BULL. AUSTRAL. MATH. SOC. 30D55
VOL. 38 (1988) [357–364]

FRACTIONAL INTEGRATION AND THE HYPERBOLIC DERIVATIVE

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We improve S. Yamashita's hyperbolic version of the well-known Hardy-Littlewood theorem. Let f be holomorphic and bounded by one in the unit disc D. If $\left(f^{\#}\right)^p$ has a harmonic majorant in D for some p, $p > 0$, then so does $\sigma(f)^q$ for all q, $0 < q < \infty$. Here

$$f^{\#} = \left|f'\right| / \left(1 - |f|^2\right) \text{ and } \sigma(f) = \tanh^{-1} |f|.$$

1. INTRODUCTION

The disc $D = \{|z| < 1\}$ is endowed with the non-Euclidean hyperbolic distance (Poincaré metric)

$$\sigma(z, w) = \frac{1}{2} \log \frac{|1 - \overline{z}w| + |z - w|}{|1 - \overline{z}w| - |z - w|}, \ (z, w \in D).$$

Let B be the family of all functions f holomorphic and bounded, $|f| < 1$, in D. For $f \in B$, we let, following Yamashita [6],

$$\sigma(f) = \sigma(f, 0) = 2^{-1} \log\{(1 + |f|)/(1 - |f|)\},$$

and

$$f^{\#} = |f'| / \left(1 - |f|^2\right).$$

These are hyperbolic counterparts of $|f|$ and $|f'|$, and $\sigma(f)^p$, $\left(f^{\#}\right)^p$ $(0 < p < \infty)$ are subharmonic in D if $f \in B$. Set

$$M_p(r, h) = \int_0^{2\pi} \left|h\left(re^{i\theta}\right)\right|^p d\theta/2\pi, \quad (0 < p < \infty),$$

for h subharmonic in D. Then h has a harmonic majorant in D if and only if $\sup_{0 \leqslant r < 1} M_1(r, h) < \infty$.

Received 5 February 1988.
Partially supported by K.M.E.

The Hardy class $H^p (0 < p < \infty)$ consists of those f holomorphic in D for which the subharmonic functions $|f|^p$ have harmonic majorants in D: the class H^∞ consists of all bounded holomorphic functions in D. Analogously, Yamashita [6] defined the hyperbolic Hardy class $H^p_\sigma (0 < p < \infty)$ as the class of those $f \in B$ for which $\sigma(f)^p$ have harmonic majorants in D, and H^∞_σ as that of those $f \in H^\infty$ bounded by a constant strictly less than one. He observed the following (II) [5, Theorem 2] in connection with (I) [3, Theorem 33] or see [1, Theorem 5.12]

(I) If $f' \in H^p$ for some p, $0 < p < 1$, then $f \in H^q$ with $q = p/(1-p)$.

(II) If $f \in B$ and if $(f^\#)^p$ for some p, $0 < p < 1$, has a harmonic majorant in D, then $f \in H^q_\sigma$ with $q = p/(1-p)$.

The index q in (I) cannot be replaced by a larger one [1, p. 90]. The question as to whether q in (II) is sharp is our starting point. One of the main differences between $f^\#$ and f' may be that $f^\#(z) \leqslant (1-|z|)^{-1}$ (by Pick's invariant form of Schwarz lemma), while $f'(z)$ is of $O(1-|z|)^{-1/p}$ [1, Theorem 5.9]. This fact leads us to introduce the concept of fractional integration [3 or 2]: If $f(z)$ is holomorphic in D, the fractional integral of f of order β, $\beta > 0$, is defined by Flett [2] as

$$I^\beta f(z) = \Gamma(\beta)^{-1} \int_0^1 (-\log t)^{\beta-1} f(tz)\,dt, \quad (z \in D). \tag{1.1}$$

The following (III), that extends (I), and (IV), which is a consequence of (III), are observed in [4, Theorem 2.1 and Remark 2.7].

