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PARTIAL REGULARITY AND APPLICATIONS

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Dedicated to Professor Sigeru MIZOHATA on his sixtieth birthday

§1. Introduction

The problem to determine the Gevrey index of solutions of a given hypoelliptic partial differential equation seems to be not yet well investigated. In this paper, we shall show the Gevrey indices of solutions of the equations of Grushin type, [6], are determined by a rather simple application of a straightforward extension of the results given in [7], [8] and [13]. For simplicity to construct left parametrices in the operator valued sense, we shall consider the equations under the stronger condition than that of [6] (cf. Condition 1 of Section 3). Typical examples of Grushin type are given by $L_1 = D_y^2 + y^2 D_x^2$, $L_2 = D_y^2 + (x^2 + y^2) D_x^2, \cdots$, which will be discussed in Section 4. We remark that our approach may be compared with the one to a similar problem discussed in [17] by using suitable L_2-estimates constructed in [16].

In Section 2, we prepare some direct extension of the results given in [13] on partial regularity of the distributions and those on pseudodifferential operators given in [7]. In Section 3, we shall establish a method to treat the equations of Grushin type. Finally, Section 4 will be devoted to a discussion on typical examples of Grushin type and to a brief comment on the application of our method for more general class of hypoelliptic partial differential equations.

§2. Partial regularity and a class of pseudodifferential operators

In this Section, we shall give some refinement of the results in [7] and [13]. Let Ω be an open subset of R^N whose point is denoted by $x = (x_1, \cdots, x_N)$. Let $q = (q_1, \cdots, q_N)$ be a N-tuple of real numbers $q_j \geqq 1$, $j = 1, \cdots, N$. We use general notations such as $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\langle \xi \rangle = \langle \xi \rangle_q = 1 + |\xi_1|^{1/q_1} + \cdots + |\xi_N|^{1/q_N}$ and $\langle \alpha, q \rangle = \alpha_1 q_1 + \cdots + \alpha_N q_N$.

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DEFINITION 2.1. Let $u \in C^\infty(\Omega)$, then we say that u is in $G^q(\Omega)$ if for any compact set K of Ω there are positive constants C_0 and C_1 such that

$$\sup_{x \in K} |D^\alpha u(x)| \leqq C_0 C_1^{|\alpha|} |\alpha|^{\langle \alpha, q \rangle}, \qquad \alpha \in Z_+^N. \tag{2.1}$$

PROPOSITION 2.1. *Let $u \in \mathscr{D}'(\Omega)$. Then $u \in G^q$ in a neighborhood of $x_0 \in \Omega$ if and only if for some neighborhood U of x_0 there is a bounded sequence $u_j \in \mathscr{E}'(\Omega)$, $j = 1, 2, \cdots$, which is equal to u in U and satisfies the estimates*

$$|\hat{u}_j(\xi)| \leqq C_0 C_1^j j!\, \langle \xi \rangle_q^{-j}, \qquad j = 1, 2, \cdots, \tag{2.2}$$

for some constants C_0 and $C_1 > 0$.

Proof. Necessity. Let $u \in G^q$ in $\{|x - x_0| \leqq 3\delta\}$, $\delta > 0$. We can find the functions $\chi_j(x)$, $j = 1, 2, \cdots$, such that $\chi_j \in C_0^\infty(|x - x_0| < 2\delta)$, equal to 1 when $|x - x_0| \leqq \delta$ and

$$|D^{\alpha+\beta} \chi_j| \leqq C_\beta C^{|\alpha|} j^{|\alpha|} \qquad \text{if} \quad |\alpha| \leqq j. \tag{2.3}$$

Here C depends only on N and δ, and C_β depends only on N, δ and β (cf. [11], Lemma 2.2). Then $u_j = \chi_j u$ is bounded in $\mathscr{E}'$. By assumption we have for some constant C_1

$$\sup_{|x - x_0| \leqq 3\delta} |D^\alpha u| \leqq C_1^{1+|\alpha|} |\alpha|^{\langle \alpha, q \rangle}. \tag{2.4}$$

