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Two-Weight Estimates for Singular Integrals Defined on Spaces of Homogeneous Type

D. E. Edmunds, V. Kokilashvili and A. Meskhi

Abstract. Two-weight inequalities of strong and weak type are obtained in the context of spaces of homogeneous type. Various applications are given, in particular to Cauchy singular integrals on regular curves.

Introduction

The question of the boundedness of integral transforms defined on spaces of homogeneous type (SHT) arises naturally when studying boundary-value problems for partial differential equations with variable coefficients. For example, when the underlying domain is strongly pseudo-convex, one is led to use the concept of the Heisenberg group (and more general structures) as a model for the boundary of the domain in the theory of functions of several complex variables. Such problems indicate a strong need for structures more general than spaces of functions on Euclidean space. The space domain might, for instance, be most conveniently endowed with a quasimetric induced by a differential operator or tailored to suit the kernel of a given integral operator (see [19], Chapters I, XII and XIII).

On the other hand, it is well-known that the solubility of boundary-value problems for elliptic partial differential equations in domains with non-smooth boundaries depends crucially on the geometry of the boundary. In [9], Chapter IV it is shown that the presence of angular points (involving cusps) can result in non-existence or non-uniqueness of solutions of Dirichlet and Neumann problems for harmonic functions from Smirnov classes and boundary functions in appropriate Lebesgue spaces. In this connection, two-weight inequalities for singular integrals with pairs of weights like those considered in the sequel enable one to identify, for the boundary functions, the weighted Lebesgue spaces for which the problem becomes soluble.

Two-weight inequalities of strong type with monotonic weights for Hilbert transforms have been established in [15]. Analogous problems for singular integrals in Euclidean spaces were considered in [7] and were generalised in [6] for singular integrals on Heisenberg groups. For Calderón-Zygmund singular integrals, conditions for a pair of radial weights ensuring the validity of two-weight inequalities of strong type have been obtained by the first two authors [4] (see also [17]) and generalised for homogeneous groups and on spaces of homogeneous type with some additional assumptions by the last two authors [11], [12]. Moreover, these last papers contain weak type inequalities as well.

In this paper we derive various two-weight inequalities of strong and weak type. The sufficient conditions given are optimal in the sense that for Hilbert transforms these are also

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necessary, see for example [15] and [4]. Some very special cases of the problems discussed in this paper were studied in [5], Chapter IX.

The paper is organised as follows: In Section 1 we establish two-weight criteria for Hardy-type transforms, while Section 2 is devoted to the various two-weight strong type inequalities in an SHT setting. Section 3 contains two-weight weak type inequalities for singular integrals. In Section 4 applications to Cauchy singular integrals on regular curves and some examples are given. Constants (often different constants in the same series of inequalities) will generally be denoted by b or c.

1 Basic Ingredients

In this section we are going to establish two-weight criteria for some extensions of Hardy-type transforms defined on an arbitrary measure space (X, μ) with σ-finite measure μ. Let us suppose there is a function $d\colon X \times X \to [0, \infty)$ such that there exists some $x_0 \in X$ for which $d(x_0, \cdot)\colon X \to [0, \infty)$ is μ-measurable and

$$0 < \mu\{x : t_1 < d(x_0, x) < t_2\}$$

whenever $0 < t_1 < t_2 < a$, where $a := \sup\{d(x_0, x) : x \in X\}$.

Given a μ-measurable function $w\colon X \to \mathbb{R}$ which is positive μ-a.e., and $p \in [1, \infty)$, we denote by $L^p_w(X)$ the space of μ-measurable functions $f\colon X \to \mathbb{R}$ with finite norm

$$\|f|L^p_w(X)\| = \Big(\int_X |f(x)|^p w(x)\, d\mu\Big)^{1/p}.$$

We shall prove the following two theorems:

Theorem 1.1 *Let $1 < p \le q < \infty$, $\mu\{x \in X : d(x_0, x) = 0\} = 0$ and let v (resp. w) be a.e. positive and measurable on $(0, a)$ (resp. μ-a.e. positive and μ-measurable on X). Then the inequality*

$$(1.1)\quad \Big(\int_0^a v(t)\Big|\int_{\{x:d(x_0,x)<t\}} f(x)\, d\mu\Big|^q dt\Big)^{1/q} \le c\Big(\int_X |f(x)|^p w(x)\, d\mu\Big)^{1/p}, \quad f \in L^p_w(X)$$

holds with some $c > 0$ independent of f if, and only if,

$$(1.2)\qquad B := \sup_{0<t<a}\Big(\int_t^a v(\tau)\, d\tau\Big)^{1/q}\Big(\int_{\{x:d(x_0,x)<t\}} w^{1-p'}(x)\, d\mu\Big)^{1/p'} < \infty.$$

In addition, if c is the best constant in (1.1), then

$$B \le c \le 4B.$$

Theorem 1.2 *Let $1 < p \le q < \infty$, $\mu\{x \in X : d(x_0, x) = a\} = 0$ and let v and w be as in Theorem 1.1. Then the inequality*

$$(1.3)\quad \Big(\int_0^a v(t)\Big|\int_{\{x:d(x_0,x)>t\}} f(x)\, d\mu\Big|^q dt\Big)^{1/q} \le c\Big(\int_X |f(x)|^p w(x)\, d\mu\Big)^{1/p}, \quad f \in L^p_w(X)$$

holds with a constant c independent of f if, and only if,

$$B_1 := \sup_{0<t<a} \Big(\int_0^t v(\tau)\,d\tau\Big)^{1/q} \Big(\int_{\{x:d(x_0,x)>t\}} w^{1-p'}(x)\,d\mu\Big)^{1/p'} < \infty. \tag{1.4}$$

Moreover, for the best constant c in (1.3) we have

$$B_1 \le c \le 4B_1.$$

The proofs of these theorems are based on the two following simple lemmas.

Lemma 1.1 *Let $1 < p \le q < \infty$, and let v and w be as in Theorem 1.1. Then the following statements are equivalent:*

(i) there is a constant $c > 0$ such that for all $f \in L^p_w(X)$,

$$\Big(\int_0^a v(t)\Big|\int_{\{x:d(x_0,x)<t\}} f(x)\,d\mu\Big|^q dt\Big)^{1/q} \le c\Big(\int_X |f(x)|^p w(x)\,dx\Big)^{1/p};$$

(ii) there is a constant $c > 0$ such that for all $g \in L^{q'}_{v^{1-q'}}(0,a)$,

$$\Big(\int_X w^{1-p'}(x)\Big|\int_{d(x_0,x)}^a g(t)\,dt\Big|^{p'} d\mu\Big)^{1/p'} \le c\Big(\int_0^a |g(t)|^{q'} v^{1-q'}(t)\,dt\Big)^{1/q'}.$$

Lemma 1.2 *Let $1 < p \le q < \infty$ and let v and w be as in Theorem 1.1. Then the following two assertions are equivalent:*

(i) there exists a constant $c > 0$ such that

$$\Big(\int_0^a v(t)\Big|\int_{\{x:d(x_0,x)>t\}} f(x)\,d\mu\Big|^q dt\Big)^{1/q} \le c\Big(\int_X |f(x)|^p w(x)\,d\mu\Big)^{1/p}$$

for arbitrary $f \in L^p_w(X)$;

(ii) there exists a constant $c > 0$ such that

$$\Big(\int_X w^{1-p'}\Big|\int_0^{d(x_0,x)} g(t)\,dt\Big|^{p'} d\mu\Big)^{1/p'} \le c\Big(\int_0^a |g(t)|^{q'} v^{1-q'}(t)\,dt\Big)^{1/q'}$$

for any $g \in L^{q'}_{v^{1-q'}}(0,a)$.

Proof of Lemma 1.1 We shall show that (i) implies (ii). We have

$$A := \Big(\int_X w^{1-p'}(x)\Big|\int_{d(x_0,x)}^a g(t)\,dt\Big|^{p'} d\mu\Big)^{1/p'} = \sup\Big|\int_X \Big(\int_{d(x_0,x)}^a g(t)\,dt\Big) f(x)\,d\mu\Big|,$$

where the supremum is taken over all f for which

$$\int_X |f(x)|^p w(x)\,dx \le 1.$$

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Then change of order of integration, Hölder's inequality and condition (i) give us:

$$
\begin{aligned}
A &\le \sup\Big(\int_0^a |g(t)|\Big(\int_{\{x:d(x_0,x)<t\}} |f(x)|\,d\mu\Big)\,dt\Big)\\
&\le \sup\Big(\int_0^a |g(t)|^{q'} v^{1-q'}(t)\,dt\Big)^{1/q'}\Big(\int_0^a v(t)\Big(\int_{d(x_0,x)<t} |f(x)|\,d\mu\Big)^q\,dt\Big)^{1/q}\\
&\le c\Big(\int_0^a |g(t)|^{q'} v^{1-q'}(t)\,dt\Big)^{1/q'}.
\end{aligned}
$$

Analogously we can show that (ii) implies (i).

As the proof of Lemma 1.2 is similar we omit it.

Proof of Theorem 1.1 First we prove that (1.2) implies (1.1). Thanks to Lemma 1.1 it is sufficient to prove that under condition (1.2) we have the inequality

$$
\Big(\int_X w^{1-p'}(x)\Big|\int_{d(x_0,x)}^a g(t)\,dt\Big|^{p'}\,d\mu\Big)^{1/p'} \le c\Big(\int_0^a |g(t)|^{q'} v^{1-q'}(t)\,dt\Big)^{1/q'}
$$

for any $g \in L^{q'}_{v^{1-q'}}(0,a)$ with a constant c independent of g.

Let $g \ge 0$, $g \in L^{q}_{v^{1-q'}}(0,a)$. Then for arbitrary $t \in (0,a)$ we have

$$
\int_t^a g(\tau)\,d\tau \le \Big(\int_t^a |g(\tau)|^{q'} v^{1-q'}(\tau)\,d\tau\Big)^{1/q'}\Big(\int_t^a v(\tau)\,d\tau\Big)^{1/q} < \infty.
$$

Then the function

$$
I(t) = \int_t^a g(\tau)\,d\tau
$$

is continuous and decreasing on $(0,a)$. Moreover, $I(t) \to 0$ as $t \to a$.

Now suppose

$$
\int_0^a g(\tau)\,d\tau \in (2^m, 2^{m+1}]
$$

for some integer m. Then for each integer $k \le m$ there exists $t_k \in (0,a)$ such that

$$
2^k = \int_{t_{k+1}}^{t_k} g(\tau)\,d\tau \quad \text{for } k \le m-1
$$

and

$$
2^m = \int_{t_m}^a g(\tau)\,d\tau.
$$

The sequence $\{t_k\}_{-\infty}^m$ decreases; let $\alpha = \lim_{k\to-\infty} t_k$. Then we have $[0,a] = \bigcup_{k=-\infty}^m [t_{k+1}, t_k) \cup [\alpha, a]$, where $t_{m+1} = 0$. Thus

$$\begin{aligned} X &= \{x : 0 \le d(x_0, x) \le a\} \\ &= \bigcup_{k=-\infty}^{m} \{x \in X : t_{k+1} \le d(x_0, x) < t_k\} \cup \{x : \alpha \le d(x_0, x) \le a\}. \end{aligned}$$

If

$$\int_0^a g(\tau)\,d\tau = \infty$$

then in this case $m = \infty$ and

$$X = \bigcup_{k\in\mathbb{Z}} \{x : t_{k+1} \le d(x_0, x) < t_k\} \cup \{\alpha \le d(x_0, x) \le a\}.$$

Now let $t \in [t_{k+1}, t_k)$; then $I(t) = \int_t^a g(\tau)\,d\tau \le \int_{t_{k+1}}^a g(\tau)\,d\tau = 2^{k+1}$ as $k \le m$. For $t \in [\alpha, a]$ we have

$$\int_t^a g(\tau)\,d\tau \le \int_\alpha^a g(\tau)\,d\tau \le \int_{t_k}^a g(\tau)\,d\tau = 2^k$$

for arbitrary $k \le m$, and consequently $I(t) = 0$. Further

$$\begin{aligned} &\Big(\int_X w^{1-p'}(x)\Big(\int_{d(x_0,x)}^a g(t)\,dt\Big)^{p'} d\mu\Big)^{q'/p'} \\ &\qquad = \Big(\int_{0\le d(x_0,x)\le a} w^{1-p'}(x)\Big(\int_{d(x_0,x)}^a g(t)\,dt\Big)^{p'} d\mu\Big)^{q'/p'} \\ &\qquad = \Big(\sum_{k\le m} \int_{\{t_{k+1}\le d(x_0,x)<t_k\}} w^{1-p'}(x)\Big(\int_{d(x_0,x)}^a g(t)\,dt\Big)^{p'} d\mu\Big)^{q'/p'} \\ &\qquad \le \Big(\sum_{k\le m} 2^{(k+1)p'}\Big(\int_{\{x:t_{k+1}\le d(x_0,x)<t_k\}} w^{1-p'}(x)\,d\mu\Big)\Big)^{q'/p'}. \end{aligned}$$

This last expression can be estimated from above by

$$\begin{aligned} &4^{q'} \sum_{k\le m} 2^{(k-1)q'}\Big(\int_{\{x:t_{k+1}\le d(x_0,x)<t_k\}} w^{1-p'}(x)\,d\mu\Big)^{q'/p'} \\ &\quad = 4^{q'} \sum_{k\le m}\Big(\int_{t_k}^{t_{k-1}} g(\tau)\,d\tau\Big)^{q'}\Big(\int_{\{x:t_{k+1}\le d(x_0,x)<t_k\}} w^{1-p'}(x)\,d\mu\Big)^{q'/p'} \\ &\quad \le 4^{q'} \sum_{k\le m}\Big(\int_{t_k}^{t_{k-1}} g^{q'}(\tau)v^{1-q'}(\tau)\,d\tau\Big)\Big(\int_{t_k}^a v(\tau)\,d\tau\Big)^{\frac{q'}{q}}\Big(\int_{\{x:d(x_0,x)<t_k\}} w^{1-p'}(x)\,d\mu\Big)^{\frac{q'}{p'}} \\ &\quad \le 4^{q'}B^{q'} \sum_{k\le m}\int_{t_k}^{t_{k-1}} g^{q'}(\tau)v^{1-q'}(\tau)\,d\tau = 4^{q'}B^{q'}\int_0^a g^{q'}(\tau)v^{1-q'}(\tau)\,d\tau. \end{aligned}$$

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This proves the sufficiency part of Theorem 1.1.