(III) If $f \in H^p (0 < p < \infty)$ and $f(z) = O(1-|z|)^{-\gamma}$ with $0 < \gamma \leqslant 1/p$, then $I^\beta f \in H^q$ with $q = p\gamma/(\gamma - \beta)$, where $0 < \beta < \gamma$.

(IV) If f is a Bloch function (that is to say $f'(z) = O(1-|z|)^{-1}$) and $f' \in H^p (0 < p < 1)$, then $f \in H^q$ for all q, $0 < q < \infty$: $q = \infty$ cannot be allowed in the conclusion.

To show the hyperbolic counterparts of (III) and (IV) is our goal. We adopt, for calculational simplicity, a definition "equivalent to (1.1)":

$$I^\beta h(z) = \Gamma(\beta)^{-1} \int_0^1 (1-t)^{\beta-1} h(tz)\,dt, \quad (z \in D, \beta > 0),$$

for subharmonic $h = (f^\#)^p \quad (0 < p < \infty)$ in D. Since

$$\sigma(z, w) = \inf \int_c |dz| / \left(1 - |z|^2\right), \quad (z, w \in D),$$

where c runs through any arc in D joining z and w, it follows that

$$\sigma(f(z), f(0)) \leqslant \int_0^{|z|} f^\#\left(re^{i\theta}\right) dr, \quad \left(z = |z|\, e^{i\theta}\right),$$

https://doi.org/10.1017/S0004972700027714 Published online by Cambridge University Press

so that

$$\begin{aligned}\sigma(f(z)) &\leqslant \sigma(f(z), f(0)) + \sigma(f(0)) \\ &\leqslant |z| \int_0^1 f^{\#}(tz)dt + \sigma(f(0)) \\ &= |z|\, I^1 f^{\#}(z) + \sigma(f(0)).\end{aligned}$$

We shall show the following:

THEOREM 1. *Let* $f \in B$. *If* $(f^{\#})^p$ $(0 < p < \infty)$ *has a harmonic majorant in* D *and* $f^{\#}(z) = 0(1 - |z|)^{-\gamma}$ *with* $0 < \gamma \leqslant 1$, *then* $\sup_{0 \leqslant r < 1} M_q(r, I^\beta f^{\#}) < \infty$ *with* $q = p\gamma/(\gamma - \beta)$, *where* $0 < \beta < \gamma$.

THEOREM 2. *If* $f \in B$ *and* $(f^{\#})^p$ *admits a harmonic majorant in* D *for some* p, $0 < p < 1$, *then* $f \in H^q_\sigma$ *for all* q, $0 < q < \infty$.

Theorem 2 improves (II) significantly and shows that q in (II) is not sharp. The bound on q in Theorem 2 is sharp in the sense that there is a function $f \in B$ such that $(f^{\#})^p$ admits harmonic majorants in D for arbitrary p, $0 < p < 1$, but $f \notin H^\infty_\sigma$:even more f need not be a function which is hyperbolically Dirichlet finite, that is,

$$\iint_D (f^{\#})^2(z)dxdy = \infty.$$

2. PROOF OF THEOREM 1

Our results depend on the following two lemmata.

LEMMA 1. [5, Lemma 2]. *Let* $f \in B$. *Let* u *be one of* $f^{\#}$ *or* $\sigma(f)$. *Set* $M(\theta) = M(u, \theta) = \sup\{u(re^{i\theta}) : 0 \leqslant r < 1\}$. *If* $u^p (0 < p < \infty)$ *admits harmonic majorants in* D, *then*

$$\int_0^{2\pi} M(\theta)^p d\theta \leqslant C_p \int_0^{2\pi} u^*(\theta)^p d\theta,$$

where $u^*(\theta) = \lim_{r \to 1} u(re^{i\theta})$ *(which exists a.e.) and* C_p *is a positive constant depending only on* p.