It follows that

$$|D^\alpha(\chi_j u)| \leqq C C_{q_0} (C + C_1)^{|\alpha|} j^{\langle \alpha, q \rangle}, \qquad \langle \alpha, q \rangle \leqq j + q_0,$$

where $q_0 = \max(q_1, \cdots, q_N) \geqq 1$, from which we have

$$|\xi^\alpha \widehat{\chi_j u}(\xi)| \leqq C_2^{|\alpha|+1} j^j, \qquad \langle \alpha, q \rangle \leqq j + q_0.$$

On the other hand we have

$$\sum_{\langle \alpha, q \rangle \leqq j + q_0} |\xi^\alpha| \geqq C_3^j \xi_q^j, \qquad j = 1, 2, \cdots$$

for a constant C_3 independent of j, then we conclude that the estimates (2.2) hold.

Sufficiency. Since we have

$$|\xi^\alpha| \leqq \langle \xi \rangle_q^j, \quad \langle \alpha, q \rangle \leqq j, \quad j = 1, 2, \cdots$$

the estimates of type (2.1) in $|x - x_0| \leqq \delta$ are almost evident by using the Fourier inversion formula and (2.2).

Now we shall use a partition of the variable $x = (x', x'')$, $x' = (x_1, \cdots, x_p)$, $x'' = (x_{p+1}, \cdots, x_N)$, $1 \leqq p \leqq N-1$. We also use the partition of the multi-index $\alpha = (\alpha', \alpha'')$, $\alpha' = (\alpha_1, \cdots, \alpha_p)$, $\alpha'' = (\alpha_{p+1}, \cdots, \alpha_N)$. We recall that $u \in \mathscr{E}'(\Omega)$ is (partially) regular with respect to x' if for any $s > 0$ there exist numbers $t = t(s) \in R$ and $C = C(s)$ such that

$$(2.5) \qquad |\hat{u}(\xi)| \leqq C(1 + |\xi'|)^{-s}(1 + |\xi''|)^t, \quad \xi \in R^N. \quad \text{(cf. [5])}$$

DEFINITION 2.2. (cf. [13], Def. 3.2). Let $u \in \mathscr{D}'(\Omega)$. We say u is in $G_{x'}^{q'}$, $q' = (q_1, \cdots, q_p)$, $q_j \geqq 1$, $j = 1, \cdots, p$, in a neighborhood of $x_0 \in \Omega$ if for some neighborhood U of x_0 there is a bounded sequence $u_j \in \mathscr{E}'(\Omega)$, $j = 1, 2, \cdots$, which is equal to u in U and satisfies the estimates

$$(2.6) \qquad |\hat{u}_j(\xi)| \leqq C_0 C_1^j j! \langle \xi' \rangle^{-j} (1 + |\xi''|)^k, \qquad j = 1, 2, \cdots$$

for some constants C_0, $C_1 > 0$ and $k \in R$. Here we have denoted by $\langle \xi' \rangle = 1 + |\xi_1|^{1/q_1} + \cdots + |\xi_p|^{1/q_p}$. We define quite similarly, $u \in G_{x''}^{q''}$, $q'' = (q_{p+1}, \cdots, q_N)$.

We can see that by the same method of the proof of Proposition 3.1 of [13] we have its refininement as follows:

PROPOSITION 2.2. *Let $u \in \mathscr{D}'(\Omega)$. Then $u \in G^q$ in a neighborhood of $x_0 \in \Omega$ if and only if $u \in G_{x'}^{q'}$ and $u \in G_{x''}^{q''}$ in a neighborhood of $x_0 \in \Omega$.*

For the proof we only replace $|\xi'|$ by $\langle \xi' \rangle_{q'}$ and $|\alpha'|$ by $\langle \alpha', q' \rangle$ etc., in the proof of Proposition 3.1 of [13].