Now it remains to prove that (1.1) implies (1.2). Let $s \in (0,a)$. If in (1.1) we take $f = w^{1-p'}\chi_{\{x:d(x_0,x)<s\}}$ then we have

$$\begin{aligned} c\Big(\int_{\{x:d(x_0,x)<s\}} w^{1-p'}(x)\,d\mu\Big)^{1/p} &\geq \Big(\int_0^a v(t)\Big(\int_{\{x:d(x_0,x)<t\}} f(x)\,d\mu\Big)^q dt\Big)^{1/q} \\ &\geq \Big(\int_s^a v(t)\Big(\int_{\{x:d(x_0,x)<t\}} f(x)\,d\mu\Big)^q dt\Big)^{1/q} \\ &\geq \Big(\int_{\{x:d(x_0,x)<s\}} w^{1-p'}(x)\,d\mu\Big)\Big(\int_s^a v(t)\,dt\Big)^{1/q}. \end{aligned}$$

Finally, from the above we obtain

$$\Big(\int_s^a v(\tau)\,dt\Big)^{1/q}\Big(\int_{\{x:d(x_0,x)<s\}} w^{1-p'}(x)\,d\mu\Big)^{1/p'} \leq c$$

for any $s \in (0,a)$ and the proof is complete. ■

Proof of Theorem 1.2 To show that (1.4) implies (1.3), by Lemma 1.2 it is sufficient to prove that (1.4) guarantees the validity of the inequality

$$\Big(\int_X w^{1-p'}(x)\Big|\int_0^{d(x_0,x)} g(t)\,dt\Big|^{p'} d\mu\Big)^{1/p'} \leq c\Big(\int_0^a |g(t)|^{q'} v^{1-q'}(t)\,dt\Big)^{1/q'}$$

for arbitrary $g \in L^{q'}_{v^{1-q'}}(0,a)$ with a constant c independent of g.

Let $g \geq 0$, $g \in L^{q'}_{v^{1-q'}}(0,a)$. Then it is easy to see that

$$J(t) = \int_0^t g(\tau)\,d\tau < \infty$$

for any $t \in (0,a)$.

The function J is continuous, increases on $(0,a)$ and $\lim_{t\to 0} J(t) = 0$.

Now let

$$\int_0^a g(\tau)\,d\tau \in (2^m, 2^{m+1}]$$

for some integer m. Then by continuity of J, given any integer k, $k \leq m$, there exists t_k, $t_k \in (0,a)$, such that

$$2^k = \int_0^{t_k} g(\tau)\,d\tau = \int_{t_k}^{t_{k+1}} g(\tau)\,d\tau \quad \text{for } k \leq m-1$$

and

$$2^m = \int_0^{t_m} g(\tau)\,d\tau.$$

The sequence $\{t_k\}_{k=-\infty}^{m}$ increases; let $\lim_{k\to-\infty} t_k = \beta$. Then

$$\begin{aligned} X &= \{x : 0 \le d(x_0, x) \le a\} \\ &= \{x : 0 \le d(x_0, x) \le \beta\} \cup \Big(\bigcup_{j=-\infty}^{m} \{x : t_j < d(x_0, x) \le t_{j+1}\} \Big), \end{aligned}$$

where $t_{m+1} = a$.

If $t \in [0, \beta]$ then $I(t) = \int_0^t g(\tau)\,d\tau \le \int_0^{t_k} g(\tau)\,d\tau = 2^k$ for arbitrary k, $k \le m$, and so $I(t) = 0$. If $t \in (t_j, t_{j+1}]$ for $j \le m$ then

$$I(t) \le \int_0^{t_{j+1}} g(\tau)\,d\tau = 2^{j+1}.$$

Now we have

$$\begin{aligned} &\Big(\int_X w^{1-p'}(x) \Big(\int_0^{d(x_0,x)} g(t)\,dt \Big)^{p'} d\mu \Big)^{q'/p'} \\ &\qquad = \Big(\int_{0\le d(x_0,x)\le a} w^{1-p'}(x) \Big(\int_0^{d(x_0,x)} g(t)\,dt \Big)^{p'} d\mu \Big)^{q'/p'} \\ &\qquad = \Big(\sum_{j\le m} \int_{\{x: t_j < d(x_0,x) \le t_{j+1}\}} w^{1-p'}(x) \Big(\int_0^{d(x_0,x)} g(\tau)\,d\tau \Big)^{p'} d\mu \Big)^{q'/p'} \\ &\qquad \le \sum_{j\le m} \Big(\int_{\{x: t_j < d(x_0,x) \le t_{j+1}\}} w^{1-p'}(x) \Big(\int_0^{d(x_0,x)} g(\tau)\,d\tau \Big)^{p'} d\mu \Big)^{q'/p'}. \end{aligned}$$

This last expression can be estimated from above by

$$\begin{aligned} &\sum_{j\le m} \Big(\int_0^{t_{j+1}} g(\tau)\,d\tau \Big)^{q'} \Big(\int_{\{x: t_j < d(x_0,x) \le t_{j+1}\}} w^{1-p'}(x)\,d\mu \Big)^{q'/p'} \\ &= \sum_{j\le m} 2^{(j+1)q'} \Big(\int_{\{x: t_j < d(x_0,x) \le t_{j+1}\}} w^{1-p'}(x)\,d\mu \Big)^{q'/p'} \\ &\le 4^{q'} \sum_{j\le m} \Big(\int_{t_{j-1}}^{t_j} g^{q'}(\tau) v^{1-q'}(\tau)\,d\tau \Big) \Big(\int_{t_{j-1}}^{t_j} v(\tau)\,d\tau \Big)^{\frac{q'}{q}} \Big(\int_{\{x: t_j < d(x_0,x) \le t_{j+1}\}} w^{1-p'}(\tau)\,d\mu \Big)^{\frac{q'}{p'}} \\ &\le 4^{q'} B^{q'} \Big(\int_0^a g^{q'}(\tau) v^{1-q'}(\tau)\,d\tau \Big). \end{aligned}$$

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This shows that (1.4) implies (1.3). For the proof of the necessity part it is sufficient to take $f = w^{1-p'}\chi_{\{x:d(x_0,x)>s\}}$ in (1.3). ■

We remark that Theorems 1.1 and 1.2 extend results contained in [13], [14] and [10].

We conclude this section with two results dealing with limiting cases.

Proposition 1.1 *Let $a = \infty$ and suppose that v and w are as in Theorem 1.1. If*

$$\sup_{0<t<\infty}\Big(\int_t^\infty v(\tau)\,d\tau\Big)\operatorname*{ess\,sup}_{\{x:d(x_0,x)<2t\}}\frac{1}{w(\tau)} < \infty,$$

then there exists $c > 0$ such that for arbitrary $f \in L^1_w(X)$ we have

$$\int_0^\infty v(t)\Big|\int_{\{x:d(x_0,x)<t\}} f(x)\,d\mu\Big|\,dt \le \int_X |f(x)|w(x)\,d\mu.$$

Proof Change of order of integration and simple estimations give us

$$\begin{aligned}\int_0^\infty v(t)\Big|\int_{\{x:d(x_0,x)<t\}} f(x)\,d\mu\Big|\,dt \\ &\le \int_0^\infty v(t)\Big(\int_{\{x:d(x_0,x)<t\}}|f(x)|\,d\mu\Big)\,dt \\ &= \int_X |f(x)|\Big(\int_{d(x_0,x)}^\infty v(t)\,dt\Big)\frac{w(x)}{w(x)}\,d\mu \\ &\le \int_X |f(x)|\Big(\int_{d(x_0,x)}^\infty v(t)\,d\mu\Big)\Big(\operatorname*{ess\,sup}_{\{y:d(x_0,y)<2d(x_0,x)\}}\frac{1}{w(y)}\Big)w(x)\,d\mu \\ &\le c\int_X |f(x)|w(x)\,d\mu.\end{aligned}$$

Analogously we can prove

Proposition 1.2 *Let v and w be as in Theorem 1.1. If*

$$\sup_{0<t<a}\Big(\int_0^t v(\tau)\,d\tau\Big)\operatorname*{ess\,sup}_{\{x:d(x_0,x)>t/2\}}\frac{1}{w(x)} < \infty$$

then we have, for all $f \in L^1_w(X)$,

$$\int_0^a v(t)\Big|\int_{\{x:d(x_0,x)>t\}} f(x)\,d\mu\Big|\,dt \le c\int_X |f(x)|w(x)\,d\mu$$

with a constant c independent of f.

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2 Singular Integrals on SHT: Strong-Type Inequalities

A space of homogeneous type (SHT) (X, d, μ) is a topological space X, endowed with a complete measure μ, such that (a) the space of compactly supported continuous functions is dense in $L^1(X, \mu)$, and (b) there exists a non-negative real-valued function (quasimetric) $d\colon X \times X \to \mathbb{R}$ satisfying

(i) $d(x, x) = 0$ for all $x \in X$;
(ii) $d(x, y) > 0$ for all $x \neq y$, $x, y \in X$;
(iii) there is a constant $a_0 > 0$ such that $d(x, y) \leq a_0 d(y, x)$ for all $x, y \in X$;
(iv) there is a constant $a_1 \geq 1$ such that $d(x, y) \leq a_1\big(d(x, z) + d(z, y)\big)$ for all $x, y, z \in X$;
(v) for every neighbourhood V of x in X there exists $r > 0$ such that the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ is contained in V;
(vi) the balls $B(x, r)$ are measurable for every $x \in X$ and every $r > 0$;
(vii) there is a constant $b > 0$ such that $\mu\big(B(x, 2r)\big) \leq b\mu\big(B(x, r)\big) < \infty$ for every $x \in X$ and every r, $0 < r < \infty$.

In addition we shall suppose that there exists a point $x_0 \in X$ such that if $a = \sup\{d(x_0, x) : x \in X\}$, then for arbitrary t_1 and t_2, $0 < t_1 < t_2 < a$, we have

$$\mu\big(B(x_0, t_2) \setminus B(x_0, t_1)\big) > 0.$$

For the definition, various examples and properties of SHT see [2], [20]. There are numerous interesting examples of SHT, such as Euclidean space with an anisotropic distance and Lebesgue measure, any compact C^∞ Riemannian manifold with the Riemannian metric and volume, and the boundary of any bounded Lipschitz domain in $\mathbb{R}^n$ with the induced Euclidean metric and Lebesgue measure. From now on in this section, X will stand for an SHT with the properties listed above.

We shall also need the Muckenhoupt class $A_p(X)$. A μ-measurable, locally integrable function $w\colon X \to \mathbb{R}$ which is positive μ-a.e. is called a weight. If $1 < p < \infty$, then $A_p(X)$ is the set of all weights w such that

$$\sup\left(\frac{1}{\mu(B)}\int_B w(x)\, d\mu\right)\left(\frac{1}{\mu(B)}\int_B w^{-1/(p-1)}(x)\, d\mu\right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset X$. The Muckenhoupt class $A_1(X)$ is the set of all weights w such that

$$\sup\left(\frac{1}{\mu(B)}\int_B w(x)\, d\mu\right)\left(\operatorname*{ess\,sup}_{x\in B}\frac{1}{w(x)}\right) < \infty,$$

where again the supremum is taken over all balls B in X. Moreover, the maximal operator M is defined by

$$(Mf)(x) = \sup\big(\mu(B)\big)^{-1}\int_B |f(y)|\, d\mu(y), \quad x \in X,$$

where the supremum is taken over all balls B in X containing x.