LEMMA 2. *Let* $f \in B$, $0 < \beta, \gamma < \infty$, *and* $0 < \alpha = \beta/\gamma < p < \infty$. *Then*

$$I^\beta\left((f^{\#})^p\right)(z) \leqslant C M(z)^{p-\alpha} M(z, \gamma)^\alpha, \quad (z \in D), \tag{2.1}$$

where $M(z) = M(f^{\#}, z) = \sup\{f^{\#}(tz) : 0 \leqslant t < 1\}$, $M(z, \gamma) = M(f^{\#}, z, \gamma) = \sup\{(1 - t)^\gamma f^{\#}(tz) : 0 \leqslant t < 1\}$, $C = C_{p, \beta, \gamma}$ *is a positive constant depending only on* p, β *and* γ.

Proof: Fix $z \in D$. We may assume that $0 < M(z, \gamma)$ and $M(z) < \infty$. If we set $s = 1 - \{M(z, \gamma)/M(z)\}^{1/\gamma} \in [0, 1)$ then we have

$$
\begin{aligned}
(2.2) \qquad I^{\beta}\Big(\big(f^{\#}\big)^{p}\Big)(z) &\leqslant \Gamma(\beta)^{-1} \int_0^s + \int_s^1 (1-t)^{\beta-1}\big(f^{\#}\big)^{p}(tz)\,dt \\
&\leqslant \Gamma(\beta)^{-1}\Big\{M(z, \gamma)^{p} \int_0^s (1-t)^{\beta-1-p\gamma}dt \\
&\qquad + M(z)^{p} \int_s^1 (1-t)^{\beta-1}dt\Big\} \\
&\leqslant C_{p,\beta,\gamma} M(z)^{p-\alpha} M(z, \gamma)^{\alpha},
\end{aligned}
$$

whence (2.1) follows. ∎

We prove Theorem 1: Set $p = 1$ in (2.1) and integrate the q-th power of both sides with respect to $d\theta/2\pi$, then

$$M_q\big(r, I^{\beta} f^{\#}\big) \leqslant C \int_0^{2\pi} M(z)^{p} M(z, \gamma)^{q-\alpha} d\theta, \quad \big(z = re^{i\theta}\big),$$

because $q(1-\alpha) = p$. It then follows from the condition

$$K := \sup_{z \in D} (1 - |z|)^{\gamma} f^{\#}(z) < \infty$$

that

$$M_q\big(r, I^{\beta} f^{\#}\big) \leqslant C K^{q-\alpha} \int_0^{2\pi} M(\theta)^{p} d\theta,$$

where $M(\theta) = \sup\{f^{\#}\big(\mathrm{r}e^{i\theta}\big) : 0 \leqslant r < 1\}$. Now, Lemma 1 with $u = f^{\#}$ gives that

$$\int_0^{2\pi} M(\theta)^{p} d\theta \leqslant C_p \int_0^{2\pi} u^{*}(\theta)^{p} d\theta.$$

But then this last integral is dominated by $\lim_{r \to 1} M_p\big(r, f^{\#}\big)$, by Fatou's Lemma. Gathering up, we have

$$(2.3) \qquad M_q\big(r, I^{\beta} f^{\#}\big) \leqslant C K^{q-\alpha} \lim_{r \to 1} M_p\big(r, f^{\#}\big),$$

where $C = C_{p,q,\beta,\gamma}$. Note that $\lim_{r \to 1} M_p\big(r, f^{\#}\big) = \sup_r M_p\big(r, f^{\#}\big)$ by the subharmonicity of $\big(f^{\#}\big)^{p}$. Therefore, we get the desired conclusion if we take the supremum for r, $0 \leqslant r < 1$, on the left hand side of (2.3).

