t_1^l} y_k(\mathbf{t}) = (\alpha_k)^l \left[ e^{\sum_{i=1}^{\infty} t_i \alpha_k^i} + (-1)^l \gamma_k e^{\sum_{i=1}^{\infty} t_i (-\alpha_k)^i} \right] \tag{1.2.4}$$

$$= 2(\alpha_k)^l \gamma_k^{1/2} e^{\sum_{i=1}^{\infty} t_{2i} \alpha_k^{2i}} \begin{cases} \cosh\left( \sum_{i=0}^{\infty} t_{2i+1} \alpha_k^{2i+1} - \frac{1}{2} \log \gamma_k \right) & l \text{ even} \\ \sinh\left( \sum_{i=0}^{\infty} t_{2i+1} \alpha_k^{2i+1} - \frac{1}{2} \log \gamma_k \right) & l \text{ odd.} \end{cases} \tag{1.2.5}$$

Now define the $\tau$-function $\tau^{(N)}_{\alpha_1,\dots,\alpha_N,\gamma_1,\dots,\gamma_N}(\mathbf{t})$ as the Wronskian determinant

$$\tau^{(N)}_{\alpha_1,\dots,\alpha_N,\gamma_1,\dots,\gamma_N}(\mathbf{t}) := \begin{vmatrix} y_1(\mathbf{t}) & y_2(\mathbf{t}) & \cdots & y_N(\mathbf{t}) \\ y_1'(\mathbf{t}) & y_2'(\mathbf{t}) & \cdots & y_N'(\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(N-1)}(\mathbf{t}) & y_2^{(N-1)}(\mathbf{t}) & \cdots & y_N^{(N-1)}(\mathbf{t}) \end{vmatrix} \tag{1.2.6}$$

$$= e^{\sum_{i=1}^{\infty} \sum_{k=1}^{N} \alpha_k^{2i} t_{2i}} \begin{vmatrix} y_1(\mathbf{t}_0) & y_2(\mathbf{t}_0) & \cdots & y_N(\mathbf{t}_0) \\ y_1'(\mathbf{t}_0) & y_2'(\mathbf{t}_0) & \cdots & y_N'(\mathbf{t}_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(N-1)}(\mathbf{t}_0) & y_2^{(N-1)}(\mathbf{t}_0) & \cdots & y_N^{(N-1)}(\mathbf{t}_0) \end{vmatrix},$$

where $\mathbf{t}_0 := (t_1, 0, t_3, 0, \dots)$, and the derivatives $\{{y_i}' \dots {y_i}^{(N-1)}\}$ are taken with respect to $x = t_1$. The function $\tau^{(N)}_{\alpha_1,\dots,\alpha_N,\gamma_1,\dots,\gamma_N}(\mathbf{t})$ has the remarkable property that twice its second logarithmic derivative, evaluated at the parameter values $(t_1 = x, t_2 = 0, t_3 = t,\ t_i = 0,\ i > 3)$

$$u(x,t) := 2\frac{\partial^2}{\partial x^2} \log \tau^{(N)}_{\alpha_1,\dots,\alpha_N,\gamma_1,\dots,\gamma_N}(x, 0, t, 0, 0 \dots) \tag{1.2.7}$$

satisfies the KdV equation (1.1.40). Solutions of this form are called *standard $N$-soliton solutions* to the KdV equation.

More generally, if we choose $3N$ complex constants

$$\{\alpha_k, \beta_k, \gamma_k\}_{k=1,\dots,N} \tag{1.2.8}$$

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with $\alpha_k, \beta_k$'s all distinct, $\gamma_k \neq 0$, and define the functions

$$y_k(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i \alpha_k^i} + \gamma_k e^{\sum_{i=1}^{\infty} t_i \beta_k^i}, \quad k = 1, \dots, N, \tag{1.2.9}$$

we arrive at the more general Wronskian determinant

$$\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(\mathbf{t}) := \begin{vmatrix} y_1(\mathbf{t}) & y_2(\mathbf{t}) & \cdots & y_N(\mathbf{t}) \\ y_1'(\mathbf{t}) & y_2'(\mathbf{t}) & \cdots & y_N'(\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(N-1)}(\mathbf{t}) & y_2^{(N-1)}(\mathbf{t}) & \cdots & y_N^{(N-1)}(\mathbf{t}) \end{vmatrix}. \tag{1.2.10}$$

The function

$$u(x, y, t) := 2\frac{\partial^2}{\partial x^2} \log\left(\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(x, y, t, t_4, \dots)\right) \tag{1.2.11}$$

can then be shown to satisfy the $2+1$ dimensional nonlinear partial differential equation

$$3u_{yy} = (4u_t - 6uu_x - u_{xxx})_x\,, \tag{1.2.12}$$

known as the *Kadomtsev–Petviashvili* (KP) *equation* (which plays a prominent rôle in plasma physics and in the study of shallow water ocean waves), together with an infinite set of further nonlinear autonomous PDEs, each involving partial derivatives of finite order with respect to a finite number of the KP flow parameters $\mathbf{t} = (t_1, t_2, \dots)$. These are collectively known as the *KP hierarchy.* They may all be deduced from a single family of bilinear relations known as the Hirota bilinear equations, (see Section 1.10 below), satisfied by the $\tau$-function $\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(\mathbf{t})$, and by all solutions of the KP hierarchy. Solutions of the form (1.2.10) are referred to as *standard $N$-soliton solutions* of the KP hierarchy in Wronskian form.

If $\beta_j = -\alpha_j$ for all $j$, the standard KP-solitons are independent of $y = t_2$ and all further even flow parameters $\{t_{2i}\}$ and reduce to KdV $N$-solitons, since

$$e^{x\alpha_k + y\alpha_k^2 + t\alpha_k^3} + \gamma_k e^{-x\alpha_k + y\alpha_k^2 - t\alpha_k^3} = e^{y\alpha_k^2}\left[e^{x\alpha_k + t\alpha_k^3} + \gamma_k e^{-x\alpha_k - t\alpha_k^3}\right], \tag{1.2.13}$$

and the second logarithmic derivative in $x$ eliminates the $y$-dependence.