Now we pass to the definition of singular integrals on SHT (see for example [2]). Let $k\colon (X \times X) \setminus \{(x,x) : x \in X\} \to \mathbb{R}$ be a measurable function satisfying the conditions:

$$|k(x,y)| \le \frac{c}{\mu\big(B(x,d(x,y))\big)}, \quad \text{for all } x,y \in X,\ x \neq y,$$

and

$$|k(x_1,y) - k(x_2,y)| + |k(y,x_1) - k(y,x_2)| \le c\omega\left(\frac{d(x_2,x_1)}{d(x_2,y)}\right)\frac{1}{\mu\Big(B\big(x_2,d(x_2,y)\big)\Big)},$$

for every $x_1, x_2, y \in X$ such that $d(x_2,y) > bd(x_1,x_2)$. Here ω is a positive, non-decreasing function on $(0,\infty)$, satisfying the well-known Δ_2-condition (that is, $\omega(2t) \le c\omega(t)$ for all $t > 0$ and some $c > 0$ independent of t) and the Dini condition

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty.$$

We assume as well that for some p_0, $1 < p_0 < \infty$, and all $f \in L^{p_0}_\mu(X)$ the limit

$$Kf(x) = \lim_{\varepsilon \to 0+} \int_{X \setminus B(x,\varepsilon)} k(x,y) f(y)\,d\mu$$

exists a.e. and that the operator K is bounded in $L^{p_0}_\mu(X)$. For the definition of singular integrals and other remarks see [19], Chapter I, pp. 29–36 and also [5], p. 295. The boundedness of the operator K under the above conditions for some $p_0 \in (1,\infty)$ guarantees the existence of the principal value Kf a.e. for all $f \in L^p_\mu(X)$ and the boundedness of K in $L^p_\mu(X)$ for all $p \in (1,\infty)$ (see, for example, [19], Chapter V, 6.17, p. 223 and also [8]).

Lemma 2.1 *Let $1 < p < \infty$, suppose that $\mu\{x_0\} = 0$, let w be a weight function on X, let $\rho \in A_p(X)$ and suppose that the following conditions are satisfied:*

(1) there exists an increasing function σ on $(0, 4a_1a)$, such that for some positive constant c_1,

$$\rho(x)\sigma\big(2a_1 d(x_0,x)\big) \le c_1 w(x) \quad \text{a.e.};$$

(2) for arbitrary t, $0 < t < a$, we have that

$$\int_{B(x_0,t)} w^{1-p'}(x)\,d\mu < \infty.$$

Then $K\phi(x)$ exists μ-a.e. for any $\varphi \in L^p_w(X)$.

Proof Let $0 < \alpha < \frac{a}{a_1}$ and put

$$S_\alpha = \Big\{x \in X : d(x_0,x) \ge \frac{\alpha}{2}\Big\}.$$

Let us suppose $\phi \in L^p_w(X)$. Then

$$\phi(x) = \phi_1(x) + \phi_2(x), \tag{2.1}$$

where $\phi_1 = \phi \chi_{S_\alpha}$ and $\phi_2 = \phi - \phi_1$. Using condition (1) it is easy to see that

$$\begin{aligned}\int_X |\phi_1(x)|^p \rho(x)\, d\mu &= \frac{\sigma(\frac{\alpha}{2})}{\sigma(\frac{\alpha}{2})} \int_{S_\alpha} |\phi(x)|^p \rho(x)\, d\mu \\ &\leq \frac{1}{\sigma(\frac{\alpha}{2})} \int_{S_\alpha} |\phi(x)|^p \rho(x) \sigma\big(2a_1 d(x_0, x)\big)\, d\mu \\ &\leq \frac{c_1}{\sigma(\frac{\alpha}{2})} \int_{S_\alpha} |\phi(x)|^p w(x)\, d\mu < \infty\end{aligned}$$

for arbitrary α, $0 < \alpha < \frac{a}{a_1}$. Consequently for such α, $K\phi_1 \in L^p_\rho(X)$ (see, for example, [8] and [19], Section 6.13, p. 21) and $K\phi_1(x)$ exists μ-a.e. on X.

Now let x be such that $d(x_0, x) > \alpha a_1$ (the constant a_1 appears in the definition of X). If $y \in X$ and $d(x_0, y) < \frac{\alpha}{2}$ then

$$d(x_0, x) \leq a_1\big(d(x_0, y) + d(y, x)\big) \leq a_1\big(d(x_0, y) + a_0 d(x, y)\big).$$

Hence

$$d(x, y) \geq \frac{1}{a_0 a_1} d(x_0, x) - \frac{1}{a_0} d(x_0, y) \geq \frac{\alpha}{a_0} - \frac{\alpha}{2a_0} = \frac{\alpha}{2a_0}.$$

In addition

$$\mu\Big(B\big(x_0, d(x, y)\big)\Big) \leq c\mu\Big(B\big(x, d(x, y)\big)\Big).$$

In fact for $z \in B\big(x_0, d(x, y)\big)$ we have

$$d(x, z) \leq a_1\big(d(x, x_0) + d(x_0, z)\big) \leq a_1\big(d(x, x_0) + d(x, y)\big).$$

On the other hand

$$\begin{aligned}d(x, x_0) &\leq a_1\big(d(x, y) + d(y, x_0)\big) \leq a_1\big(d(x, y) + a_0 d(x_0, y)\big) \\ &\leq a_1\Big(d(x, y) + \frac{a_0 \alpha}{2}\Big) \leq a_1\big(d(x, y) + a_0^2 d(x, y)\big) = a_1(1 + a_0^2)\, d(x, y)\end{aligned}$$

and

$$d(x, z) \leq a_1\big(a_1(1 + a_0^2)\, d(x, y) + d(x, y)\big) = a_1\big(1 + a_1(1 + a_0^2)\big)\, d(x, y).$$

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Hence

$$B\big(x_0, d(x,y)\big) \subset B\Big(x, a_1(1+a_1(1+a_0^2))\, d(x,y)\Big).$$

By the doubling condition (vii) above we conclude that

$$\mu\Big(B\big(x_0, d(x,y)\big)\Big) \le c\mu\Big(B\big(x, d(x,y)\big)\Big). \tag{2.2}$$

For $K\phi_2$ by (2.2) and the Hölder inequality we have

$$\begin{aligned}
|K\phi_2(x)| &= \Big|\int_X \phi_2(y)k(x,y)\,d\mu\Big| \le c\int_{B(x_0,\frac{\alpha}{2})} \frac{|\phi(y)|}{\mu\Big(B\big(x,d(x,y)\big)\Big)}\,d\mu \\
&\le c\int_{B(x_0,\frac{\alpha}{2})} \frac{|\phi(y)|}{\mu\Big(B\big(x_0,d(x,y)\big)\Big)}\,d\mu \le \frac{c}{\mu\big(B(x_0,\frac{\alpha}{2a_0})\big)}\int_{B(x_0,\frac{\alpha}{2})}|\phi(y)|\,d\mu \\
&\le \frac{c}{\mu B(x_0,\frac{\alpha}{2a_0})}\Big(\int_{B(x_0,\frac{\alpha}{2})}|\phi(y)|^p w(y)\,d\mu\Big)^{1/p}\Big(\int_{B(x_0,\frac{\alpha}{2})} w^{1-p'}(y)\,d\mu\Big)^{1/p'} \\
&< \infty.
\end{aligned}$$

Thus $K\phi(x)$ is absolutely convergent for arbitrary x such that $d(x_0,x) > \alpha a_1$. We can take α arbitrarily small and as $\mu\{x_0\} = 0$ we conclude that $K\phi_2(x)$ converges absolutely μ-a.e. on X. By (2.1), $K\phi(x)$ exists a.e. on X.

Theorem 2.1 *Let $1 < p < \infty$, suppose that $\mu\{x_0\} = 0$, let σ be a positive continuous increasing function on $(0, 4a_1a)$, let $\rho \in A_p(X)$ and suppose that w is a weight function on X. Let $v(x) = \sigma\big(d(x_0,x)\big)\rho(x)$ and suppose that the following conditions are fulfilled:*

(i) there exists $c > 0$ such that

$$\sigma\big(2a_1 d(x_0,x)\big)\rho(x) \le c w(x) \quad \mu\text{-a.e.}; \tag{2.3}$$

(ii)

$$\sup_{0<t<a}\Bigg(\int_{X\setminus B(x_0,t)} \frac{v(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^p}\Bigg)\Big(\int_{B(x_0,t)} w^{1-p'}(x)\,d\mu\Big)^{p-1} < \infty. \tag{2.4}$$

Then there exists a constant $c > 0$ such that for any $f \in L^p_w(X)$ we have

$$\int_X |Kf(x)|^p v(x)\,d\mu \le c\int_X |f(x)|^p w(x)\,d\mu. \tag{2.5}$$

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Proof Without loss of generality we can suppose that σ may be represented by

$$\sigma(t) = \sigma(0+) + \int_0^t \phi(\tau)\, d\tau, \quad \phi \geq 0.$$

In fact there exists a sequence of absolutely continuous functions σ_n such that $\sigma_n(t) \leq \sigma(t)$ and $\lim_{n\to\infty} \sigma_n(t) = \sigma(t)$ for any $t \in (0, 4a_1 a)$. For such functions we may take

$$\sigma_n(t) = \sigma(0+) + n \int_0^t \Big[\sigma(\tau) - \sigma\Big(\tau - \frac{1}{n}\Big)\Big]\, d\tau.$$

We have

$$\begin{aligned}\int_X |Kf(x)|^p v(x)\, d\mu &= \sigma(0+) \int_X |Kf(x)|^p \rho(x)\, d\mu \\ &\quad + \int_X |Kf(x)|^p \rho(x) \Big(\int_0^{d(x_0,x)} \phi(t)\, dt\Big)\, d\mu = I_1 + I_2.\end{aligned}$$

If $\sigma(0+) = 0$ then $I_1 = 0$. If $\sigma(0+) \neq 0$ by the boundedness of K in $L^p_\rho(X)$ thanks to (2.3)

$$\begin{aligned}I_1 &\leq c\sigma(0+) \int_X |f(x)|^p \rho(x)\, d\mu \\ &\leq c \int_X |f(x)|^p \rho(x) \sigma\big(2a_1 d(x_0, x)\big)\, d\mu \leq c \int_X |f(x)|^p w(x)\, dx.\end{aligned} \tag{2.6}$$

After changing the order of integration in I_2 we have

$$\begin{aligned}I_2 &= \int_0^a \phi(t) \Big(\int_{\{x:d(x_0,x)>t\}} |Kf(x)|^p \rho(x)\, d\mu\Big)\, dt \\ &\leq c \int_0^a \phi(t) \Big(\int_{\{x:d(x_0,x)>t\}} \rho(x) \Big| \int_{\{y:d(x_0,y)>\frac{t}{2a_1}\}} f(y) k(x,y)\, d\mu\Big|^p d\mu\Big)\, dt \\ &\quad + c \int_0^a \phi(t) \Big(\int_{\{x:d(x_0,x)>t\}} \rho(x) \Big| \int_{\{y:d(x_0,y)\leq\frac{t}{2a_1}\}} f(y) k(x,y)\, d\mu\Big|^p d\mu\Big)\, dt \\ &= I_{21} + I_{22}.\end{aligned}$$

Using the boundedness of K in $L^p_\rho(x)$ we obtain

$$\begin{aligned}I_{21} &\leq c \int_0^a \phi(t) \Big(\int_{\{y:d(x_0,y)>\frac{t}{2a_1}\}} |f(y)|^p \rho(y)\, d\mu\Big)\, dt \\ &\leq c \int_X |f(y)|^p \rho(y) \Big(\int_0^{2a_1 d(x_0,y)} \phi(t)\, dt\Big)\, d\mu \\ &\leq c \int_X |f(y)|^p \rho(y) \sigma\big(2a_1 d(x_0, y)\big)\, d\mu \\ &\leq c \int_X |f(y)|^p w(y)\, d\mu.\end{aligned} \tag{2.7}$$