DEFINITION 2.3 (Generalization of [7], Def. 4.1). Let $-\infty < m < \infty$; $0 \leqq \delta < \rho \leqq 1$; $s \geqq 1$; $q = (q_1, \cdots, q_N)$, $q_j \geqq 1$, $j = 1, \cdots, N$. We denote by $S_{\rho,\delta,s}^{m,q}(\Omega \times R^N)$ the set of all $a(x, \xi) \in C^\infty(\Omega \times R^N)$ such that for every compact set K of Ω there are positive constants C_0, C_1 and B such that

$$(2.7) \qquad \sup_{x \in K} |a_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_0 C_1^{|\alpha+\beta|} \alpha! \beta!^s \langle \xi \rangle_q^{m-\rho|\alpha|+\delta|\beta|} \langle \xi \rangle_q \geqq B|\alpha|^\theta,$$

where $\theta = s/(\rho - \delta)$.

We associate with such a symbol $a(x, \xi)$ a pseudo-differential operator as usual:

$$a(x, D)u(x) = (2\pi)^{-N} \iint e^{i\langle x-y, \xi \rangle} a(x, \xi) u(y) dy d\xi, \qquad u \in C_0^\infty(\Omega).$$

Let $K(x, y) \in \mathscr{D}'(\Omega \times \Omega)$ be the distribution kernel of $a(x, D)$ expressed by the oscillatory integral:

$$K(x, y) = (2\pi)^{-N}\int e^{i\langle x-y,\xi\rangle}a(x, \xi)d\xi\,.$$

Then we get the following theorem by a slight modification of the proof of [7], Theorem 1.1.

THEOREM 2.1. *Let* $a(x, \xi) \in S^{m,q}_{\rho,\delta,s}(\Omega \times R^N)$. *Then we have the following*:

(i) $K(x, y) \in G^{\theta q}_{x,y}(\Omega \times \Omega \backslash \Delta)$, $\Delta = \{(x, x); x \in \Omega\}$, $\theta = s/(\rho - \delta)$.

(ii) The operator $a(x, D)$ is $G^{\theta' q}$-pseudolocal i.e., for any $\theta' \geqq \theta$ and $u \in \mathscr{E}'(\Omega)$ which is in $G^{\theta' q}$ in a neighborhood of $x_0 \in \Omega$ we have $a(x, D)u \in G^{\theta' q}$ in the same neighborhood of $x_0 \in \Omega$.

§ 3. Partial differential equations of Grushin type

In the following, we shall use the same notation of [6]. Let Ω be an open set of R^N whose point is denoted by $x = (x_1, \cdots, x_N)$. Let there be given rational numbers $\rho_j \geqq 1$, and $\sigma_j \geqq 0$, $1 \leqq j \leqq N$, such that for any j, $1 \leqq j \leqq N$, one of the following three relations is satisfied:

$$\text{(3.1)} \qquad \text{a) } \rho_j = \sigma_j = 1 \quad \text{b) } \rho_j > \sigma_j > 0 \quad \text{c) } \sigma_j = 0\,.$$

Let y denote the family of variables x_j for which property a) holds. Let x' be the set of remaining variables, so that x has representation $x = (x', y)$, $x' = (x_1, \cdots, x_k)$, $y = (y_1, \cdots, y_n)$, $k + n = N$. In turn x' is represented in the form $x' = (x'', x''')$, where b) holds for x'' and c) for x'''.

Let m be a positive integer and set

$$\text{(3.2)} \qquad \mathscr{M} = \{(\gamma, \alpha);\ |\alpha| \leqq m,\ \langle\rho, \alpha\rangle \geqq \langle\sigma, \gamma\rangle \geqq \langle\rho, \alpha\rangle - m\}\,,$$

$$\text{(3.3)} \qquad \mathscr{M}_0 = \{(\gamma, \alpha);\ |\alpha| \leqq m,\ \langle\sigma, \gamma\rangle = \langle\rho, \alpha\rangle - m\}\,,$$

where (γ, α) is a pair of multi-indices of dimension N with nonnegative integers such that $\gamma_j = 0$ for j if $\sigma_j = 0$ $(1 \leqq j \leqq k)$.