The Wronskian formula (1.2.10) for the $\tau$-function $\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(\mathbf{t})$ can also be rewritten in a more general determinantal form [102, 104] (detailed in Section 6.1) as

$$\tau^{(N)}_{\vec{\alpha},\vec{\beta},\vec{\gamma}}(\mathbf{t}) = \det(A e^{\sum_{i=1}^{\infty} t_i B^i} C^T), \tag{1.2.14}$$

where, for (1.2.10), $A$ is the $N \times 2N$ double Vandermonde-type matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_N & \beta_1 & \beta_2 & \cdots & \beta_N \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{N-1} & \alpha_2^{N-1} & \cdots & \alpha_N^{N-1} & \beta_1^{N-1} & \beta_2^{N-1} & \cdots & \beta_N^{N-1} \end{bmatrix}, \tag{1.2.15}$$

$B$ is the diagonal $2N \times 2N$ matrix

$$B = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N, \beta_1, \beta_2, \ldots, \beta_N) \tag{1.2.16}$$

and $C$ is also $N \times 2N$, with the special form

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 & \gamma_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & \gamma_N \end{bmatrix}. \tag{1.2.17}$$

General $N$-soliton solutions of the KP hierarchy are of the form

$$\tau^{(N)}_{A,B,C}(\mathbf{t}) = \det(Ae^{\sum_{i=1}^{\infty} t_i B^i} C^T), \tag{1.2.18}$$

where $A$ and $B$ are as above, but $C$ can be an arbitrary $N \times 2N$ complex matrix of full rank. Such solutions were introduced and systematically studied by Matveev, Zakharov and others [102, 104, 142, 290, 293]. The necessary and sufficient conditions that they be nonsingular for arbitrary real values of the flow parameters $(t_1, t_2, \cdots)$ were derived by Kodama and Williams [46, 170–173].

## 1.3 Schur functions

Recall that, for a set of indeterminates $\mathbf{x} = (x_1, \ldots, x_n)$, the $n \times n$ *Vandermonde matrix* $V(x_1, \ldots, x_n)$ is defined as

$$V_{ij}(x_1, \ldots, x_n) := x_j^{n-i}, \quad 1 \le i, j \le n. \tag{1.3.1}$$

Its determinant, denoted

$$\Delta(x_1, \ldots, x_n) := \det(x_i^{n-j}) = \prod_{i<j}(x_i - x_j), \tag{1.3.2}$$

is called the *Vandermonde determinant*.

Consider an integer partition (see Appendix A)

$$\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}), \quad \lambda_1 \ge \cdots \ge \lambda_{\ell(\lambda)}, \quad \lambda_i \in \mathbf{N}^+ \tag{1.3.3}$$

of length $\ell(\lambda)$ and weight $|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i$. The Young diagram of $\lambda$ consists of $\ell(\lambda)$ vertically stacked, left-aligned rows of $\lambda_i$ square boxes, $i = 1, \ldots, \ell(\lambda)$, weakly descending downward. Figure 1.3 gives an illustration for the partition $\lambda = (3, 1)$.

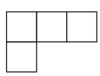

Fig. 1.3. Young diagram of the partition $(3, 1)$

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In the following, we use the notation $[x]$ for the special infinite sequence defined by

$$[x] := \left(x, \frac{x^2}{2}, \frac{x^3}{3}, \dots\right). \tag{1.3.4}$$

More generally, for a set of $n$ indeterminates $\mathbf{x} = (x_1, x_2, \dots, x_n)$ we associate the following infinite sequence:

$$[\mathbf{x}] = \sum_{k=1}^{n}[x_k] = \left(\sum_{k=1}^{n} x_k, \frac{1}{2}\sum_{k=1}^{n} x_k^2, \frac{1}{3}\sum_{k=1}^{n} x_k^3, \dots\right), \tag{1.3.5}$$

which are, up to a normalization factor, the power sum symmetric functions (Appendix D, eq. (D.1.20))

$$t_i = p_i/i, \quad p_i = \sum_{k=1}^{n} x_k^n. \tag{1.3.6}$$

The simplest definition of the *Schur polynomial* $s_\lambda([\mathbf{x}])$ in the indeterminates $(x_1, \dots, x_n)$ associated to a partition $\lambda$ is given by Jacobi's *bialternant formula*

$$s_\lambda([\mathbf{x}]) = \frac{\det_{1\le i,j\le n}(x_j^{\lambda_i - i + n})}{\det_{1\le i,j\le n}(x_j^{n-i})}. \tag{1.3.7}$$

(See Appendices A and D for notation and definitions regarding partitions and symmetric functions.) Due to cancellations of zeros, this is actually a homogeneous symmetric polynomial in $\mathbf{x}$ of degree $|\lambda|$ (see Appendix D.1).

Alternatively, viewing the Schur functions $s_\lambda(\mathbf{t})$ as functions of the infinite sequence of parameters $\mathbf{t} = (t_1, t_2, \dots)$ defined in (1.3.6), they can be expressed by the *Jacobi-Trudi determinant formula* (see Appendix D.1.2, eq. (D.1.42)).

$$s_\lambda(\mathbf{t}) = \det_{1\le i,j\le \ell(\lambda)}(h_{\lambda_i - i + j}(\mathbf{t})), \tag{1.3.8}$$

where $h_j(\mathbf{t}) := s_{(j)}(\mathbf{t})$ are the complete symmetric functions, which are graded polynomials in $(t_1, t_2, \dots)$ of weight $j$, with $t_i$ assigned the weight $i$. These are obtained from the generating function formula

$$\prod_{a=1}^{n}(1 - x_a z)^{-1} = e^{\sum_{k=1}^{\infty} t_k z^k} = \sum_{j=0}^{\infty} h_j(\mathbf{t}) z^j. \tag{1.3.9}$$

The bialternant formula (1.3.7) is equivalent to the Jacobi–Trudi formula (1.3.8) if we set $\mathbf{t} = [\mathbf{x}]$.