Now we estimate I_{22}. When $d(x_0,x) > t$ and $d\Big(x_0,y\Big) \le \frac{t}{2a_1}$ we have

$$\begin{aligned} d(x_0,x) &\le a_1\big(d(x_0,y)+d(y,x)\big) \le a_1\big(d(x_0,y)+a_0 d(x,y)\big) \\ &\le a_1\Big(\frac{t}{2a_1}+a_0 d(x,y)\Big) \le a_1\Big(\frac{d(x_0,x)}{2a_1}+a_0 d(x,y)\Big). \end{aligned}$$

Hence

$$\frac{d(x_0,x)}{2a_1a_0} \le d(x,y)$$

and

$$\mu\Big(B\big(x,d(x_0,x)\big)\Big) \le b\mu\Big(B\Big(x,\frac{d(x_0,x)}{2a_1a_0}\Big)\Big) \le b\mu\Big(B\big(x,d(x,y)\big)\Big). \tag{2.8}$$

As in the preceding Lemma 2.1 we conclude that

$$\mu\Big(B\big(x_0,d(x_0,x)\big)\Big) \le b\mu\Big(B\big(x,d(x_0,x)\big)\Big)$$

and therefore from (2.8) we have

$$\mu\Big(B\big(x_0,d(x_0,x)\big)\Big) \le b\mu\Big(B\big(x,d(x,y)\big)\Big). \tag{2.9}$$

Using (2.9) we derive the inequalities

$$\begin{aligned} I_{22} &\le c\int_0^a \phi(t)\Big(\int_{\{x:d(x_0,y)>t\}} \rho(x)\Big(\int_{\{y:d(x_0,y)\le\frac{t}{2a_1}\}} \frac{|f(y)|}{\mu\Big(B\big(x,d(x,y)\big)\Big)}\Big)^p d\mu\Big)\,dt \\ &\le c\int_0^a \phi(t)\Big(\int_{\{x:d(x_0,x)>t\}} \frac{\rho(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^p}\Big)\Big(\int_{B(x_0,t)} |f(y)|\,d\mu\Big)^p\,dt. \end{aligned}$$

It is easy to see that for any s, $0 < s < a$, we have

$$\begin{aligned} &\int_s^a \phi(t)\Big(\int_{\{x:d(x_0,x)>t\}} \frac{\rho(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^p}\,d\mu\Big)\,dt \\ &\quad\le \int_{\{x:d(x_0,x)\ge s\}} \frac{\rho(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^p}\cdot\Big(\int_s^{d(x_0,x)} \phi(t)\,dt\Big)\,d\mu \\ &\quad\le \Big(\int_{\{x:d(x_0,x)\ge s\}} \frac{\rho(x)\sigma\big(d(x_0,x)\big)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^p}\Big)\,d\mu. \end{aligned}$$

Now applying Theorem 1.1 we conclude that

$$I_{22} \le c \int_X |f(x)|^p w(x)\, d\mu.$$

Finally from the last estimate and (2.7) we obtain (2.5). ■

Corollary 2.1 *Let $1 < p < \infty$, suppose that $\mu\{x_0\} = 0$, let σ and u be positive, increasing functions on $(0, 4a_1a)$, let $\rho \in A_p(X)$ and put $v(x) = \sigma\big(d(x_0,x)\big)\rho(x)$, $w(x) = u\big(d(x_0,x)\big)\rho(x)$. Then the inequality (2.5) is valid under the following two conditions:*

there exists a positive number b such that

$$\sigma\Big(2a_1 t\Big) \le bu(t) \tag{2.10}$$

for any $t \in (0, 2a)$;

$$\sup_{0<t<a} \left(\int_{X\setminus B(x_0,t)} \frac{v(x)}{\Big(\mu\Big(B\big(x_0, d(x_0,x)\big)\Big)\Big)^p}\, d\mu\right)\left(\int_{B(x_0,t)} w^{1-p'}(x)\, d\mu\right)^{p-1} < \infty. \tag{2.11}$$

Now we are going to consider the case when the weight on the left side is decreasing. We shall assume that throughout the rest of Section 2 and in Section 3,

$$a = \sup\{d(x_0,x) : x \in X\} = \infty.$$

Lemma 2.2 *Let $1 < p < \infty$, let σ be a positive decreasing function on $(0,\infty)$, let $\rho \in A_p$ and let w be a weight function. Suppose the following conditions are fulfilled:*

(i) there exists a positive constant b such that

$$\sigma\left(\frac{d(x_0,x)}{2a_1}\right)\rho(x) \le bw(x) \quad \text{a.e.} \tag{2.12}$$

(ii)

$$\int_{X\setminus B(x_0,t)} w^{1-p'}(x)\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{-p'} d\mu < \infty \tag{2.13}$$

for any $t > 0$.

Then $K\phi(x)$ exists μ-a.e. for arbitrary $\phi \in L^p_w(X)$.

Proof Fix arbitrarily $\alpha > 0$ and let

$$S_\alpha = \{x : d(x_0,x) \ge \alpha\}.$$

Write

$$\phi(x) = \phi_1(x) + \phi_2(x),$$

where $\phi_1(x) = \phi(x)\chi_{S_\alpha}(x)$ and $\phi_2(x) = \phi(x) - \phi_1(x)$.

For ϕ_2 we have

$$\begin{aligned}\int_X |\phi_2(x)|^p \rho(x)\, d\mu &= \frac{\sigma(\alpha)}{\sigma(\alpha)} \int_{B(x_0,\alpha)} |\phi(x)|^p \rho(x)\, d\mu \\ &\le \frac{1}{\sigma(\alpha)} \int_{B(x_0,\alpha)} |\phi(x)|^p \rho(x) \sigma\big(d(x_0,x)\big)\, d\mu \\ &\le \frac{c_1}{\sigma(\alpha)} \int_{B(x,\alpha)} |\phi(x)|^p w(x)\, d\mu < \infty.\end{aligned}$$

Hence $\phi_2 \in L^p_\rho(X)$ and so $K\phi_2 \in L^p_\rho(X)$. From this we see that $K\phi_2(x)$ exists almost everywhere.

Now let $x \in X$ be such that $d(x_0,x) < \frac{\alpha}{2a_1}$. If $d(x_0,y) \ge \alpha$ then

$$d(x_0,y) \le a_1\big(d(x_0,x) + d(x,y)\big)$$

and so

$$d(x,y) \ge \frac{1}{a_1} d(x_0,y) - d(x_0,x) \ge \frac{\alpha}{a_1} - \frac{\alpha}{2a_1} = \frac{\alpha}{2a_1}.$$

Moreover, it is easy to prove that

$$\mu\Big(B\big(x_0, d(x_0,y)\big)\Big) \le b\mu\Big(B\big(x, d(x,y)\big)\Big).$$

From these inequalities we obtain estimates for ϕ_1:

$$\begin{aligned}|K\phi_1(x)| &\le c \int_{S_\alpha} \frac{|\phi(y)|}{\mu\Big(B\big(x,d(x,y)\big)\Big)}\, d\mu \le c \int_{S_\alpha} \frac{|\phi(y)|}{\mu\Big(B\big(x_0,d(x_0,y)\big)\Big)}\, d\mu \\ &\le c\Big(\int_{S_\alpha} |\phi(y)|^p w(y)\, d\mu\Big)^{1/p} \Big(\int_{S_\alpha} w^{1-p'} \Big(\mu\Big(B\big(x_0, d(x_0,y)\big)\Big)\Big)^{-p'} d\mu\Big)^{1/p'} \\ &< \infty.\end{aligned}$$

As we may take α arbitrarily large we conclude that $K\phi_1(x)$ and consequently $K\phi(x)$, exist a.e.

Theorem 2.2 *Let $1 < p < \infty$, suppose that $\mu\{x_0\} = 0$, let σ be a positive continuous decreasing function on $(0,\infty)$, let $\rho \in A_p(X)$, $v(x) = \sigma\big(d(x_0,x)\big)\rho(x)$ and suppose that w is a weight function. Assume that the following two conditions are fulfilled:*

(i) there exists a positive c such that

$$\sigma\left(\frac{d(x_0,x)}{2a_1}\right)\rho(x) \le cw(x); \tag{2.14}$$

(ii)

$$\sup_{t>0}\left(\int_{B(x_0,t)} v(x)\,d\mu\right)\left(\int_{X\setminus B(x_0,t)} w^{1-p'}(x)\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{-p'}\,d\mu\right)^{p-1} < \infty. \tag{2.15}$$

Then the inequality (2.5) is valid.

Proof Without loss of generality we suppose that σ is representable as

$$\sigma(t) = \sigma(+\infty) + \int_t^\infty \phi(\tau)\,d\tau.$$

In fact there exists a sequence of decreasing absolutely continuous functions such that $\sigma_n(t) \le \sigma(t)$ and $\lim_{n\to\infty}\sigma_n(t) = \sigma(t)$ for any t. For example, we may take

$$\sigma_n(t) = \sigma(+\infty) + n\int_t^\infty \left[\sigma(\tau) - \sigma\Big(\tau + \frac{1}{n}\Big)\right] d\tau.$$

It is easy to see that

$$\sigma_n(t) = n\int_t^{t+\frac{1}{n}} \sigma(\tau)\,d\tau.$$

Moreover $\lim_{n\to\infty}\sigma_n(t) = \sigma(t)$ for any t by virtue of the continuity of σ. On the other hand $\sigma_n(t) \le \sigma(t)$ for any $t > 0$. Hence

$$\begin{aligned}\int_X |Kf(x)|^p v(x)\,dx &= \sigma(+\infty)\int_X |Kf(x)|^p \rho(x)\,d\mu \\ &\quad + \int_X |Kf(x)|^p\left(\int_{d(x_0,x)}^\infty \phi(t)\,dt\right) d\mu = I_1 + I_2.\end{aligned}$$

If $\sigma(+\infty) = 0$ then $I_1 = 0$. But if $\sigma(+\infty) \ne 0$ by virtue of the boundedness of K in $L^p_\rho(X)$ we have

$$\begin{aligned}I_1 &\le c\sigma(+\infty)\int_X |f(x)|^p\rho(x)\,d\mu \\ &\le c\int_X |f(x)|^p\rho(x)\sigma\left(\frac{d(x_0,x)}{2a_1}\right)d\mu \le c\int_X |f(x)|^p w(x)\,d\mu.\end{aligned} \tag{2.16}$$

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Now we pass to I_2:

$$
\begin{aligned}
I_2 &\le \int_0^\infty \phi(t)\Big(\int_{B(x_0,t)} |Kf(x)|^p \rho(x)\,d\mu\Big)\,dt \\
&\le c\int_0^\infty \phi(t)\Big(\int_{B(x_0,t)} \rho(x)\Big|\int_{B(x_0,2a_1t)} f(y)k(x,y)\,d\mu\Big|^p\,d\mu\Big) \\
&\quad + c\int_0^\infty \phi(t)\Big(\int_{B(x_0,t)} \rho(x)\Big|\int_{X\setminus B(x_0,2a_1t)} f(y)k(x,y)\,d\mu\Big|^p\,d\mu\Big)\,dt = I_{21}+I_{22}.
\end{aligned}
$$

Again since K is bounded in $L^p_\rho(X)$ we obtain:

$$
\begin{aligned}
I_{21} &\le c\int_0^\infty \phi(t)\Big(\int_{B(x_0,2a_1t)} |f(y)|^p\rho(y)\,d\mu\Big)\,dt \\
&= c\int_X |f(y)|^p\rho(y)\Big(\int_{d(x_0,x)}^\infty \phi(t)\,dt\Big)\,d\mu \le c\int_X |f(y)|^p w(y)\,d\mu.
\end{aligned}
\tag{2.17}
$$

It remains to estimate I_{22}. When $x \in B(x_0,t)$ and $y \in X\setminus B(x_0,2a_1t)$ we have

$$\mu\Big(B\big(x_0,d(x_0,y)\big)\Big) \le b\mu\Big(B\big(x,d(x,y)\big)\Big).$$

In fact,

$$d(x_0,y) \le a_1\big(d(x_0,x)+d(x,y)\big) \le a_1\big(t+d(x,y)\big) \le a_1\left(\frac{d(x_0,y)}{2a_1}+d(x,y)\right).$$

Hence

$$\frac{d(x_0,y)}{2a_1} \le d(x,y)$$

and

$$\mu\Big(B\big(x,d(x_0,y)\big)\Big) \le b\mu\Big(B\big(x,d(x,y)\big)\Big). \tag{2.18}$$

In addition

$$\mu\Big(B\big(x_0,d(x_0,y)\big)\Big) \le b\mu\Big(B\big(x,d(x_0,y)\big)\Big). \tag{2.19}$$

For if $z \in B\big(x_0,d(x_0,y)\big)$, then

$$
\begin{aligned}
d(x,z\big) &\le a_1\big(d(x,x_0)+d(x_0,z)\big) \le a_1\big(a_0 d(x_0,x)+d(x_0,y)\big) \\
&\le a_1\big(a_0 t + d(x_0,y)\big) \le a_1\Big(a_0\frac{d(x_0,y)}{2a_1}+d(x_0,y)\Big) \\
&= a_1\Big(\frac{a_0}{2a_1}+1\Big)\,d(x_0,y).
\end{aligned}
$$