Now we study differential operators introduced in [6] of the form

$$\text{(3.4)} \qquad L(x, D) = \sum_{\mathscr{M}} a_{\alpha\gamma}(x)x^\gamma D^\alpha\,, \qquad a_{\alpha\gamma}(x) \in C^\infty(R^N)\,.$$

Associating with (3.4) we shall consider the operator

$$\text{(3.5)} \qquad L_0(x'', y, D) = \sum_{\mathscr{M}} a_{\alpha\gamma}(0)x^\gamma D^\alpha\,.$$

CONDITION 1. $L_0(x'', y, D)$ is strongly elliptic of even degree m for $|x''| + |y| = 1$.

CONDITION 2. The differential equation

$$L_0(x'', y, \xi, D_y)v(y) = 0 \tag{3.6}$$

has no non-trivial solution in $\mathscr{S}(R_y^n)$ for any fixed $\xi \in R^k$, $\xi \neq 0$ and x''.

Remark 3.1. Condition 1 is stronger than that of [6] in which the operator $L_0(x'', y, D)$ is merely supposed to be elliptic for $|x''| + |y| = 1$. We can replace Condition 1 by the original one if we apply the investigation of Beals, [2], Section 6 in the proof of Theorem 3.2 below.

THEOREM 3.1. *Under the conditions* 1 *and* 2, *the operator L is partially hypoelliptic in y in a neighborhood of the original in the following sense*:

(i) *There exists an open set $U \ni 0$ such that if $u \in \mathscr{E}'(U)$ and $L(x, D)u$ is regular with respect to y in U then u is also regular with respect to y in U.*

(ii) *If the coefficients $a_{\alpha\gamma}(x)$ are in $G^s(U)$, $s \geqq 1$, and if $u \in \mathscr{E}'(U)$, $L(x, D)u \in G_y^s$ in an open subset of U, then $u \in G_y^s$ in the same set.*

Proof. We shall investigate how the assumptions of [13], Theorems 4.3 and 4.4 are satisfied for the characteristic polynomial $L(x', y, \xi, \eta)$. Following [6] we set

$$\begin{aligned}
|x|_\sigma &= |x_1|^{1/\sigma_1} + \cdots + |x_N|^{1/\sigma_N}, \\
|x'|_\sigma &= |x_1|^{1/\sigma_1} + \cdots + |x_k|^{1/\sigma_k}, \\
|\xi|_\rho &= |\xi_1|^{1/\rho_1} + \cdots + |\xi_k|^{1/\rho_k}, \\
h(x'', y, \xi) &= |x|_\sigma^{\rho_k - 1}|\xi_1| + \cdots + |x|_\sigma^{\rho_k - 1}|\xi_k|,
\end{aligned}$$

where the summation for $|x|_\sigma$ and $|x'|_\sigma$ is only over the indices for which $\sigma_j \neq 0$. Then by Lemma 3.3 of [6], there exist a neighborhood U of $0 \in R^N$ and positive constants B and C such that

$$|L(x', y, \xi, \eta)| \geqq C \sum_{\substack{|\beta| \leqq m \\ \beta = (\beta_1, \cdots, \beta_n)}} h^{m - |\beta|}(x'', y, \xi)|\eta^\beta|, \qquad x = (x', y) \in U, \quad |\eta| \geqq B. \tag{3.7}$$

From this we have particularly

$$|L(x', y, \xi, \eta)| \geqq C(1 + |\eta|)^m, \qquad |\eta| \geqq B. \tag{3.7$'$}$$

This shows that L is partially elliptic in y since the degree of L is m and Hypothesis (H-1)$_\infty$ of Theorem 4.4, [13] is satisfied with respect to y taking $m_0 = m$. Furthermore (H-2)$_\infty$ of [13] is also satisfied in the following form: There are positive constants C_0, C_1 and B such that

$$|L_{(\beta)}^{(\alpha)}(x', y, \xi, \eta)| \leqq C_0 C_1^{|\alpha+\beta|} \alpha! \beta!^s |L| (1+|\eta|)^{-|\alpha|}(1+|\xi|)^m, \qquad (x', y, \xi, \eta) \in U \times \{|\eta| \geqq B|\alpha|\}. \tag{3.8}$$