Schur functions, viewed as depending on an infinite set of unconstrained flow parameters $\mathbf{t} = (t_1, t_2, \dots)$, are all $\tau$-functions of the KP hierarchy. (See Section 1.10 below and Sections 3.2 and 3.6 for full definitions and details.) In particular, the function

$$u^{(\lambda)}(x, y, t) := 2\frac{\partial^2}{\partial x^2}\ln\left(s_\lambda(x, y, t, 0, 0, \dots)\right) \tag{1.3.10}$$

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is a rational solution of (1.2.12). For example, choosing $\lambda$ to be a *staircase partition* (see Figure 1.4), i.e., of the form

$$\lambda = (m, m-1, \ldots, 2, 1), \tag{1.3.11}$$

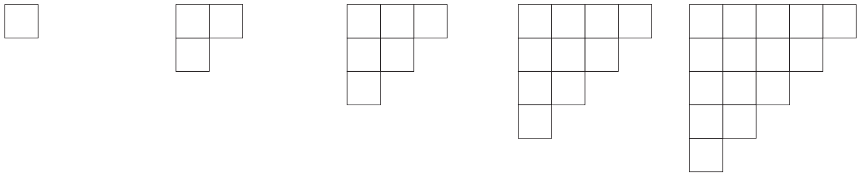

Fig. 1.4. Young diagrams of the staircase partitions $n = 1, 2, 3, 4, 5$

and setting $y = 0$, the rational function

$$u_m(x, t) := u^{(m,m-1,\ldots,2,1)}(x, 0, t) \tag{1.3.12}$$

satisfies the KdV equation (1.1.40). It is well-known [13, 16] that these are the most general solutions of the KdV hierarchy that are rational in $x$ for all values of the parameters $\mathbf{t}$ and vanish as $x \to \infty$.

## 1.4 Rational solutions of the KP equation and the Calogero–Moser system

There exist further solutions of the KP equation (1.2.12) that are rational in $x$ and vanish as $|x| \to \infty$ for all values of the variables $y$ and $t$. The partial fraction decomposition of such functions can be shown to have a rather special form [16, 179, 180, 282].

**Exercise 1.1.** Show that if $u(x, y, t)$ is a solution of (1.2.12) that is rational in the variable $x$ and vanishes as $|x| \to \infty$, it must be of the form

$$u(x, y, t) = -2 \sum_{j=1}^{n} \frac{1}{(x - x_j(y, t))^2}, \tag{1.4.1}$$

for some positive integer $n$. That is, $u$ may only have second order poles in $x$, and each double pole has the same constant coefficient $-2$.

The evolution of such solutions of the KP equation is therefore reduced to the analysis of the motion of the poles in terms of the variables $y$ and $t$. It was shown by Krichever [179, 180] that the dependence of the poles $\{x_k(y, t)\}_{k=1}^{n}$ on the

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$y$ variable is determined by Hamilton's equations for the *rational Calogero–Moser many-body system* [43, 203], governed by the Hamiltonian

$$H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{i<j} \frac{1}{(q_i - q_j)^2}, \tag{1.4.2}$$

where

$$q_k := -ix_k, \; k = 1, \dots, n, \tag{1.4.3}$$

$\{p_k\}_{k=1}^n$ are the momenta canonically conjugate to the position variables $\{q_k\}_{k=1}^n$ and $y$ is the time variable.

This is a completely integrable Hamiltonian system on the $2n$-dimensional phase space $\mathbf{C}^n \oplus \mathbf{C}^{*n}$ with symplectic form

$$\omega = \sum_{j=1}^{n} dq_i \wedge dp_i \tag{1.4.4}$$

since it possesses $n$ functionally independent Poisson commuting invariants. Calogero [43] and Moser [203] showed that the equations of motion can be interpreted as *isospectral deformations* of the $n \times n$ Hermitian matrix

$$L := \begin{bmatrix} p_1 & \frac{i}{q_1 - q_2} & \cdots & \frac{i}{q_1 - q_n} \\ \frac{i}{q_2 - q_1} & p_2 & \cdots & \frac{i}{q_2 - q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{i}{q_n - q_1} & \frac{i}{q_n - q_2} & \cdots & p_n \end{bmatrix} \tag{1.4.5}$$

i.e., deformations that preserve the eigenvalues of $L$. The family of Poisson commuting invariants $\{H_1, \dots, H_n\}$ can be chosen as the trace invariants

$$H_j := (-1)^j \mathrm{tr}(L^j), \quad j = 1, \dots, n. \tag{1.4.6}$$

In fact, extending this definition to all positive integer values $j \in \mathbf{N}^+$, these all Poisson commute (although they are not functionally independent). Their Hamiltonian flows therefore generate an infinite abelian group action

$$\begin{aligned} f(\mathbf{t}) : \mathbf{C}^n \oplus \mathbf{C}^{*n} &\to \mathbf{C}^n \oplus \mathbf{C}^{*n} \\ f(\mathbf{t}) : (q_1, \cdots q_n, p_1, \dots, p_n) &\mapsto (q_1(\mathbf{t}), \dots, q_n(\mathbf{t}), p_1(\mathbf{t}), \dots, p_n(\mathbf{t})), \\ f(\mathbf{0}) &= \mathrm{Id}, \quad f(\mathbf{t}) \cdot f(\tilde{\mathbf{t}}) = f(\mathbf{t} + \tilde{\mathbf{t}}) \end{aligned} \tag{1.4.7}$$

on the phase space, where $\mathbf{t} = (t_1, t_2, \dots)$ are the commuting time flow parameters corresponding to the Hamiltonians $(H_1, H_2, \dots)$.

The flow associated to $H_1$ corresponds to simultaneous translation of all the $x_k$'s. The first nontrivial Hamiltonian $H_2$ coincides with $-2H$ in (1.4.2). The third Hamiltonian $H_3 = \mathrm{tr}(-L^3)$ governs the $t$-dependence of the pole locations of $u(x, y, t)$. The higher Hamiltonians of the Calogero–Moser system can be shown to correspond to the higher KP flows [254], which effectively embeds the simultaneous solutions of finite dimensional Calogero–Moser system into the infinite dimensional KP hierarchy.

The $\tau$-function associated to rational solutions of the KP equation corresponding to solutions of the Calogero–Moser system can be derived most naturally in the context of the symmetry reduction construction of Kazhdan, Kostant and Sternberg [166]. This was further extended to complex solutions of KP by Wilson [282], who complexified the phase space to include the cases of colliding particles.