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From this by the doubling condition we obtain (2.19). Using the inequalities obtained we derive the estimates:

$$I_{22} \le c \int_0^\infty \phi(t)\Big(\int_{B(x_0,t)} \rho(x)\,d\mu\Big)\Big(\int_{X\setminus B(x_0,2a_1t)} |f(y)|\mu\Big(B\big(x_0,d(x_0,y)\big)\Big)\,d\mu\Big)^p\,dt$$
$$\le c \int_0^\infty \phi(t)\Big(\int_{B(x_0,t)} \rho(x)\,d\mu\Big)\Big(\int_{\{y:d(x_0,y)>t\}} |f(y)|\mu\Big(B\big(x_0,d(x_0,y)\big)\Big)\,d\mu\Big)^p\,dt.$$

In addition

$$\int_0^s \phi(t)\Big(\int_{B(x_0,t)} \rho(x)\,d\mu\Big)\,dt = \int_{B(x_0,s)} \rho(x)\Big(\int_{d(x_0,x)}^s \phi(t)\,dt\Big)\,d\mu.$$

Now application of Theorem 1.2 gives the desired inequality (2.5). ■

Corollary 2.2 *Let $1 < p < \infty$, $\mu\{x_0\} = 0$, let σ and u be positive decreasing functions on $(0,\infty)$ with σ continuous, let $\rho \in A_p(X)$, put $v(x) = \sigma\big(d(x_0,x)\big)\rho(x)$, $w(x) = u\big(d(x_0,x)\big)\rho(x)$ and suppose that the following two conditions are fulfilled:*

(i) there exists a positive number b_1 such that

$$\sigma\Big(\frac{t}{2a_1}\Big) \le b_1 u(t) \tag{2.20}$$

for any $t > 0$;
(ii)

$$\sup_{t>0}\Big(\int_{B(x_0,t)} v(x)\,d\mu\Big)\Big(\int_{X\setminus B(x_0,t)} w^{1-p'}(x)\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{-p'}\,d\mu\Big)^{p-1} < \infty. \tag{2.21}$$

Then (2.5) is valid.

In the sequel we investigate the cases when the condition (2.11) ((2.21)) implies (2.10) ((2.20)).

We prove a preliminary

Lemma 2.3 *Let $1 < p < \infty$, let $\rho \in A_p(X)$ and suppose $0 < c_1 \le c_2 < c_3 < \infty$. Then there exists a positive number c such that for any $t > 0$ we have*

$$\int_{B(x_0,c_3t)\setminus B(x_0,c_2t)} \rho(x)\,d\mu \le c\int_{B(x_0,c_1t)} \rho(x)\,d\mu.$$

Proof By the definition of the maximal function M and the doubling condition we have

$$M\phi(x) \ge \Big(\frac{1}{\mu\big(B(x_0,c_3t)\big)}\int_{B(x_0,c_3t)} |\phi(y)|\,d\mu\Big)\chi_{B(x_0,c_3t)\setminus B(x_0,c_2t)}(x) \tag{2.22}$$
$$\ge \Big(\frac{b_1}{\mu\big(B(x_0,c_1t)\big)}\int_{B(x_0,c_1t)} |\phi(y)|\,d\mu\Big)\chi_{B(x_0,c_3t)\setminus B(x_0,c_2t)}(x)$$

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for any $\phi \in L^p_\rho(X)$. From (2.22), in view of the boundedness of the operator M in $L^p_\rho(X)$ (see [19]) we obtain

$$\int_{B(x_0,c_3t)\setminus B(x_0,c_2t)} \left(\frac{1}{\mu\big(B(x_0,c_1t)\big)} \int_{B(x_0,c_1t)} |\phi(y)|\, d\mu \right)^p \rho(x)\, d\mu \le c \int_X |\phi(y)|^p \rho(y)\, d\mu.$$

If in the last inequality we take $\phi(y) = \chi_{B(x_0,c_1t)}(y)$, the desired estimate follows. ■

Definition Let $a = \infty$. A measure μ satisfies the reverse doubling condition ($\mu \in \mathrm{RD}$) if there exist constants $\eta_1 > 1$ and $\eta_2 > 1$ such that

$$\mu\big(B(x,\eta_1 r)\big) \ge \eta_2 \mu\big(B(x,r)\big)$$

for any $x \in X$ and all $r > 0$.

For this definition and its connection with the doubling condition see, for example, [18] and also [20], p. 11. As a measure with the doubling condition satisfies the reverse doubling condition as well therefore we are able to show that from (2.11) ((2.21)) automatically follows (2.10) ((2.20)).

Theorem 2.3 *Let $1 < p < \infty$, suppose that $\mu\{x_0\} = 0$, let σ and u be positive increasing functions on $(0,\infty)$ with σ continuous, let $\rho \in A_p(X)$, put $v(x) = \sigma\big(d(x_0,x)\big)\rho(x)$, $w(x) = u\big(d(x_0,x)\big)\rho(x)$ and suppose that*

$$\sup_{t>0} \left(\int_{X\setminus B(x_0,t)} \frac{v(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^p}\, d\mu \right) \left(\int_{B(x_0,t)} w^{1-p'}(x)\, d\mu \right)^{p-1} < \infty. \tag{2.23}$$

Then (2.5) holds.

Proof By Corollary 2.1 it is sufficient to prove that (2.23) implies that, given $\beta > 1$, there is a positive constant b such that

$$\sigma(\beta t) \le b u(t) \tag{2.24}$$

for all $t > 0$.

Let $\eta \ge \eta_1 > 0$, where η_1 is as in the definition of the (RD) condition. In view of (RD) we have

$$\begin{aligned}
\mu\big(B(x_0,\eta\beta t)\setminus B(x_0,\beta t)\big) &= \mu\big(B(x_0,\eta\beta t)\big) - \mu\big(B(x_0,\beta t)\big)\\
&\ge \mu\big(B(x_0,\eta\beta t)\big) - \frac{1}{\eta_2}\mu\big(B(x_0,\eta\beta t)\big)\\
&\ge \left(1 - \frac{1}{\eta_2}\right)\mu\big(B(x_0,\eta\beta t)\big).
\end{aligned}$$

Hence

$$\mu\big(B(x_0,\eta\beta t)\setminus B(x_0,\beta t)\big) \ge b\mu\big(B(x_0,\eta\beta t)\big). \tag{2.25}$$

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Since σ and u are monotone, we have

$$
\begin{aligned}
(2.26)\qquad \Big(\int_{X\setminus B(x_0,t)} v(x)\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{-p}\,d\mu\Big)
&\geq \int_{X\setminus B(x_0,\beta t)} \sigma\big(d(x_0,x)\big)\rho(x)\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{-p}\,d\mu \\
&\geq \int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)} \frac{\sigma\big(d(x_0,x)\big)\rho(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p}}\,d\mu \\
&\geq \sigma(\beta t)\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)} \frac{\rho(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p}}\,d\mu
\end{aligned}
$$

and

$$
\begin{aligned}
(2.27)\qquad \Big(\int_{B(x_0,t)} w^{1-p'}(x)\,d\mu\Big)^{p-1} &= \Big(\int_{B(x_0,t)} u^{1-p'}\big(d(x_0,x)\big)\rho^{1-p'}(x)\,d\mu\Big)^{p-1} \\
&\geq \frac{1}{u(t)}\Big(\int_{B(x_0,t)} \rho^{1-p'}(x)\,d\mu\Big)^{p-1}.
\end{aligned}
$$

Using Hölder's inequality, Lemma 2.3, (2.25), (2.26) and (2.27) it follows that

$$
\begin{aligned}
\frac{\sigma(\beta t)}{u(t)} &= \frac{\sigma(\beta t)}{u(t)}\Big(\frac{1}{\mu\big(B(x_0,\eta\beta t)\setminus B(x_0,\beta t)\big)}\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)} \rho^{1/p}(x)\rho^{-1/p}(x)\,d\mu\Big)^{p} \\
&\leq \frac{\sigma(\beta t)}{u(t)\Big(\mu\big(B(x_0,\eta\beta t)\setminus B(x_0,\beta t)\big)\Big)^{p}}\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)} \rho(x)\,d\mu \\
&\qquad\cdot\Big(\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)} \rho^{1-p'}(x)\,d\mu\Big)^{p-1} \\
&\leq b\frac{\sigma(\beta t)}{u(t)\Big(\mu\big(B(x_0,\eta\beta t)\big)\Big)^{p}}\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)} \rho(x)\,d\mu\Big(\int_{B(x_0,t)} \rho^{1-p'}(x)\,d\mu\Big)^{p-1} \\
&\leq b\frac{\sigma(\beta t)}{u(t)}\Big(\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)} \rho(x)\Big(\mu\big(B(x_0,\eta\beta t)\big)\Big)^{-p}\,d\mu\Big)\Big(\int_{B(x_0,t)} \rho^{1-p'}(x)\,d\mu\Big)^{p-1} \\
&\leq b\frac{\sigma(\beta t)}{u(t)}\Big(\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)} \rho(x)\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{-p}\,d\mu\Big) \\
&\qquad\cdot\Big(\int_{B(x_0,t)} \rho^{1-p'}(x)\,d\mu\Big)^{p-1} \\
&\leq b\Big(\int_{X\setminus B(x_0,t)} \frac{v(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p}}\,d\mu\Big)\Big(\int_{B(x_0,t)} w^{1-p'}(x)\,d\mu\Big)^{p-1} \leq c.
\end{aligned}
$$

Finally by Corollary 2.1 we obtain (2.5). ■

Analogously we can prove

Theorem 2.4 *Let $1 < p < \infty$, suppose $\mu\{x_0\} = 0$, let σ and u be positive decreasing functions, σ be continuous, $\rho \in A_p(X)$ and put $v(x) = \sigma\big(d(x_0, x)\big)\rho(x)$, $w(x) = u\big(d(x_0, x)\big)\rho(x)$. Suppose that*

$$\sup_{t>0}\Big(\int_{B(x_0,t)} v(x)\,d\mu\Big)\Big(\int_{X\setminus B(x_0,t)} \frac{w^{1-p'}(x)}{\Big(\mu\Big(B\big(x_0, d(x_0,x)\big)\Big)\Big)^{p'}}\,d\mu\Big)^{p-1} < \infty. \tag{2.28}$$

Then (2.5) holds.

We shall now discuss the following question: if a pair (σ, u) of positive increasing (decreasing) functions satisfies the condition (2.23) ((2.28)) with $\rho \equiv 1$, then for which functions $\rho \in A_p(X)$ does (2.5) remain valid? It is evident that not all ρ in $A_p(X)$ have this property. Nevertheless we have

Theorem 2.5 *Let $1 < p < \infty$, let $\mu\{x_0\} = 0$. Let σ and u be positive increasing functions on $(0,\infty)$, with σ continuous. If*

(2.29)

$$\sup_{t>0}\Big(\int_{X\setminus B(x_0,t)} \frac{\sigma\big(d(x_0,x)\big)}{\Big(\mu\Big(B\big(x_0, d(x_0,x)\big)\Big)\Big)^{p}}\,d\mu\Big)\Big(\int_{B(x_0,t)} u^{1-p'}\big(d(x_0,x)\big)\,d\mu\Big)^{p-1} < \infty$$

and $\rho \in A_1(X)$ then we have

$$\int_X |Kf(x)|^p \sigma\big(d(x_0,x)\big)\rho(x)\,d\mu \le c\int_X |f(x)|^p u\big(d(x_0,x)\big)\rho(x)\,d\mu. \tag{2.30}$$

Proof Let η_1 and η_2 be as in the definition of condition (RD). Then we have

$$\mu\big(B(x_0, \eta_1^{k+1}t) \setminus B(x_0, \eta_1^k t)\big) \ge (\eta_2 - 1)\mu\big(B(x_0, \eta_1^k t)\big)$$

for any non-negative integer k.