This means we can take $\rho=1$, $\delta=0$ in $(\text{H-2})_\infty$. To prove (3.8) it is nearly sufficient to verify that we have the simple estimate of the form

$$|D_x^\nu D_{\xi,\eta}^\pi x^\gamma \xi^\alpha \eta^\beta| \leqq C(1+|\eta|)^{m-|\pi|}(1+|\xi|)^m, \qquad x \in U$$

for $(\gamma, \alpha+\beta) \in \mathscr{M}$, $\alpha = (\alpha_1, \cdots, \alpha_k, 0, \cdots, 0)$, $\beta = (0, \cdots, \beta_1, \cdots, \beta_n)$ and $\nu \leqq \gamma$, $\pi \leqq \alpha+\beta$. Thus we have the assertion of Theorem 3.1 by Theorems 4.3 and 4.4 of [13]. We remark that the term $(1+|\xi|)^m$ has not appeared in the Hypothesis $(\text{H-2})_\infty$ of [13] but this does not demand any change of the proof.

Next we shall study the partial regularity with respect to x' for the solutions of the equation

$$L(x, D)u(x) = f(x).$$

Let

$$\rho_0 = \min_{1\leqq j\leqq k} \rho_j, \quad \rho^0 = \max_{1\leqq j\leqq k} \rho_j, \quad \sigma_0 = \min_{1\leqq j\leqq k} \sigma_j, \quad \sigma^0 = \max_{1\leqq j\leqq k} \sigma_j.$$

If $\rho_0 > \sigma^0$, setting $q' = (\rho_1/\rho_0, \cdots, \rho_k/\rho_0)$ and $\delta = \sigma^0/\rho_0$, we have $q_j \geqq 1$, $j = 1, \cdots, k$, and $0 \leqq \delta < 1$.

THEOREM 3.2. *Under the Conditions* 1 *and* 2 *and* $\rho_0 > \sigma^0$ *we have the following;*

(i) *The operator* $L(x, D)$ *is hypoelliptic in a neighborhood of the origin.*

(ii) *If the coefficients* $a_{\alpha\gamma}(x)$ *are in* $G^s(\Omega)$, $s \geqq 1$, $\Omega \ni 0$, *then there exists an open set* $U \ni 0$ *such that if* $u \in \mathscr{E}'(\Omega)$, $L(x, D)u \in G^s(\Omega)$ *then* $u \in G_{x,y}^{\theta q', s}(U)$, *where* $\theta = s/(1-\delta)$.

Proof. We need to recall some fundamental results of Grushin, [6] in a slightly modified form as treated in [8], Chapter II. Let B_μ, $\mu > 0$, be the ball $\{|y| < \mu\}$ in R_y^n and $\mathscr{D}_\mu = H_0^{m/2}(B_\mu) \cap H^m(B_\mu)$, be the Sobolev space of order m with Dirichlet boundary condition. Suppose $\Omega = \Omega' \times B_\mu$, where Ω' is a neighborhood of the origin of $R_{x'}^k$. As in [6] and [8], we consider $L(x, D)$ as a pseudo-differential operator in the region Ω' with the operator valued symbol

$$p(x', \xi) = L(x', y, \xi, D_y) \in \mathscr{L}(\mathscr{D}_\mu, L_2(B_\mu)). \tag{3.9}$$

The symbol $p(x', \xi)$ is in $S_{1,0}^m(\Omega' \times R_\xi^k)$ in this sense.

We state a straightforward extension of the results of [8] and [6] without proof.

LEMMA 3.1 (cf. [8], Lemma 6.1 and [6], Lemma 3.5). *If the Hypotheses of Theorem* 3.2 *are satisfied, there exist positive numbers* A, C, μ *and a neighborhood* Ω' *of* $0 \in R^k$ *such that for all* $\xi \in R^k$, $|\xi| \geqq A$, $x' \in \Omega'$ *and* $v(y) \in \mathscr{D}\mu$ *we have*

$$\sum_{|\beta| \leqq m} \int |(|\xi|_\rho + h(x'', y, \xi))^{m-|\beta|} D_y^\beta v(y)|^2 dy \leqq C \int |L(x', y, \xi, D_y)v(y)|^2 dy . \tag{3.10}$$