Consider pairs of complex $n \times n$ matrices $X$ and $Z$ such that the rank-1 condition

$$\operatorname{rank}(XZ - ZX - \mathbf{I}_n) = 1 \tag{1.4.8}$$

holds, where $\mathbf{I}_n$ is the $n \times n$ identity matrix. For any pair of such matrices, there exists a pair of nonzero column vectors $v, w \in \mathbf{C}^n$ such that

$$XZ - ZX - \mathbf{I}_n = vw^t. \tag{1.4.9}$$

Let $W_n$ denote the set of quadruples $(X, Z, v, w)$

$$W_n = \{(X, Z, v, w) \in \mathfrak{gl}(n, \mathbf{C}) \times \mathfrak{gl}(n, \mathbf{C}) \times \mathbf{C}^n \times \mathbf{C}^n\}, \tag{1.4.10}$$

viewed as a complex $2n(n+1)$-dimensional phase space with symplectic structure

$$\omega = \operatorname{tr}(idX \wedge dZ + dv \wedge dw^t), \tag{1.4.11}$$

and let $\bar{C}_n \subset W_n$ denote the submanifold consisting of $\{(X, Z, v, w)\}$ satisfying (1.4.9)

$$\bar{C}_n = \{(X, Z, v, w) \in \mathfrak{gl}(n, \mathbf{C}) \times \mathfrak{gl}(n, \mathbf{C}) \times \mathbf{C}^n \times \mathbf{C}^n \,:\, XZ - ZX - \mathbf{I}_n = vw^t\}. \tag{1.4.12}$$

This is actually the condition

$$J = \mathbf{I} \tag{1.4.13}$$

for the moment map $J : W_n \to \mathfrak{gl}^*(n, \mathbf{C})$ defined by

$$J = XZ - ZX - vw^t, \tag{1.4.14}$$

which generates the Hamiltonian action

$$\begin{aligned} \mathrm{GL}(n, \mathbf{C}) \times W_n &\to W_n \\ (g, (X, Z, v, w)) &\mapsto (gXg^{-1}, gZg^{-1}, gv, (g^t)^{-1}w), \quad g \in \mathrm{GL}(n, \mathbf{C}). \end{aligned} \tag{1.4.15}$$

The Poisson commuting Hamiltonians $\{H_k\}_{k=1,\dots,n}$ defined in (1.4.6) are invariant under this action. Therefore, the moment map (1.4.14) is constant under their flows, and we may pass to the quotient space $C_n = \bar{C}_n/\mathrm{GL}(n, \mathbf{C})$.

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If the matrix $X$ in a quadruple $(X, Z, v, w) \in \bar{C}_n$ is diagonalizable, it can be shown [166,282] that $X$ has distinct eigenvalues and the equivalence class defined by the $\mathrm{GL}(n, \mathbf{C})$ orbits has a representative of the form $(iD, L, e, -e)$, where

$$D := \mathrm{diag}(q_1, \cdots, q_n), \quad e := \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{1.4.16}$$

and $L$ is of the form (1.4.5). Letting $g \in \mathrm{GL}(n, \mathbf{C})$ be the unique element that diagonalizes $L$ and stabilizes $e$:

$$L = g^{-1} Z_0 g, \quad ge = e, \tag{1.4.17}$$

where

$$Z_0 := \mathrm{diag}(\beta_1, \ldots, \beta_n), \tag{1.4.18}$$

and defining

$$X := igDg^{-1}, \tag{1.4.19}$$

it follows (cf. Chapter 6, Section 6.2.4) that the integrated flow can be expressed as

$$X(\mathbf{t}) = X_0 - \sum_{k=1}^{\infty} k t_k (Z_0)^{k-1}. \tag{1.4.20}$$

The $\tau$-function corresponding to the pair $(X_0, Z_0)$ is defined to be [282]

$$\tau_{(X,Z)}(\mathbf{t}) = \det \left( -X_0 + \sum_{k=1}^{\infty} k t_k Z_0^{k-1} \right), \tag{1.4.21}$$

where the matrix argument is given by the integrated Calogero–Moser flows. Identifying $x \equiv t_1$ and applying the formula

$$u(\mathbf{t}) = 2 \frac{\partial^2 \ln(\tau(\mathbf{t}))}{\partial x^2} \tag{1.4.22}$$

then gives the rational solution (1.4.1) of the KP hierarchy. In particular, the corresponding rational solution of the KP equation is

$$u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \left( \det \left( X + xZ - yZ^2 + tZ^3 \right) \right). \tag{1.4.23}$$

Note that, similarly to the soliton solutions of KdV and KP discussed in Section 1.2, the $\tau$-function $\tau_{(X,Z)}(\mathbf{t})$ in (1.4.21) can be written in the equivalent form [102, 104]

$$\tau_{(X,Z)}(\mathbf{t}) = \det(A_{X_0} e^{\sum_{i=1}^{\infty} t_i B^i} C^T), \tag{1.4.24}$$

with the $N \times 2N$ dimensional matrices $A_{X_0}$ and $C$ defined as

$$A_{X_0} = \left( \mathbf{I}_n, \quad -X_0 - \mathbf{I}_n \right), \quad C = \left( \mathbf{I}_n, \quad \mathbf{I}_n \right), \tag{1.4.25}$$

where $X_0$ is any matrix $X_0 \in \mathrm{Mat}^{n\times n}$ that satisfies

$$[X_0, Z_0] = \mathbf{I}_n - ee^t, \tag{1.4.26}$$

and

$$B_{Z_0} = \begin{pmatrix} Z_0 & \mathbf{I} \\ 0 & Z_0 \end{pmatrix}. \tag{1.4.27}$$

## 1.5 KP $\tau$-functions associated to algebraic curves

Let $X$ be a compact Riemann surface of genus $g$ and fix a canonical homology basis

$$a_1, \dots, a_g, b_1, \dots, b_g \tag{1.5.1}$$

of $H_1(X, \mathbf{Z})$ with intersection numbers

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}, \quad 1 \le i, j \le g. \tag{1.5.2}$$

Let $\{\omega_i\}_{i=1,\dots,g}$ be a basis for the space $H^1(X)$ of holomorphic differentials satisfying the standard normalization conditions

$$\oint_{a_i} \omega_j = \delta_{ij}, \quad \oint_{b_j} \omega_j = B_{ij}, \tag{1.5.3}$$

where $B$ is the *Riemann matrix* of periods. The matrix $B$ belongs to the *Siegel upper half space* (see Appendix F and [256]),

$$\mathbf{S}_g = \left\{B \in \mathrm{Mat}_{g\times g}(\mathbf{C}) \;:\; B^T = B,\ \Im(B) \text{ is positive definite}\right\}. \tag{1.5.4}$$