3. Proof of Theorem 2

We may assume $q > p$. Fix such a q. It was observed in Section 1 that

$$\sigma(f(z)) \leqslant |z|\, I^1 f^{\#}(z) + \sigma(f(0)) :$$

and it is obvious from the definition that

$$I^1 f^{\#}(z) \leqslant \Gamma(\beta) I^{\beta} f^{\#}(z), \quad (z \in D),$$

for $0 < \beta < 1$. Therefore

$$\begin{aligned}(3.1) \qquad \{\sigma(f(z))\}^q &\leqslant \{I^1 f^{\#}(z) + \sigma(f(0))\}^q \\ &\leqslant 2^q\{\Gamma(\beta)^q \big(I^{\beta} f^{\#}(z)\big)^q + \sigma(f(0))^q\}, \quad (0 < \beta < 1).\end{aligned}$$

Now, take $\beta < 1$ so that $q = p/(1-\beta)$. It then follows from Theorem 1 with $\gamma = 1$ that $\sup_{0 \leqslant r < 1} M_q\big(r, I^{\beta} f^{\#}\big) < \infty$, so that $\{\sigma(f(z))\}^q$ has a harmonic majorant by (3.1). Since this is true for any $q > p$, the conclusion follows.

4. An example

There is a function $f \in B$ such that $\big(f^{\#}\big)^p$ admits harmonic majorants in D for arbitrary p, $0 < p < 1$, but f is not hyperbolically Dirichlet finite.

Let

$$f(z) = e^{-g(z)}, \quad (z \in D),$$

where

$$g(z) = \exp\{\frac{1}{2} \log \left(\frac{1-z}{2}\right)\},\ g(0) = \sqrt{1/2}, \quad (z \in D).$$

Set

$$\phi(z) = \cos\{\frac{1}{2}\, \mathrm{Arg}\left(\frac{1-z}{2}\right)\}, \quad (z \in D),$$

for simplicity. Then after a simple calculation, we have $\sqrt{1/2} \leqslant \phi(z) \leqslant 1$, so that

$$\begin{aligned}|f(z)| &= \exp\left(-\mathrm{Re}\ g(z)\right) \\ &= \exp\left(-\,|g(z)|\,\phi(z)\right) \\ &< 1.\end{aligned}$$

Therefore $f \in B$. If we note that for $0 \leqslant \theta < 2\pi$,

$$|g^*(\theta)| = \sqrt{\sin(\theta/2)},$$

https://doi.org/10.1017/S0004972700027714 Published online by Cambridge University Press

and

$$|f^*(\theta)| = \exp\{-\sqrt{\sin(\theta/2)}\cos\left(\frac{\pi-\theta}{4}\right)\} \leqslant 1 \tag{4.1}$$

with equality in (4.1) only at $\theta = 0$, we can conclude by a routine calculation that

$$|g^*(\theta)|^{-1}\cdot\left(1-|f^*(\theta)|^2\right)^{-1} \sim \begin{cases} \theta & \text{near } \theta = 0 \\ 2\pi - \theta & \text{near } \theta = 2\pi \end{cases}.$$

and the left hand side of (4.2) is bounded away from zero elsewhere on $[0, 2\pi)$. (As usual, $F^*(\theta) = \lim_{r\to 1-} F(re^{i\theta})$, and "$F(\theta) \sim G(\theta)$ near $\theta = a$" means that there exist positive constants C_1 and C_2 satisfying $C_1 < F(\theta)/G(\theta) < C_2$ in a neighbourhood of $\theta = a$). Therefore

$$\int_0^{2\pi} \{|g^*(\theta)|\left(1-|f^*(\theta)|^2\right)\}^{-p} d\theta < \infty$$

for all p, $0 < p < 1$. It now follows from Lemma 1 and the fact

$$\left|f'^*(\theta)\right| = |g^*(\theta)|^{-1}\exp\{-|g^*(\theta)|\,\phi^*(\theta)\} \leqslant |g^*(\theta)|^{-1}$$

that

$$\begin{aligned}\sup_r M_p(r, f^\#) &\leqslant C_p\int_0^{2\pi} u^*(\theta)^p d\theta \\ &\leqslant C_p\int_0^{2\pi}\{|g^*(\theta)|\left(1-|f^*(\theta)|^2\right)\}^{-p}d\theta \\ &< \infty, \qquad (0 < p < 1),\end{aligned}$$

where $u = f^\#$.