We may assume that there are constants C_0 and C_1 such that

$$\sup_{x \in \Omega' \times B_\mu} \sum_{|\alpha| \leqq m} |D^\beta a_{\alpha\gamma}(x)| \leqq C_0 C_1^{|\beta|} \beta!^s , \qquad \beta \in Z_+^N . \tag{3.11}$$

Then from the estimate (3.10) we can find another couple of constants C_0 and C_1 such that

$$\|p_{(\beta_1)}^{(\alpha_1)}(x', \xi)v\|_{L_2(B_\mu)} \leqq C_0 C_1^{|\alpha_1+\beta_1|} \alpha_1! \beta_1!^s \|p(x', \xi)v\|_{L_2(B_\mu)} \langle \xi \rangle_q^{-|\alpha_1|+\delta|\beta_1|} \tag{3.12}$$

for all $|\xi| \geqq A$, $x' \in \Omega'$ and $v = v(y) \in \mathscr{D}_\mu$, where $p(x', \xi)$ is defined by (3.9) and α_1, β_1 are arbitrary multi-indices of dimension k. Since $p_{(\beta_1)}^{(\alpha_1)}(x', \xi)v(y)$ is a sum of the terms

$$(a_{\alpha\gamma}(x)x^\gamma)^{(\beta_1)}(\xi^{\alpha'})^{(\alpha_1)} D_y^\beta v(y) , \quad (\alpha, \gamma) \in \mathscr{M}, \quad \alpha = (\alpha', \beta) ,$$

it is sufficient to prove the estimate of the form

$$\|(a_{\alpha\gamma}(x)x^\gamma)^{(\beta_1)}(\xi^{\alpha'})^{(\alpha_1)} D_y^\beta v(y)\|_{L_2(B_\mu)} \leqq C_0 C_1^{|\alpha_1+\beta_1|} \alpha_1! \beta_1!^s |\xi|_\rho^{-\rho 0|\alpha_1|+\sigma 0|\beta_1|} \|p(x', \xi)v\|_{L_2(B_\mu)} . \tag{3.13}$$

We note that

$$|\xi|_\rho^{-\rho 0|\alpha_1|+\sigma 0|\beta_1|} = (|\xi_1|^{1/\rho_1} + \cdots + |\xi_k|^{1/\rho_k})^{-\rho 0(|\alpha_1|-\delta|\beta_1|)}$$

which is equivalent to

$$(|\xi_1|^{1/q_1} + \cdots + |\xi_k|^{1/q_k})^{-|\alpha_1|+\delta|\beta_1|} .$$

Thus (3.13) follows from the estimate of the form

$$|x^{\gamma-\beta_1} \xi^{\alpha'-\alpha_1} \eta^\beta| \leqq C|\xi|_\rho^{-\langle\rho, \alpha_1\rangle+\langle\sigma, \beta_1\rangle}(|\xi|_\rho + h(x'', y, \xi)^{m-|\beta|}|\eta^\beta|) \tag{3.14}$$

for $(\alpha, \gamma) \in \mathscr{M}$, $\alpha = (\alpha', \beta)$, which is established by observing the quasi-homogeneity property of both sides in the sense of [6], that is, with positive parameter λ, make substitution $x'' \to \lambda^{-\sigma} x''$, $y \to \lambda^{-1} y$, $\xi \to \lambda^{\rho}\xi$, $\eta \to \lambda\eta$ then the left hand side of (3.4) is of degree $\leqq m - \langle \rho, \alpha_1 \rangle + \langle \sigma, \beta_1 \rangle$ in λ while the right hand side is just of degree $m - \langle \rho, \alpha_1 \rangle + \langle \sigma, \beta_1 \rangle$ in λ.

We take as the left inverse of $p(x', \xi)$ by

$$p^{-1}; L_2(B_\mu) \longrightarrow \mathscr{D}_\mu = H_0^{m/2}(B_\mu) \cap H^m(B_\mu)\,, \tag{3.15}$$

which is defined in $L_2(B_\mu)$ and $\|p^{-1}\|_{\mathscr{L}(L_2(B_\mu), \mathscr{D}_\mu)}$ is uniformly bounded in $(x', \xi) \in \Omega' \times R_\xi^k$ (cf. (3.7)) and (3.10).)