The *Riemann $\theta$ function* on $\mathbf{C}^g$ corresponding to the period matrix $B$ is defined to be

$$\theta(Z|B) := \sum_{N\in\mathbf{Z}^g} e^{i\pi(N,BN)+2i\pi(N,Z)}. \tag{1.5.5}$$

Choose a point $p_\infty \in X$, a local parameter $\zeta$ in a neighbourhood of $p_\infty$ with $\zeta(p_\infty) = 0$ and a positive divisor of degree $g$

$$\mathcal{D} := \sum_{i=1}^{g} p_i, \quad p_i \in X. \tag{1.5.6}$$

For any positive integer $k \in \mathbf{N}^+$ let $\Omega_k$ be the unique meromorphic differential of the second kind characterized by the following conditions:

(a) the only singularity of $\Omega_k$ is a pole of order $k+1$ at $p = p_\infty$ with vanishing residue,
(b) the expansion of $\Omega_k$ around $p = p_\infty$ is of the form

$$\Omega_k = d(\zeta^{-k}) + \sum_{j=1}^{\infty} Q_{ij}\zeta^j d\zeta, \tag{1.5.7}$$

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(c) $\Omega_k$ is normalized to have vanishing $a$-cycles:

$$\oint_{a_i} \Omega_j = 0. \tag{1.5.8}$$

Denote by $\mathbf{U}_k \in \mathbf{C}^g$ the vector of $b$-cycles of $\Omega_k$:

$$(\mathbf{U}_k)_j := \oint_{b_j} \Omega_k. \tag{1.5.9}$$

Denote the image of $\mathcal{D}$ under the Abel map $\mathcal{A}: \ S^g(X) \to \mathbf{C}^g$

$$\mathbf{E} := \mathcal{A}(\mathcal{D}) \in \mathbf{C}^g, \quad \mathbf{E}_j = \mathcal{A}_j(\mathcal{D}) := \sum_{j=1}^{g} \int_{p_0}^{p_i} \omega_j \tag{1.5.10}$$

with arbitrary base point $p_0$.

Define the $\tau$-function associated to the data $(X, \mathcal{D}, p_\infty, \zeta)$ as

$$\tau_{(X,\mathcal{D},p_\infty,\zeta)}(\mathbf{t}) := e^{-\frac{1}{2}\sum_{ij} Q_{ij} t_i t_j} \theta \left( \mathbf{E} + \sum_{k=1}^{\infty} t_k \mathbf{U}_k \middle| B \right). \tag{1.5.11}$$

As was shown by Krichever [178], following earlier work of Its and Matveev [141,142] on quasi-periodic solutions of the KdV equation, the second logarithmic derivative of $\tau_{(X,\mathcal{D},p_\infty,\zeta)}(\mathbf{t})$ with respect to $t_1 = x$ is a solution of the KP equation (1.2.12). Moreover, $\tau_{(X,\mathcal{D},p_\infty,\zeta)}(\mathbf{t})$ satisfies the infinite sequence of bilinear Hirota equations of the KP hierarchy.

This property of $\theta$ functions associated to algebraic curves is also the key to solving the *Schottky problem*: characterize the subset of the Siegel upper half space $\mathbf{S}_g$ formed by the period matrices associated to compact algebraic curves of genus $g$ over $\mathbf{C}$. Novikov conjectured and Mulase [205] and Shiota [253] proved that if a $\theta$-function $\theta(Z|B)$ for a matrix $B$ in the Siegel upper half space gives a solution to the KP hierarchy with a suitably chosen quadratic form $Q$, there exists a compact algebraic curve of genus $g$ whose period matrix is $B$.

## 1.6 Matrix model integrals

Let $d\mu_0(M)$ be the Lebesgue measure on the $N^2$ dimensional space $\mathbf{H}^{N\times N}$ of $N \times N$ complex Hermitian matrices. Let $\rho(M)$ be a conjugation invariant integrable density function

$$\rho(UMU^\dagger) = \rho(M), \quad U \in U(N). \tag{1.6.1}$$

Define a deformation family of measures

$$d\mu_{N,\rho}(\mathbf{t}) := e^{\mathrm{Tr}(\sum_{i=1}^{\infty} t_i M^i)} \rho(M) d\mu_0(M) \tag{1.6.2}$$

for small $\mathbf{t} = (t_1, t_2, \cdots)$ and let

$$\tau_{N,\rho}(\mathbf{t}) := \int_{\mathbf{H}^{N\times N}} d\mu_{N,\rho}(\mathbf{t}). \tag{1.6.3}$$

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Then $\tau_{N,\rho}(\mathbf{t})$ also satisfies the infinite hierarchy of bilinear Hirota equations of the KP hierarchy [167].

## 1.7 The Toda lattice: bilinear equations and multisoliton solutions

The Toda lattice [268] is a chain of interacting point particles on the real line with exponential nearest neighbour interactions. Denoting the particle positions $\{q_n\}_{n\in\mathbf{Z}}$ and the momenta $\{p_n\}_{n\in\mathbf{Z}}$, the equations of motion are

$$\frac{d^2 q_n}{dt^2} = \ddot{q}_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}. \tag{1.7.1}$$

This is a Hamiltonian system generated by the Hamiltonian

$$H_2 = \sum_{m\in\mathbf{Z}} \left( \frac{p_n^2}{2} + e^{q_n - q_{n-1}} \right) \tag{1.7.2}$$

with respect to the canonical Poisson bracket structure

$$\{f(\mathbf{q},\mathbf{p}),\ g(\mathbf{q},\mathbf{p})\} := \sum_i \left( \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right). \tag{1.7.3}$$