Next, we show that $f^\#$ is not hyperbolically Dirichlet finite. Let

$$h(z) = \frac{|f(z)g(z)|}{1-|f(z)|^2}, \quad (z \in D).$$

Then

$$\begin{aligned} h(z) &= |g(z)|\{\exp(\operatorname{Re} g(z)) - \exp(-Reg(z))\}^{-1} \\ &\geqslant 2^{-1}\operatorname{Re} g(z)\ \operatorname{cosech}(\operatorname{Re} g(z)) \\ &\geqslant 2^{-1}\ \operatorname{cosech}(1) \\ &> 0, \end{aligned}$$

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because $0 <$ Re $g(z) < 1$ and the function x cosech(x) is decreasing for $x \geqslant 0$. If we note that $f^{\#}(z) = |2/(1-z)|\, h(z)$, it is now apparent that

$$\iint_D \left(f^{\#}\right)^2(z)\,dx\,dy = \infty.$$

5. A remark

We can say that the result of Theorem 2 is sharp in the other sense, that is, our example in Section 4 illustrates the sharpness of the following

(5.1) "If $f^{\#}$ has a harmonic majorant in D, then $f \in H_\sigma^\infty$"

The result (5.1) follows from the inequality [**7**, Theorem 3]

$$\lim_{r \to 1-} \sigma\left(f\left(re^{i\theta}\right), f(0)\right) \leqslant \int_0^1 f^{\#}\left(xe^{i\theta}\right)dx \leqslant (2)^{-1} \int_0^{2\pi} \left(f^{\#}\right)^*(t)\,dt.$$

6. One more theorem

Let f, p, β and γ be as in Lemma 2. If

$$K := \sup_{z \in D} (1-|x|)^\gamma \left|f^{\#}(z)\right| < \infty,$$

then by (2.2)

$$\int_0^1 (1-t)^{\beta-1} \left(f^{\#}\right)^p \left(tre^{i\theta}\right)dt \leqslant CK^\alpha M(z)^\delta,$$

so that

$$\begin{aligned}\int_0^1 \int_0^{2\pi} &(1-t)^{\beta-1} \left(f^{\#}\right)^p \left(tre^{i\theta}\right)d\theta\,dt \\ &\leqslant CK^\alpha \int_0^{2\pi} \sup\{\left|f^{\#}\left(tre^{i\theta}\right)\right|^\delta : 0 \leqslant r \leqslant 1\}\,d\theta \\ &\leqslant CK^\alpha \int_0^{2\pi} M(\theta)^\delta\,d\theta,\end{aligned}$$

where $\alpha = \beta/\gamma$, $\delta = p - \alpha$, $C = C_{p,\gamma,\beta}$ and $M(\theta) = \sup\{\left|f^{\#}\left(re^{i\theta}\right)\right| : 0 < r < 1\}$. Thus

$$\int_0^1 \int_0^{2\pi} (1-r)^{\beta-1} \left(f^{\#}\right)^p \left(re^{i\theta}\right)d\theta\,dr \leqslant CK^\alpha \int_0^{2\pi} M(\theta)^\delta\,d\theta \tag{6.1}$$

by the monotone convergence theorem. Now, Lemma 1 followed by Fatou's lemma and the subharmonicity of $\left(f^{\#}\right)^\delta$ makes the right hand side of (6.1) dominated by

$$C_{p,\gamma,\beta} K^\alpha \sup\{M_\delta\left(r, f^{\#}\right) : 0 \leqslant r < 1\}.$$

We state this:

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THEOREM 3. *Suppose that* $f \in B$ *and* $f^{\#}(z) = 0(1-|z|)^{-\gamma}$, *and let* $0 < \gamma \leqslant 1$, $0 < p < q < \infty$. *If* $\left(f^{\#}\right)^{p}$ *admits a harmonic majorant in* D *then*

$$\int_0^1 \int_0^{2\pi} (1-r)^{\gamma(q-p)-1} \left(f^{\#}\right)^{q} \left(re^{i\theta}\right) dr d\theta < \infty.$$

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