Now in order to construct a left parametrix of $p(x', D)$, determine recursively the symbols b_j by means of the relations

$$b_0(x', \xi) = p^{-1}(x', \xi) \in (L_2(B_\mu), \mathscr{D}_\mu) \tag{3.16}$$

and for $j = 1, 2, \cdots$

$$b_j(x', \xi) = -\left[\sum_{1 \leqq |\alpha| \leqq j} \frac{1}{\alpha!} \partial_\xi^\alpha b_{j-|\alpha|} D_{x'}^\alpha p\right] b_0\,. \tag{3.17}$$

We note that we have

$$D_{x'}^\alpha \partial_\xi^\beta b_0 = -\, b_0(D_{x'}^\alpha \partial_\xi^\beta p) b_0 \in \mathscr{L}(L_2(B_\mu)), \mathscr{D}_\mu)$$

if $|\alpha + \beta| = 1$ keeping in mind that $pb_0 = \mathrm{Id}$ in $L_2(B_\mu)$ and $p_0 b = \mathrm{Id}$ in $\mathscr{D}_\mu$. By induction, $D_x^\alpha \partial_\xi^\beta b_0$ for any α and $\beta \in Z_+^k$ is a linear combination of the monomials

$$b(\alpha(1), \cdots, \alpha(h); \beta(1), \cdots, \beta(h)) = b_0 \prod [(D_{x'}^{\beta(j)} \partial_\xi^{\alpha(j)} p) b_0]$$

with $\alpha = \sum \alpha(j)$, $\beta = \sum \beta(j)$. Then by using (3.12), we can see that $b_j(x', \xi) \in \mathscr{L}(L_2(B_\mu), \mathscr{D}_\mu)$ and there are constants C_0 and C_1 such that

$$\begin{aligned} &\sup_{x' \in \Omega'} \|b_{j(\beta_1)}^{(\alpha_1)}(x', \xi)\|_{\mathscr{L}(L_2(B_\mu), \mathscr{D}_\mu)} \\ &\qquad \leqq C_0 C_1^{|\alpha_1 + \beta_1|}(|\beta_1| + j)!^s \alpha_1! \langle \xi \rangle_q^{-|\alpha_1| + \delta|\beta_1|}\,, \\ &\qquad\qquad \alpha_1\,, \quad \beta_1 \in Z_+^k\,, \quad |\xi| \geqq A\,. \end{aligned} \tag{3.18}$$

As in [7], we prepare a series of cut-off functions $\phi_j(\xi) \in C(R_\xi^k)$, $j = 0, 1, \cdots$, satisfying

$$\begin{aligned} &0 \leqq \phi_j(\xi) \leqq 1 \quad \text{and} \quad \phi_j(\xi) = 0 \\ &\text{if} \quad \langle \xi \rangle_q \leqq 2R \sup(j^\theta, 1) \quad \text{and} \quad \phi_j(\xi) = 1 \\ &\text{for} \quad \langle \xi \rangle_q \geqq 3R \sup(j^\theta, 1)\,, \quad \theta = s/(1-\delta)\,, \quad R > 0\,; \end{aligned} \tag{3.19}$$

$$(3.20) \qquad |D_\xi^\alpha \phi_j| \leqq (C/(Rj^{\theta-1}))^{|\alpha|} \qquad \text{if } |\alpha| \leqq 2j\,.$$

Taking R sufficiently large we can see that

$$b(x', \xi) \equiv \sum_{j=0}^{\infty} \phi_j(\xi) b_j(x', \xi) \in S_{1,\delta,s}^{0,q}(\Omega' \times R_\xi^k)$$

in the operator $\mathscr{L}(L_2(B_\mu), \mathscr{D}_\mu)$-valued sense. We can apply essentially the same method of the proof of [7], Theorem 3.1 and have the relation

$$(3.21) \qquad b(x', D)p(x', D)v = v + Fv\,, \qquad v \in \mathscr{D}_\mu\,,$$

where F is an integral operator with kernel $F(x', y') \in \mathscr{L}(\mathscr{D}_\mu, \mathscr{D}_\mu)$ such that we have the estimate of the form

$$(3.22) \qquad \sup_{x', y' \in \Omega'} \|D_{x'}^\alpha D_{y'}^\beta F(x', y')\|_{\mathscr{L}} \leqq C_0 C_1^{|\alpha+\beta|} \alpha!^\theta \beta!^\theta\,, \qquad \alpha, \beta \in Z_+^k\,.$$