It may be viewed either as a doubly infinite lattice, with $\infty < n < \infty$, a semi-infinite one with $0 \le n < \infty$, a finite one $1 \le n \le N$, for some positive integer $N \in \mathbf{N}^+$ or a periodic one, with $q_{i+N} = q_i$. In all cases, there is a Lax equation representation [80] of the equations of motion (1.7.1) as infinitesimal isospectral deformations

$$\frac{dL}{dt} = [A, L]. \tag{1.7.4}$$

The Lax pair $(L, A)$ consists either of doubly or singly infinite matrices, or finite $N \times N$ matrices, with $L$ symmetric and $A$ skew symmetric, each of tridiagonal (Jacobi matrix) type, with $3 \times 3$ blocks along the principal diagonal of the following form

$$L_3^{(n)} = \begin{pmatrix} b_{n-1} & a_{n-1} & 0 \\ a_{n-1} & b_n & a_n \\ 0 & a_n & b_{n+1} \end{pmatrix}, \tag{1.7.5}$$

$$A_3^{(n)} = \frac{1}{2}\begin{pmatrix} 0 & a_{n-1} & 0 \\ -a_{n-1} & 0 & a_n \\ 0 & -a_n & 0 \end{pmatrix}, \tag{1.7.6}$$

where

$$a_n := \frac{1}{2} e^{\frac{q_{n+1}-q_n}{2}}, \quad b_n := \frac{1}{2} p_n. \tag{1.7.7}$$

(For the periodic case, there is an equivalent Lax pair consisting of $N \times N$ matrices $(L(z), A(z))$ depending in a simple way [14,15] on an auxiliary spectral

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parameter $z$.) In terms of the coordinates $\{a_n, b_n\}$, the equations of motion (1.7.1) are

$$\dot{a}_n = a_n(b_{n+1} - b_n), \quad \dot{b}_n = 2(a_n^2 - a_{n-1}^2). \tag{1.7.8}$$

These different variants of the Toda lattice, as well as its generalizations for all simple Lie algebras, have been studied in great detail [14,15,80,166,177] as typical models of completely integrable Hamiltonian systems having Lax representations as isospectral flows. From the infinitesimal form (1.7.4) of the equations of motion, it follows that the spectral invariants of $L$

$$H_i = \frac{1}{i}\mathrm{tr}\left(L^i\right), \quad i \in \mathbf{N}^+ \tag{1.7.9}$$

are conserved quantities which, moreover, can be verified to Poisson commute:

$$\{H_i,\ H_j\} = 0. \tag{1.7.10}$$

For the finite case, only the first $N$ of these are independent; for the $N$-periodic case these are replaced by $\operatorname*{res}_{z=0}\left(\mathrm{tr}L^i(z)\right)$; for the doubly infinite one, the positive powers in the trace invariants may be supplemented by negative ones.

The general solutions for the finite [166, 177] and periodic [14, 15] cases are explicitly known, and special classes of solutions (multisolitons, rational, etc.) are known for the infinite and semi-infinite ones [134, 208]. The $\tau$-function formulation of the equations of motion involves a lattice of $\tau$-functions, denoted $\{\tau_n\}_{n\in\mathbf{Z}}$. These are only projectively defined, such that their ratio is

$$\frac{\tau_{n+1}}{\tau_n} = e^{q_{n+1}}. \tag{1.7.11}$$

The coordinates $a_n$ are given in terms of these by

$$a_n = \frac{\sqrt{\tau_{n+1}\tau_{n-1}}}{\tau_n}, \tag{1.7.12}$$

where the positive square root is understood. The equations of motion (1.7.1) and (1.7.8) are easily seen to be equivalent to the following system of (Hirota) bilinear equations for the lattice of $\tau$-functions $\{\tau_n\}_{n\in\mathbf{Z}}$

$$\ddot{\tau}_n\tau_n - (\dot{\tau}_n)^2 - \tau_{n+1}\tau_{n-1} + \tau_n^2 = 0, \tag{1.7.13}$$

with similar systems for the Hamiltonian equations generated by all the commuting invariants $\{H_i\}$.

For the infinite Toda lattice, as in the case of the KP hierarchy, there exists a $3N$-parameter family of exponential type (multisoliton) solutions of (1.7.13), defined for each positive integer $N$, and set of $3N$ complex numbers $\{\alpha_i, \beta_i, \gamma_i\}_{i=1,\dots,N}$, in the form of an $N \times N$ Wronskian determinant [208]

$$\tau_n = \det\left(\phi_i(n + j - i)\right)|_{1\le i,j\le N}, \tag{1.7.14}$$

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where for successive values of $n$ the functions $\phi_i(n)$ are time derivatives of those to the left

$$\phi_i(n+1) = \dot{\phi}_i(n) \tag{1.7.15}$$

and satisfy the linear constant coefficient homogeneous system

$$\ddot{\phi}_i(n) - 2(\cosh \alpha_i)\dot{\phi}_i(n) + \phi_i(n) = 0. \tag{1.7.16}$$

The explicit general solution of eqs. (1.7.15) and (1.7.16) is

$$\phi_i(n) = \beta_i e^{\alpha_i n} e^{e^{\alpha_i t}} + \gamma_i e^{-\alpha_i n} e^{e^{-\alpha_i t}}. \tag{1.7.17}$$

## 1.8 Generating function for intersection indices (Kontsevich integral)

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne–Mumford compactification of the moduli space of compact Riemann surfaces $X$ of genus $g$, with $n$ marked points $\{p_i\}_{i=1,\dots,n}$. Let $\pi_i : \mathcal{L}_i \to \overline{\mathcal{M}}_{g,N}$ be the line bundle whose fibre at $X \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space at $p_i$:

$$\pi_i^{-1}(X) = T^* X_{p_i} \quad i = 1, \dots, n, \tag{1.8.1}$$

and let $c_1(\mathcal{L}_i)$ denote the first Chern class of $\mathcal{L}_i$.

The *intersection indices* on the moduli space $\overline{\mathcal{M}}_{g,n}$ are defined by integrals of products of Chern classes, labelled as

$$\langle \tau_{i_1} \cdots \tau_{i_m} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^{n} c_1(\mathcal{L}_j)^{d_j}, \tag{1.8.2}$$

where $i_k$ is the number of times $k$ appears in the sequence $(d_1, \dots, d_n)$. The above integral vanishes unless

$$\sum_{j=1}^{n} d_j = \dim\left(\overline{\mathcal{M}}_{g,n}\right) = 3g - 3 + n. \tag{1.8.3}$$

These intersection indices appeared in the context of two-dimensional gravity, along with their exponential generating function, referred to as the *Kontsevich–Witten generating function* [174, 283], defined as

$$F_{KW}(\mathbf{t}) = \left\langle e^{\sum_i t_i \tau_i} \right\rangle = \sum_{m \in \mathbf{N}} \sum_{i_1, \dots, i_m} \langle \tau_{i_1} \cdots \tau_{i_m} \rangle \frac{t_1^{i_1} \cdots t_m^{i_m}}{i_1! \cdots i_m!}. \tag{1.8.4}$$

The exponentiated generating function

$$\tau_{KW}(\mathbf{t}) := e^{F_{KW}(\mathbf{t})} \tag{1.8.5}$$

is called the *Kontsevich–Witten $\tau$-function*, which can be obtained as a certain stable limit (in the sense of [149, 174]) of the formal asymptotic expansions of a sequence of Hermitian matrix integrals.