From this and the doubling condition for μ we derive

$$\begin{aligned} &\int_{B(x_0,\eta_1^{k+1}t)\setminus B(x_0,\eta_1^k t)} \rho(x)\Big(\mu\Big(B\big(x_0, d(x_0,x)\big)\Big)\Big)^{-p}\,d\mu \\ &\qquad\le \frac{b^{p-1}}{\Big(\mu\big(B(x_0,\eta_1^k t)\big)\Big)^{p-1}}\frac{1}{\mu\big(B(x_0,\eta_1^{k+1}t)\big)}\int_{B(x_0,\eta_1^{k+1}t)} \rho(x)\,d\mu. \end{aligned} \tag{2.31}$$

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Now using (2.31) and the A_1 condition for ρ we obtain the following estimates, in which

$$
\begin{aligned}
A = &\operatorname*{ess\,sup}_{x\in B(x_0,\eta_1^{k+1}t)} \frac{1}{\rho(x)} \Big(\int_{B(x_0,\eta_1^{k+1}t)} u^{1-p'}\big(d(x_0,x)\big)\,d\mu\Big)^{p-1} : \\
&\int_{X\setminus B(x_0,t)} \frac{\sigma\big(d(x_0,x)\big)\rho(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p}}\,d\mu \Big(\int_{B(x_0,t)} u^{1-p'}\big(d(x_0,x)\big)\rho^{1-p'}(x)\,d\mu\Big)^{p-1} \\
&\le A\sum_{k=0}^{\infty}\int_{B(x_0,\eta_1^{k+1}t)\setminus B(x_0,\eta_1^{k}t)} \frac{\sigma\big(d(x_0,x)\big)\rho(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p}}\,d\mu \\
&\le Ab\sum_{k=0}^{\infty}\frac{\sigma(\eta_1^{k+1}t)}{\Big(\mu\big(B(x_0,\eta_1^{k}t)\big)\Big)^{p-1}}\frac{1}{\Big(\mu\big(B(x_0,\eta_1^{k+1}t)\big)\Big)}\int_{B(x_0,\eta_1^{k+1}t)}\rho(x)\,d\mu \\
&\le b\sum_{k=0}^{\infty}\int_{B(x_0,\eta_1^{k+2}t)\setminus B(x_0,\eta_1^{k+1}t)} \frac{\sigma\big(d(x_0,x)\big)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p}}\,d\mu \\
&\qquad\cdot\Big(\int_{B(x_0,\eta_1^{k+2}t)} u^{1-p'}\big(d(x_0,x)\big)\,d\mu\Big)^{p-1} \\
&\le b.
\end{aligned}
$$

Finally with the help of Theorem 2.3 we obtain (2.30). ■

Theorem 2.6 *Let $1 < p < \infty$, $\mu\{x_0\} = 0$, let σ and u be decreasing functions on $(0,\infty)$ with σ continuous, and suppose that*

$$
\sup_{t>0}\Big(\int_{B(x_0,t)}\sigma\big(d(x_0,x)\big)\,d\mu\Big)\Big(\int_{X\setminus B(x_0,t)}\frac{u^{1-p'}\big(d(x_0,x)\big)}{\Big(\mu\big(B\big(x_0,d(x_0,x)\big)\big)\Big)^{p'}}\,d\mu\Big)^{p-1} < \infty. \tag{2.32}
$$

Then if $\rho \in A_1(X)$ we have the inequality

$$
\int_X |Kf(x)|^p\sigma\big(d(x_0,x)\big)\rho^{1-p}(x)\,d\mu \le c\int_X |f(x)|^p u\big(d(x_0,x)\big)\rho^{1-p}(x)\,d\mu \tag{2.33}
$$

with a constant c independent of f.

Proof Again let η_1 and η_2 be as in the definition of the reverse doubling condition. By the A_1 condition for ρ, the doubling and the (RD) conditions and (2.32) we obtain the

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following chain of inequalities:

$$\begin{aligned}
&\Big(\int_{B(x_0,t)} \sigma\big(d(x_0,x)\big)\rho^{1-p}(x)\,d\mu\Big)^{p'-1}\Big(\int_{X\setminus B(x_0,t)} \frac{u^{1-p'}\big(d(x_0,x)\big)\rho^{(1-p)(1-p')}(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p'}}\,d\mu\Big)\\
&\quad= \Big(\int_{B(x_0,t)} \sigma\big(d(x_0,x)\big)\rho^{1-p}(x)\,d\mu\Big)^{p'-1}\\
&\qquad\times\Big(\sum_{k=0}^{\infty}\int_{B(x_0,\eta_1^{k+1}t)\setminus B(x_0,\eta_1^{k}t)} \frac{u^{1-p'}\big(d(x_0,x)\big)\rho(x)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p'}}\,d\mu\Big)\\
&\quad\le b\Big(\int_{B(x_0,t)} \sigma\big(d(x_0,x)\big)\,d\mu\Big)^{p'-1}\sum_{k=0}^{\infty}\frac{u^{1-p'}(\eta_1^{k+1}t)}{\Big(\mu\big(B(x_0,\eta_1^{k}t)\big)\Big)^{p'-1}}\\
&\quad\le b\Big(\int_{B(x_0,t)} \sigma\big(d(x_0,x)\big)\,d\mu\Big)^{p'-1}\sum_{k=0}^{\infty}u^{1-p'}(\eta_1^{k+1}t)\\
&\qquad\times\int_{B(x_0,\eta_1^{k+2}t)\setminus B(x_0,\eta_1^{k+1}t)}\frac{d\mu}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p'}}\\
&\quad\le b\Big(\int_{B(x_0,\eta_1 t)} \sigma\big(d(x_0,x)\big)\,d\mu\Big)^{p'-1}\int_{X\setminus B(x_0,\eta_1 t)}\frac{u^{1-p'}\big(d(x_0,x)\big)}{\Big(\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)\Big)^{p'}}\,d\mu\le b.
\end{aligned}$$

As $\rho\in A_1$ it follows that $\rho^{1-p}\in A_p$, and by Theorem 2.4 the last estimation leads to the desired result.

3 Two-Weight Weak-Type Inequalities

In this section we shall establish two-weight weak-type inequalities for singular integrals defined on an SHT (X,d,μ). We need the following lemmas.

Lemma 3.1 *Let $\mu\{x_0\}=0$, let w be a weight function on X and let $\rho\in A_1(X)$. Suppose that the following conditions are fulfilled:*

(i) there exists a positive increasing function σ on $(0,\infty)$ such that for some positive constant b_1, and with a_1 as in the definition of (X,d,μ),

$$\sigma\big(2a_1 d(x_0,x)\big)\rho(x)\le b_1 w(x)\quad a.e.$$

(ii)

$$\operatorname*{ess\,sup}_{x\in B(x_0,t)}\frac{1}{w(x)}<\infty$$

for any $t>0$.

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Then $Kf(x)$ exists a.e. on X for any $\phi \in L^1_w(X)$.

Proof Fix $\alpha > 0$. Let

$$S_\alpha = X \setminus B(x_0, \frac{\alpha}{2})$$

and given $\phi \in L^1_w(X)$, put

$$\phi(x) = \phi_1(x) + \phi_2(x)$$

where $\phi_1(x) = \phi(x)\chi_{S_\alpha}(x)$, $\phi_2(x) = \phi(x) - \phi_1(x)$. For ϕ_1 we have

$$\int_X |\phi_1(x)|\rho(x)\,d\mu = \frac{\sigma(\frac{\alpha}{2})}{\sigma(\frac{\alpha}{2})}\int_{S_\alpha} |\phi(x)|\rho(x)\,d\mu \le \frac{1}{\sigma(\frac{\alpha}{2})}\int_{S_\alpha} |\phi(x)|\rho(x)\sigma\big(d(x_0,x)\big)\,d\mu$$
$$\le \frac{b}{\sigma(\alpha/2)}\int_{S_\alpha} |\phi(x)|w(x)\,d\mu < \infty.$$

Consequently $\phi_1 \in L^1_\rho(X)$ and so due to the weak-type one-weight inequality, $K\phi_1$ belongs to weak $L^1_\rho(X)$ (see, for example, [8] and also [5], p. 309). Hence $K\phi_1(x)$ exists a.e.

Now we shall show that $K\phi_2(x)$ converges absolutely on the set $\{x : d(x_0,x) > \alpha a_1\}$. For $d(x_0,y) < \frac{\alpha}{2}$ and $d(x_0,x) > \alpha$ we have $d(x,y) \ge \frac{\alpha}{2a_0}$ and

$$\mu\Big(B\big(x_0, d(x,y)\big)\Big) \le b\mu\Big(B\big(x, d(x,y)\big)\Big).$$

(See the proof of Lemma 2.1).

Then

$$|K\phi_2(x)| \le b\int_{B(x_0,\alpha/2)} \frac{|\phi(y)|}{\mu\Big(B\big(x,d(x,y)\big)\Big)}\,d\mu \le b\frac{1}{\mu\big(B(x,\alpha)\big)}\int_{B(x_0,\alpha/2)} |\phi(y)|\,d\mu$$
$$\le \frac{b}{\mu\big(B(x_0,\alpha)\big)}\int_{B(x_0,\alpha)} |\phi(y)|w(y)\,d\mu\Big(\operatorname*{ess\,sup}_{x\in B(x_0,d)} \frac{1}{w(x)}\Big) < \infty.$$

In view of the arbitrariness of α we conclude that $K\phi_2(x)$ is convergent and $K\phi(x)$ exists a.e.

Analogously we can prove

Lemma 3.2 *Let w be a weight function on X and let $\rho \in A_1(X)$. Suppose that*

(i) there exists a positive decreasing function σ on $(0,\infty)$ such that

$$\sigma\Big(\frac{d(x_0,x)}{2a_1}\Big)\rho(x) \le bw(x) \quad a.e.;$$

(ii)

$$\operatorname*{ess\,sup}_{x\in X\setminus B(x_0,t)} \frac{1}{w(x)\mu\Big(B\big(x_0, d(x_0,x)\big)\Big)} < \infty$$

for any $t > 0$.

Then $K\phi(x)$ *exists a.e. for arbitrary* $\phi \in L^1_w(X)$.

Let

$$Pf(x) = \frac{1}{\mu\Big(B\big(x_0, d(x_0,x)\big)\Big)} \int_{\{d(x_0,y)<d(x_0,x)\}} f(y)\,d\mu.$$

Lemma 3.3 *Let* $\mu\{x_0\} = 0$, v *and* w *be weight functions on* X. *If the following condition is fulfilled:*

$$\sup_{\substack{\tau,t\\ \tau>t}} \left(\frac{1}{\mu\big(B(x_0,\tau)\big)} \int_{\{x\in X: t<d(x_0,x)<\tau\}} v(x)\,d\mu\right) \operatorname*{ess\,sup}_{\{d(x_0,x)\le t\}} \frac{1}{w(x)} < \infty,$$

then there exists a positive constant c *such that for any* $\lambda > 0$ *and* $f \in L^1_w(X)$ *we have*

$$\int_{\{x:|Pf(x)|>\lambda\}} v(x)\,d\mu \le \frac{c}{\lambda}\int_X |f(x)|w(x)\,d\mu.$$

Proof Let $f \ge 0$, $f \in L^1_w(X)$; then for arbitrary $s > 0$ we have

$$\int_{B(x_0,s)} f(x)\,d\mu \le \left(\int_{B(x_0,s)} f(x)w(x)\,d\mu\right) \operatorname*{ess\,sup}_{x\in B(x_0,s)} \frac{1}{w(x)} < \infty.$$