Now if $u \in C^\infty(\Omega' \times B_\mu)$ and $L(x, D)u \in G_{x'}^\theta(\Omega' \times B_\mu)$, then by Theorem 2.1, (ii), (3.21) and (3.22) we have the partial regularity, $u \in G_{x'}^{\theta q'}$, in a neighborhood of the origin of R^N. Then by applying Theorem 3.1, (ii) and Proposition 2.2, we have finally $u \in G_{x',y}^{\theta q', s}$ in a neighborhood of the origin of R^N. Thus we have obtained the assertion (ii) of Theorem 3.2. The assertion (i) can be obtained by more rough procedure and we omit the proof (cf. [6]).

§4. Examples and comments

First we shall consider the following operators:

$$L_1 = \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x^2}\,, \qquad L_2 = \frac{\partial^2}{\partial y^2} + (x^2 + y^2) \frac{\partial^2}{\partial x^2}\,,$$
$$L_3 = \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\,.$$

(1) We see that L_1 has the form (3.4) with $\rho_2 = \sigma_2 = 1$, $\rho_1 = 2$, $\sigma_1 = 0$. Then we have $q' = 1$, $\delta = 0$ and $\theta = 1$. Thus by Theorem 3.2 we have analytic hypoellipticity of L_1 in a neighborhood of the origin of R^2.

(2) As for L^2 we have $\rho_2 = \sigma_2 = 1$, $\rho_1 = 2$, $\sigma_1 = 1$. Then we have $q' = 1$, $\delta = 1/2$ and $\theta = 2$. Thus by Theorem 3.2, we have $u \in G_{x,y}^{2,1}$ in a neighborhood of the origin of R^2 for any solution u of the equation

$$(4.1) \qquad L_2 u(x, y) = 0 \qquad \text{in } R^2\,.$$

We note that a function $u(x, y) \in G_{x,y}^{2,1}$ in a neighborhood of the origin

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satisfying (4.1) was constructed by G. Métivier, [14].

(3) L_3 has the form (3.4) with $\rho_3=\sigma_3=1$, $\rho_1=2$, $\rho_2=1$, $\sigma_1=\sigma_2=0$. Then we have $\delta=0$, $\theta=1$ and $q'=(2,1)$. Hence by Theorem 3.2 we have $u(x_1,x_2,y)\in G^{2,1,1}_{x_1,x_2,y}$ for any solution u of the equation

$$L_3u(x_1,x_2,y)=0 \tag{4.2}$$

in a neighborhood of the origin of R^3. We note that an example of the solution $u(x_1,x_2,y)\in G^{2,1,1}_{x_1,x_2,y}$ of (4.2) was constructed by M.S. Baouendi and C. Goulaouic, [1].

Our method can be applied for the operators with quasi-homogeneous principal symbols i.e., degenerate quasi-elliptic operators. For example, consider the equations

$$P_ju=\left(\frac{\partial^2}{\partial y^2}-y^j\frac{\partial}{\partial x}\right)u(x,y)=0\,,\qquad j=0,1,2,\cdots, \tag{4.3}$$

$$Q_ku=\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}+y_1^k\frac{\partial}{\partial x}\right)u(x,y_1,y_2)=0\,,\qquad k=0,1,\cdots. \tag{4.4}$$

Then we have $u\in G^{2,1}_{x,y}$ for any solution of (4.3) and $u\in G^{k+2,1,1}_{x,y_1,y_2}$ for any solution u of (4.4). We remark that relating results have been recently obtained in [15].

Finally we remark that Theorem 2.1 of this paper can be extended for a corresponding class of partially regular pseudodifferential operators as in the manner of [13], Definition 2.3 and Theorem 2.1.

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