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For each $n \in \mathbf{N}_+$ define

$$Z_n(\Lambda) := \frac{\int_{M \in \mathbf{H}^{n\times n}} e^{\frac{i}{6}\mathrm{Tr}(M^3)} e^{-\frac{1}{2}M\Lambda M} d\mu_0(M)}{\int_{M \in \mathbf{H}^{n\times n}} e^{-\frac{1}{2}M\Lambda M} d\mu_0(M)}, \tag{1.8.6}$$

where $\Lambda$ is an $n \times n$ diagonal "external coupling" matrix and $d\mu_0(M)$ is the Lebesgue measure on the space of $n \times n$ Hermitian matrices. In terms of the traces of the inverse powers of $\Lambda$

$$s_i := \mathrm{tr}(\Lambda^{-i}), \qquad i = 1, 2, \dots . \tag{1.8.7}$$

the terms of the formal asymptotic expansion of the integral $Z_n(\Lambda)$ can be expressed as polynomials with rational coefficients in the variables $s_i$:

$$Z_n(\Lambda) \sim Z_n(\mathbf{s}) \quad \text{as} \quad \mathbf{s} \to \mathbf{0}. \tag{1.8.8}$$

The asymptotic expansions $\{Z_n(\mathbf{s})\}_n$ can be shown to have a stable limit

$$Z(\mathbf{s}) := \lim_{n\to\infty} Z_n(\mathbf{s}), \tag{1.8.9}$$

in the sense that the lower order terms in the asymptotic expansion stabilize to that of a fixed asymptotic series as $n$ gets sufficiently large. It was shown by Kontsevich [174] (see also [149]) that the stable limit $Z(\mathbf{s})$ does not depend on the even variables $s_{2i}$, and if written in terms of the re-normalized and re-labelled odd variables

$$t_i := -(2i-1)!!s_{2i-1} = -(2i-1)!!\mathrm{Tr}(\Lambda^{-2i+1}), \tag{1.8.10}$$

it coincides with Witten's generating function

$$\tau_{KW}(\mathbf{t}) = Z\left(-\frac{t_1}{1!!}, 0, -\frac{t_2}{2!!}, 0, \dots\right) \tag{1.8.11}$$

and satisfies the KdV hierarchy in the variables $\mathbf{t}$. For details see Section 13.2 in Chapter 13.

## 1.9 Generating function for simple Hurwitz numbers

A special family of KP $\tau$-functions was shown by Pandharipande [230] and Okounkov [218] to serve as generating functions for simple Hurwitz numbers $H^d_{\exp}(\mu)$. The latter may be defined as the number of ways in which an element $h_\mu$ of the symmetric group $\mathcal{S}_n$ in the conjugacy class $\mathrm{cyc}(\mu)$ having cycle lengths equal to the parts of the partition $\mu$ of weight $|\mu| = n$ can be factored into a product of $d$ 2-cycles

$$h_\mu = h_1 \cdots h_d \tag{1.9.1}$$

(divided by $n!$). Alternatively, this may be interpreted as the number of $n$-sheeted branched coverings $\mathcal{C} \to \mathbf{P}^1$ of the Riemann sphere having $d+1$ branch points, one with specified ramification profile given by the partition $\mu$, while the other $d$ have simple branching (i.e., ramification profiles $(2,(1)^2)$ ), divided by the order $|\operatorname{aut}(\mathcal{C})|$ of the automorphism group of the covering.

A parametric family of KP $\tau$-functions, which depend on a pair of additional expansion parameters $(\gamma, \beta)$, may be defined by their Schur function expansion

$$\tau^{(\exp,\beta)}(\mathbf{t}) = \sum_{n=0}^{\infty} \gamma^n \sum_{\lambda, |\lambda|=n} \frac{d_\lambda}{n!} e^{\frac{\beta}{2}\sum_{i=1}^{\ell(\lambda)}(\lambda_i - 2i\lambda_i + 1)\lambda_i} s_\lambda(\mathbf{t}), \tag{1.9.2}$$

where $\{t_i = \frac{p_i}{i}\}$ are normalized power sum symmetric functions (as in eq. (1.3.6) above), which are viewed as KP flow parameters. Here $\beta$ is viewed as a (small) expansion parameter whose power is equal to the number of simple branch points $d$ and $d_\lambda$ is the dimension of the irreducible representation of $\mathcal{S}_n$ corresponding to the Young symmetry type $\lambda$. When re-expressed as a sum over the basis $\{p_\mu\}$ of power sum symmetric functions,

$$p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}, \tag{1.9.3}$$

with the $p_j$'s defined as in (1.3.6), this gives

$$\tau^{(\exp,\beta)}(\mathbf{t}) = \sum_{n=0}^{\infty} \gamma^n \sum_{\mu, \mu=n} \frac{\beta^d}{d!} H^d_{\exp}(\mu) p_\mu(\mathbf{t}), \tag{1.9.4}$$

and hence is a generating function for simple Hurwitz numbers $H^d_{\exp}(\mu)$ when expanded in the power sum basis.

## 1.10 Common features of the examples

The examples above share a number of common features:

(a) They all have finite or infinite dimensional determinantal representations.
(b) They all involve an Abelian group action with a maximal number (finite or infinite) of commuting flows.
(c) In all cases, the dynamics may be expressed in the form of bilinear, constant coefficient differential or difference equations for the $\tau$-function, or a lattice of functions.