The function

$$I(s) = \int_{B(x_0,s)} f(x)\,d\mu$$

is left-continuous and $\lim_{s\to 0} I(s) = 0$. Now suppose

$$\int_X f(x)\,d\mu \in (2^m, 2^{m+1}]$$

for some integer m and let $a_j = \sup\{s : I(s) \le 2^j\}$ for $j \le m+1$. It is easy to see that $a_{m+1} = \infty$. If $s > a_j$, then $G(s) > 2^j$. Moreover,

$$\int_{\{x:a_j\le d(x_0,x)\le a_j+1\}} f(x)\,d\mu \ge 2^j.$$

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The sequence $\{a_j\}_{j=-\infty}^{m}$ is nondecreasing; let $\alpha = \lim_{j\to-\infty} a_j$. If $J_m = \{j \le m : a_j < a_{j+1}\}$ then we have $[0,\infty) = \bigcup_{j\in J_m}(a_j, a_{j+1}] \cup [0,\alpha]$, where $(a_m, a_{m+1}] = (a_m, +\infty)$. Thus

$$X = \bigcup_{j\in J_m} E_j \cup F,$$

where $E_j = \{x \in X : a_j < d(x_0,x) \le a_{j+1}\}$ and $F = \{x \in X : 0 \le d(x_0,x) \le \alpha\}$. If

$$\int_X f(x)\,d\mu = \infty,$$

then in this case $m = \infty$ and

$$X = \bigcup_{j\in J} E_j \cup F,$$

where $J = \{j \in Z : a_j < a_{j+1}\}$. Now let $s \in (a_j, a_{j+1}]$; then $I(s) \le I(a_{j+1}) \le 2^{j+1}$ as $j \le m$. For $s \in [0,\alpha]$ we have

$$I(s) \le I(a_j) \le 2^j$$

for arbitrary $j \le m$, and consequently $I(s) = 0$. If we put

$$\sup_{\lambda>0} \lambda\Big(\int_{\{x\in X:|g(x)|>\lambda\}} v(x)\,d\mu\Big) = \|g(\cdot)\|_{L_v^{1\infty}(X)}$$

for any measurable function g, then we have the following estimates:

$$\begin{aligned}
&\Big\|\Big(\mu\Big(B\big(x_0,d(x_0,\cdot)\big)\Big)\Big)^{-1} I\big(d(x_0,\cdot)\big)\Big\|_{L_v^{1\infty}(X)}\\
&\le \sum_{j\in J}\Big\|\chi_{E_j}(\cdot)\Big(\mu\Big(B\big(x_0,d(x_0,\cdot)\big)\Big)\Big)^{-1} I\big(d(x_0,\cdot)\big)\Big\|_{L_v^{1\infty}(X)}\\
&\le \sum_{j\in J} 2^{j+1}\Big\|\chi_{E_j}(\cdot)\Big(\mu\Big(B\big(x_0,d(x_0,\cdot)\big)\Big)\Big)^{-1}\Big\|_{L_v^{1\infty}(X)}\\
&= 4\sum_{j\in J} 2^{j-1}\Big\|\chi_{E_j}(\cdot)\Big(\mu\Big(B\big(x_0,d(x_0,\cdot)\big)\Big)\Big)^{-1}\Big\|_{L_v^{1\infty}(X)}\\
&\le 4\sum_{j\in J}\Big(\int_{\{a_{j-1}\le d(x_0,x)\le a_j\}} f(x)\,d\mu\Big)\Big\|\chi_{E_j}(\cdot)\Big(\mu\Big(B\big(x_0,d(x_0,\cdot)\big)\Big)\Big)^{-1}\Big\|_{L_v^{1\infty}(X)}\\
&\le 8\big(\sup_{t>0} h(t)\big)\|f(\cdot)\|_{L_v^{1\infty}(X)},
\end{aligned}$$

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where

$$h(t) = \Big\| \chi_{\{d(x_0,x)>t\}}(\cdot)\Big(\mu\Big(B\Big(x_0, d\big(x_0,(\cdot)\big)\Big)\Big)\Big)^{-1}\Big\|_{L_\nu^{1\infty}(X)} \operatorname*{ess\,sup}_{\{d(x_0,x)\le t\}} \frac{1}{w(x)}.$$

Moreover, we have

$$\begin{aligned}
&\Big\| \chi_{\{d(x_0,x)>t\}}(\cdot)\Big(\mu\Big(B\Big(x_0, d\big(x_0,(\cdot)\big)\Big)\Big)\Big)^{-1}\Big\|_{L_\nu^{1\infty}(X)} \\
&\qquad = \sup_{\lambda<(\mu(B(x_0,t)))^{-1}} \lambda\Big(\int_{\{x:d(x_0,x)>t,(\mu(B(x_0,d(x_0,x))))^{-1}>\lambda\}} \nu(x)\,d\mu\Big) \\
&\qquad = \sup_{\lambda<(\mu(B(x_0,t)))^{-1}} G(\lambda).
\end{aligned}$$

Let $0 < \lambda < \Big(\mu\big(B(x_0,t)\big)\Big)^{-1}$; then there exists $\tau > 0$ such that $\mu\big(B(x_0,\frac{\tau}{2})\big) \le \lambda^{-1} < \mu\big(B(x_0,\tau)\big)$ and we have

$$\begin{aligned}
G(\lambda) &\le \mu\Big(B\Big(x_0,\frac{\tau}{2}\Big)\Big)^{-1}\Big(\int_{\{x:\mu(B(x_0,t))<\mu(B(x_0,d(x_0,x)))<\mu(B(x_0,\tau))\}} \nu(x)\,d\mu\Big) \\
&\le b\Big(\mu\big(B(x_0,\tau)\big)\Big)^{-1}\Big(\int_{\{t<d(x_0,x)<\tau\}} \nu(x)\,d\mu\Big) \\
&\le b\sup_{\tau>t}\Big(\mu\big(B(x_0,\tau)\big)\Big)^{-1}\Big(\int_{\{x:t<d(x_0,x)<\tau\}} \nu(x)\,d\mu\Big)
\end{aligned}$$

and finally we have

$$\sup_{t>0} h(t) \le b\sup_{\substack{\tau,t\\ \tau>t}}\Big(\frac{1}{\mu\big(B(x_0,\tau)\big)}\int_{\{x\in X:t<d(x_0,x)<\tau\}} \nu(x)\,d\mu\Big) \operatorname*{ess\,sup}_{\{d(x_0,x)\le t\}} \frac{1}{w(x)} < \infty. \qquad \blacksquare$$

Theorem 3.1 *Let $\mu\{x_0\} = 0$, σ be a positive continuous increasing function on $(0,\infty)$, let $\rho \in A_1(X)$, w be a weight function on X and put $\nu(x) = \sigma\big(d(x_0,x)\big)\rho(x)$. Suppose the following two conditions are satisfied: there exists a positive constant b_1 such that for μ-almost all $x \in X$,*

$$\rho(x)\sigma\big(2a_1 d(x_0,x)\big) \le b_1 w(x); \tag{3.1}$$

and

$$\sup_{\substack{\tau,t\\ \tau>t}}\Big(\frac{1}{\mu\big(B(x_0,\tau)\big)}\int_{\{x\in X:t<d(x_0,x)<\tau\}} \nu(x)\,d\mu\Big) \operatorname*{ess\,sup}_{\{d(x_0,x)\le t\}} \frac{1}{w(x)} < \infty. \tag{3.2}$$

Then there exists $c > 0$ such that for any $\lambda > 0$ and $f \in L_w^1(X)$ we have

$$\int_{\{x\in X:|Kf(x)|>\lambda\}} \nu(x)\,d\mu \le \frac{c}{\lambda}\int_X |f(x)|w(x)\,d\mu. \tag{3.3}$$

Proof Put $\{x \in X : |Kf(x)| > \lambda\} = H_\lambda$. Again we assume that $\sigma(t) = \sigma(0+) + \int_0^t \psi(\tau)\, d\tau$, $\psi \geq 0$. Then

$$\int_{H_\lambda} v(x)\, d\mu = \int_{H_\lambda} \sigma(0+)\rho(x)\, d\mu + \int_{H_\lambda} \rho(x)\Big(\int_0^{d(x_0,x)} \psi(t)\, dt\Big)\, d\mu = I_1 + I_2.$$

Using a weak-type one-weight inequality for K and condition (3.1) we derive

$$\begin{aligned} I_1 &\leq \frac{b\sigma(0+)}{\lambda} \int_X |f(x)|\rho(x)\, d\mu \leq \frac{b}{\lambda}\int_X |f(x)|\rho(x)\sigma\big(2a_1 d(x_0,x)\big)\, d\mu \\ &\leq \frac{b}{\lambda}\int_X |f(x)| w(x)\, d\mu. \end{aligned} \tag{3.4}$$

Now we estimate I_2. Let

$$S = \Big\{x \in X : \Big|\int_{X \setminus B(x_0, t/2a_1)} k(x,y) f(y)\, d\mu\Big| > \frac{\lambda}{2}\Big\},$$

$$S_1 = \Big\{x \in X : \Big|\int_{B(x_0, t/2a_1)} k(x,y) f(y)\, d\mu\Big| > \frac{\lambda}{2}\Big\}.$$

Then

$$\begin{aligned} I_2 &= \int_0^\infty \psi(t)\Big(\int_{\{x: d(x_0,x) > t\}} \rho(x)\chi_{H_\lambda}\, d\mu\Big)\, dt \\ &\leq \int_0^\infty \psi(t)\Big(\int_{\{x: d(x_0,x) > t\}} \rho(x)\chi_S\, d\mu\Big)\, dt \\ &\qquad + \int_0^\infty \psi(t)\Big(\int_{\{x: d(x_0,x) > t\}} \rho(x)\chi_{S_1}\, d\mu\Big)\, dt \\ &= I_{21} + I_{22}. \end{aligned}$$

Since $\rho \in A_1(X)$ we have

$$\begin{aligned} I_{21} &\leq \int_0^\infty \psi(t)\Big(\int_S \rho(x)\, d\mu\Big)\, dt \\ &\leq \frac{b}{\lambda}\int_0^\infty \psi(t)\Big(\int_{X \setminus B(x_0, t/2a_1)} |f(x)|\rho(x)\, d\mu\Big)\, dt \\ &= \frac{b}{\lambda}\int_X \rho(x)|f(x)|\Big(\int_0^{2a_1 d(x_0,x)} \psi(t)\, dt\Big)\, d\mu \\ &\leq \frac{b}{\lambda}\int_X |f(x)| w(x)\, d\mu. \end{aligned} \tag{3.5}$$

Further we note that for $d(x_0,x) > t$ and $d(x_0,x) \leq \frac{t}{2a_1}$ the inequality

$$\mu\Big(B\big(x_0, d(x_0,x)\big)\Big) \leq b\mu\Big(B\big(x, d(x,y)\big)\Big)$$

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holds (see the proof of Theorem 2.1). By virtue of the last inequality we obtain the estimates

$$\begin{aligned} I_{22} &\le \int_0^\infty \psi(t)\Big(\int_{\{d(x_0,x)>t\}} \rho(x)\chi_{\{x\in X: cPf(x)>\lambda\}}\,d\mu\Big)\,dt \\ &= \int_{\{x\in X: cPf(x)>\lambda\}} \rho(x)\Big(\int_0^{d(x_0,x)} \psi(t)\,dt\Big)\,d\mu \\ &\le \int_{\{x\in X: cPf(x)>\lambda\}} v(x)\,d\mu \end{aligned}$$

and by Lemma 3.3 we obtain

$$I_{22} \le \frac{b}{\lambda}\int_X |f(x)|w(x)\,d\mu. \tag{3.6}$$

Finally (3.4), (3.5) and (3.6) lead to (3.3).

Theorem 3.2 *Let $\mu\{x_0\} = 0$, let σ be a positive continuous decreasing function on $(0,\infty)$, let $\rho \in A_1(X)$ and let w be a weight function on X. Suppose the following two conditions hold:*

(i) there exists a positive constant b such that

$$\rho(x)\sigma\left(\frac{d(x_0,x)}{2a_1}\right) \le bw(x) \quad \text{a.e. on } X,$$

(ii)

$$\sup_{t>0}\Big(\int_{B(x_0,t)} v(x)\,d\mu\Big) \operatorname*{ess\,sup}_{x\in X\setminus B(x_0,2t)} \frac{1}{w(x)\mu\Big(B\big(x_0, d(x_0,x)\big)\Big)} < \infty.$$

Then the inequality (3.3) is true.

The proof of this theorem is based on Proposition 1.2 and some aspects of the proof of Theorem 2.2.

Corollary 3.1 *Let $\mu\{x_0\} = 0$. Let σ and u be positive increasing functions on $(0,\infty)$ with σ continuous, let $\rho \in A_1(X)$ and put $v(x) = \sigma\big(d(x_0,x)\big)\rho(x)$, $w(x) = u\big(d(x_0,x)\big)\rho(x)$. Suppose the following conditions are satisfied:*

(i) there exists a positive constant b_1 such that

$$\sigma(2a_1t) \le b_1u(t) \tag{3.7}$$

for all $t > 0$;

(ii)

$$\sup_{\substack{\tau,t\\ \tau>t}}\left(\frac{1}{\mu\big(B(x_0,\tau)\big)}\int_{\{x\in X: t<d(x_0,x)<\tau\}} v(x)\,d\mu\right) \operatorname*{ess\,sup}_{\{x: d(x_0,x)\le t\}} \frac{1}{w(x)} < \infty. \tag{3.8}$$

Then the inequality (3.3) holds.

Corollary 3.2 *Let μ, σ, u, ρ, v and w be as in Corollary 3.1 except that σ and u are decreasing rather than increasing. Suppose also that*

(i) there exists a positive constant b such that

$$\sigma\Big(\frac{t}{2a_1}\Big) \leq bu(t)$$

for all $t > 0$;
(ii)

$$\sup_{t>0}\Big(\int_{B(x_0,t)} v(x)\, d\mu\Big) \operatorname*{ess\,sup}_{x\in X\setminus B(x_0,t/2)} \frac{1}{w(x)\mu\Big(B\big(x_0, d(x_0,x)\big)\Big)} < \infty.$$

Then (3.3) holds.

Theorem 3.3 *Let $\mu\{x_0\} = 0$, let σ and u be positive increasing functions on $(0,\infty)$ with σ continuous and suppose that $\rho \in A_1(X)$. If further $v(x) = \sigma\big(d(x_0,x)\big)\rho(x)$, $w(x) = u\big(d(x_0,x)\big)\rho(x)$ and (3.8) is satisfied, then (3.3) holds.*

Proof By virtue of Corollary 3.1 it is sufficient to prove the implication (3.8) $\Longrightarrow$ (3.7).