For the examples involving $\tau$-functions of the KP hierarchy, we may express these in a uniform manner by defining an auxiliary function, the *Baker function* $\Psi^+(z, \mathbf{t})$ and its *dual Baker function* $\Psi^-(z, \mathbf{t})$ as follows:

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$$\Psi^{\pm}(z,\mathbf{t}) = e^{\pm\sum_{i=1}^{\infty} t_i z^i} \frac{\tau(\mathbf{t} \mp [z^{-1}])}{\tau(\mathbf{t})} \tag{1.10.1}$$

$$[z^{-1}] := \left(\frac{1}{z}, \frac{1}{2z^2}, \dots\right).$$

The equations of the hierarchy are then determined by the following vanishing formal residue relations, known as the Hirota bilinear residue relations (derived in detail in Section 3.5 below)

$$\operatorname*{res}_{z=0} \left(dz \Psi^{+}(z,\mathbf{t})\Psi^{-}(z,\mathbf{t}+\mathbf{s})\right) = 0, \tag{1.10.2}$$

which hold identically in the shifted flow parameters $\mathbf{s} := (s_1, s_2, \dots)$. Alternatively, by choosing four complex parameters $z_1, z_2, z_3$ and $z_4$ and defining

$$\eta_{ij}(\mathbf{t}) = (z_i - z_j)\tau(\mathbf{t} - [z_i^{-1}] - [z_j^{-1}]) \qquad \eta_{ij}(\mathbf{t}) = -\eta_{ji}(\mathbf{t}) \tag{1.10.3}$$

the residue relations (1.10.2) are equivalent to

$$\eta_{12}(\mathbf{t})\eta_{34}(\mathbf{t}) + \eta_{23}(\mathbf{t})\eta_{14}(\mathbf{t}) - \eta_{13}(\mathbf{t})\eta_{24}(\mathbf{t}) = 0, \tag{1.10.4}$$

which hold identically in $\mathbf{t}$. Through eq. (1.10.1), this is equivalent to the $\tau$-function satisfying an infinite set of constant coefficient partial differential relations, known as the *Hirota equations* (detailed in Sections 3.7 and 3.10 below), each of which involves bilinear combinations of directional derivatives in a finite number of KP flow parameters $\mathbf{t} = (t_1, t_2, \dots)$. These, in turn, are equivalent to the coefficient functions in the $z$-expansion of the Baker functions $\Psi^{\pm}(z,\mathbf{t})$ being solutions of the KP hierarchy [250], [246]. In particular, the Kadomtsev–Petviashvili equation (1.2.12) appears as the first nontrivial consequence of these relations.

For the case of the Toda lattice, the Hirota equations involve neighbouring lattice sites, as in eq. (1.7.13), and a further infinite set of similar bilinear equations, involving derivations with respect to all the flow variables.

**Exercise 1.2.** ($S$). Ptolemy's theorem states that, for a quadrilateral with consecutive vertices $A_1, A_2, A_3, A_4$ lying on a circle, the following identity holds:

$$A_1A_2 \cdot A_3A_4 + A_2A_3 \cdot A_1A_4 = A_1A_3 \cdot A_2A_4, \tag{1.10.5}$$

where $A_iA_j$ is the length of the line segment joining the vertices $A_i$ and $A_j$ (see Figure 1.5). Demonstrate this using the (trivial) KP $\tau$-function

$$\tau(t_1, t_2, t_3, \dots) = \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} t_{2k}\right) \tag{1.10.6}$$

(which just corresponds to vanishing solutions of the KP hierarchy) and the quadratic relations (1.10.4).

**Hint:** *Denoting the angles at the center* $\{\theta_i\}_{i=1,\dots,4}$, *we can parametrize these in terms of four real numbers*

$$x_1 < x_2 < x_3 < x_4 \tag{1.10.7}$$

*using the Cayley map*

$$e^{i\theta_j} = \frac{1 + ix_j}{1 - ix_j}, \quad j = 1, \dots, 4, \tag{1.10.8}$$

*which gives*

$$\sin\left(\frac{\theta_i}{2}\right) = \frac{1}{\sqrt{1 + x_i^2}}, \quad \cos\left(\frac{\theta_i}{2}\right) = \frac{x_i}{\sqrt{1 + x_i^2}}. \tag{1.10.9}$$

*If* $r$ *is the radius of the circle, the distances between the vertices are*

$$\begin{aligned} A_iA_j = 2r^2 \sin\left(\frac{\theta_j - \theta_i}{2}\right) &= 2r^2 \frac{x_j - x_i}{\sqrt{1 + x_i^2}\sqrt{1 + x_j^2}} \\ &= 2r^2 \det \begin{pmatrix} \frac{1}{\sqrt{1+x_i^2}} & \frac{x_i}{\sqrt{1+x_i^2}} \\ \frac{1}{\sqrt{1+x_j^2}} & \frac{x_j}{\sqrt{1+x_j^2}} \end{pmatrix}, \qquad 1 \le i < j \le 4. \end{aligned} \tag{1.10.10}$$

*Setting*

$$z_i = x_i, \quad i = 1, \dots, 4, \tag{1.10.11}$$

*in (1.10.3), verify that eq. (1.10.5) is equivalent to (1.10.4), and that this is just the classical* Plücker relation *(see Appendix C.5) defining the embedding of the Grassmannian* $\mathrm{Gr}_2(\mathbf{C}^4)$ *of 2-planes in* $\mathbf{C}^4$ *as a quadric in the projective space* $\mathbf{P}(\Lambda^2\mathbf{C}^4)$.

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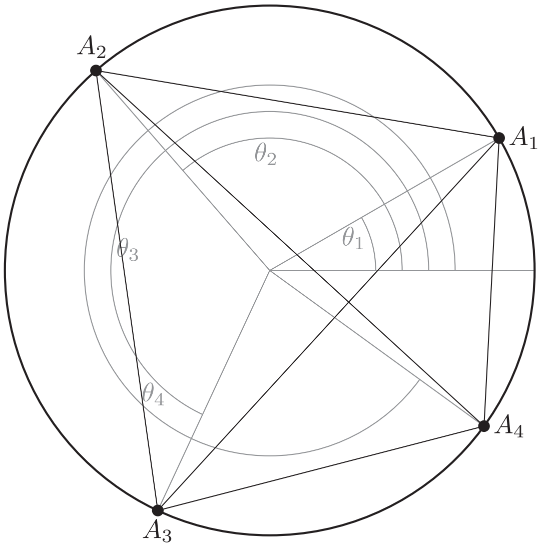


Fig. 1.5. A quadrilateral with vertices on a circle

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