Let $\beta \geq 1$, $\eta \geq \eta_1 > 1$, where η_1 is the constant from the (RD) condition. As $\rho \in A_1(X)$ we know that $\rho \in A_p(X)$ for any $p > 1$. Now using Hölder's inequality, Lemma 2.3 and the argument of the proof of Theorem 2.3 we derive the chain of inequalities

$$\begin{aligned}
&\frac{\sigma(\beta t)}{u(t)}\\
&\leq b\frac{\sigma(\beta t)}{u(t)\Big(\mu\big(B(x_0,\eta\beta t)\big)\Big)^p}\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)}\rho(x)\,d\mu \times \Big(\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)}\rho(x)^{1-p'}\,d\mu\Big)^{p-1}\\
&\leq b\frac{\sigma(\beta t)}{u(t)\Big(\mu\big(B(x_0,\eta\beta t)\big)\Big)^p}\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)}\rho(x)\,d\mu \Big(\int_{B(x_0,\frac{t}{2})}\rho(x)^{1-p'}\,d\mu\Big)^{p-1}\\
&\leq b\frac{\sigma(\beta t)}{\Big(\mu\big(B(x_0,\eta\beta t)\big)\Big)^p}\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)}\rho(x)\,d\mu \Big(\int_{B(x_0,\frac{t}{2})}w(x)^{1-p'}\,d\mu\Big)^{p-1}\\
&\leq b\frac{\sigma(\beta t)}{\Big(\mu\big(B(x_0,\eta\beta t)\big)\Big)^p}\Big(\int_{B(x_0,\eta\beta t)\setminus B(x_0,\beta t)}\rho(x)\,d\mu\Big) \times \operatorname*{ess\,sup}_{x\in B(x_0,\frac{t}{2})}\frac{1}{w(x)}\Big(\mu\big(B(x_0,\eta\beta t)\big)\Big)^{p-1}\\
&\leq b\frac{1}{\mu\big(B(x_0,\eta\beta t)\big)}\Big(\int_{\{x:\frac{t}{2}<d(x_0,x)<\eta\beta t\}}v(x)\,d\mu\Big)\operatorname*{ess\,sup}_{x\in B(x_0,\frac{t}{2})}\frac{1}{w(x)} \leq b.
\end{aligned}$$

■

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Theorem 3.4 *Let μ satisfy condition* (RD)*, suppose that $\mu\{x_0\} = 0$, let σ and u be positive decreasing functions on $(0,\infty)$ with σ continuous, let $\rho \in A_1(X)$ and put $v(x) = \sigma\big(d(x_0,x)\big)\rho(x)$ and $w(x) = u\big(d(x_0,x)\big)\rho(x)$. Suppose that*

$$\sup_{t>0}\Big(\int_{B(x_0,t)} v(x)\,d\mu\Big) \operatorname*{ess\,sup}_{x\in X\setminus B(x_0,t/2)} \frac{1}{w(x)\mu\Big(B\big(x_0,d(x_0,x)\big)\Big)} < \infty.$$

Then (3.3) holds.

4 Application to Singular Integrals on Fractal Sets; Examples

Let $\Gamma \subset \mathbb{C}$ be a connected rectifiable curve and let ν be arc-length measure on Γ. By definition, Γ is regular if

$$\nu\big(\Gamma \cap B(z,r)\big) \le cr$$

for every $z \in \mathbb{C}$ and all $r > 0$.

For r smaller than half the diameter of Γ, the reverse inequality

$$\nu\big(\Gamma \cap B(z,r)\big) \ge r$$

holds for all $z \in \Gamma$. Equipped with ν and the Euclidean metric, the regular curve becomes an SHT.

The associated kernel in which we are interested is

$$k(z,w) = \frac{1}{z-w}.$$

The Cauchy integral

$$S_\Gamma f(t) = \int_\Gamma \frac{f(\tau)}{t-\tau}\,d\nu(\tau)$$

is the corresponding singular operator.

The above-mentioned kernel in the case of regular curves is a Calderón-Zygmund kernel. As was proved by David [3], a necessary and sufficient condition for continuity of the operator S_Γ in $L_p(\Gamma)$ $(1 < p < \infty)$ is that Γ is regular.

From the results obtained in the preceding section we can derive several two-weight estimates for S_Γ.

Definition A measurable, almost everywhere positive function w on Γ is said to be in the class $A_p(\Gamma)$ if

(4.1)
$$\sup_{\substack{z\in\Gamma\\ r>0}} \frac{1}{\nu\big(B(z,r)\cap\Gamma\big)}\int_{B(z,r)\cap\Gamma} w(t)\,d\nu\bigg(\frac{1}{\nu\big(B(z,r)\cap\Gamma\big)}\int_{B(z,r)\cap\Gamma} w^{1-p'}(t)\,d\nu\bigg)^{p-1} < \infty.$$

It is known (see for example [3]) that for the continuity of S_Γ in $L^p_w(\Gamma)$, $1 < p < \infty$, when Γ is regular, it is necessary and sufficient that $w \in A_p(\Gamma)$.

Since for regular curves the measure ν satisfies the reverse doubling condition as well, we derive from Theorem 2.3

Proposition 4.1 *Let $1 < p < \infty$, let Γ be an unbounded regular curve, and let $t_0 \in \Gamma$. Let σ and u be positive increasing functions on $(0, \infty)$ with σ continuous, let $\rho \in A_p(\Gamma)$ and put $v(t) = \sigma(|t - t_0|)\rho(t)$, $w(t) = u(|t - t_0|)\rho(t)$.*

If

$$\sup_{r>0} \int_{\Gamma\setminus B(t_0,r)} \frac{v(t)}{\Big(\nu\big(B(t_0, |t - t_0|) \cap \Gamma\big)\Big)^p}\, d\nu \Big(\int_{B(t_0,r)\cap\Gamma} w^{1-p'}(t)\, d\nu\Big)^{p-1} < \infty,$$

then the inequality

$$\int_\Gamma |S_\Gamma f(t)|^p v(t)\, d\nu \le c \int_\Gamma |f(t)|^p w(t)\, d\nu$$

holds with a constant c independent of $f \in L^p_w(\Gamma)$. A corresponding version of Theorem 2.4 is valid for S_Γ.

From the results of the last section we can obtain two-weight inequalities in more general situations than the case just considered.

Let Γ be a subset of $\mathbb{R}^n$ which is an s-set $(0 \le s \le n)$ in the sense that there is a Borel measure μ in $\mathbb{R}^n$ such that (i) supp $\mu = \Gamma$; (ii) there are positive constants c_1 and c_2 such that for all $x \in \Gamma$ and all $r \in (0, 1)$,

$$c_1 r^s \le \mu\big(B(x, r) \cap \Gamma\big) \le c_2 r^s.$$

It is known (see [21], Theorem 3.4) that μ is equivalent to the restriction of Hausdorff s-measure $\mathcal{H}_s$ to Γ; we shall thus identify μ with $\mathcal{H}_s|\Gamma$.

Given $x \in \Gamma$, put $\Gamma(x, r) = B(x, r) \cap \Gamma$. By definition, if $1 < p < \infty$, $\rho \in A_p(\Gamma)$ if

$$\sup_{x\in\Gamma, r>0} \frac{1}{\mathcal{H}_s\big(\Gamma(x, r)\big)} \int_{\Gamma(x,r)} \rho(y)\, d\mathcal{H}_s(y) \Big(\frac{1}{\mathcal{H}_s\big(\Gamma(x, r)\big)} \int_{\Gamma(x,r)} \rho^{1-p'}(z)\, d\mathcal{H}_s(z)\Big)^{p-1} < \infty.$$

Let K_Γ be a Calderón-Zygmund singular integral defined on an s-set Γ by the procedure of Section 2. Since $\mathcal{H}_s|\Gamma$ satisfies condition (RD) we have, for example, the following

Proposition 4.2 *Let $1 < p < \infty$ and $x_0 \in \Gamma$. Let σ and u be positive increasing functions on $(0, \infty)$ with σ continuous. Let $\rho \in A_p(\Gamma)$ and put $v(x) = \sigma(|x - x_0|)\rho(x)$, $w(x) = u(|x - x_0|)\rho(x)$. Suppose that*

$$\sup_{r>0} \Big(\int_{\mathbb{R}^n\setminus\Gamma(x_0,r)} \frac{v(x)}{|x - x_0|^{\mathrm{sp}}}\, d\mathcal{H}_s(x)\Big) \Big(\int_{\Gamma(x_0,r)} w^{1-p'}(y)\, d\mathcal{H}_s(y)\Big)^{p-1} < \infty.$$

Then there is a constant c such that for all $f \in L^p_w(\Gamma)$,

$$\int_\Gamma |K_\Gamma f(x)|^p v(x)\, d\mathcal{H}_s(x) \le c \int_\Gamma |f(x)|^p w(x)\, d\mathcal{H}_s(x).$$

It is clear that other direct consequences of the results of previous sections may be formulated in the setting of s-sets. Note that (see [21], 4.9) since the Cantor set in $\mathbb{R}^n$ is an s-set, where

$$s = \frac{\log(3^n - 1)}{\log 3},$$

we can obtain two-weight estimates for singular integrals on a Cantor set in $\mathbb{R}^n$.

Now we provide several examples in which the conditions guaranteeing two-weight estimates for singular integrals defined on an SHT X are satisfied.

Let $x_0 \in X$ be such that $\mu\{x_0\} = 0$. Then the function

$$w(x) = \Big(\mu\big(B\big(x_0, d(x_0, x)\big)\big)\Big)^{\alpha}$$

belongs to $A_p(X)$ if, and only if, $-1 < \alpha < p - 1$ (see [5]). For this weight we have the one-weight inequality

$$\int_X |Kf(x)|^p w(x)\, d\mu \le c \int_X |f(x)|^p w(x)\, d\mu.$$

For simplicity let us consider SHT for which $\mu\big(B(x, r)\big) \sim r$. From the results of previous sections we deduce

Proposition 4.3 *Let $1 < p < \infty$. Suppose also that $a < \infty$. Then there exists a positive constant $c > 0$ such that the inequalities*

$$\int_X |Kf(x)|^p \big(d(x_0, x)\big)^{p-1}\, d\mu \le c \int_X |f(x)|^p \big(d(x_0, x)\big)^{p-1} \log^p \frac{a}{d(x_0, x)}\, d\mu$$

and

$$\int_X |Kf(x)|^p \frac{d\mu}{d(x_0, x) \log^{p-1} \frac{a}{d(x_0,x)}} \le c \int_X |f(x)|^p \frac{d\mu}{d(x_0, x)}$$

hold.

Finally note that in [4], [11] and [12] can be found several other examples of pairs of weights ensuring two-weight strong or weak-type inequalities for singular integrals defined on special SHT.

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References

[1] J. S. Bradley, *Hardy's inequalities with mixed norms*. Canad. Math. Bull. **21**(1978), 405–408.

[2] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*. Bull. Amer. Math. Soc. **83**(1977), 569–645.

[3] G. David, *Opérateurs integraux singulièrs sur certaines courbes du plan complexe*. Ann. Sci. École. Norm. Sup. (4) **17**(1984), 157–189.

[4] D. E. Edmunds and V. Kokilashvili, *Two-weighted inequalities for singular integrals*. Canad. Math. Bull. **38**(1995), 119–125.

[5] I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec, *Weight theory of integral transforms on spaces of homogeneous type*. Pitman Monograph, Surveys in Pure and Applied Mathematics **92**, Longman, 1998.

[6] V. Guliev, *Two-weight L_p inequality for singular integral operators on Heisenberg groups*. Georgian Math. J. **1**(1994), 367–376.

[7] E. G. Gusseinov, *Singular integrals in the spaces of functions summable with monotone weights (Russian)*. Mat. Sb. **132**(1977), 28–44.

[8] S. Hoffman, *Weighted norm inequalities and vector-valued inequalities for certain rough operators*. Indiana Univ. Math. J. **42**(1993), 1–14.

[9] G. Khuskivadze, V. Kokilashvili and V. Paatashvili, *Boundary value problems for analytic and harmonic functions in domains with non-smooth boundaries. Application to conformal mapping*. Memoirs of Differential Equations and Math. Phys. **14**(1998), 1–195.

[10] V. M. Kokilashvili, *On Hardy's inequalities in weighted spaces*. (Russian) Bull. Acad. Sci. Georgian SSR **96**(1979), 37–40.

[11] V. Kokilashvili and A. Meskhi, *Two-weighted estimates for singular integrals on homogeneous groups*. (Russian) Dokl. Akad. Nauk **354**(1997), 301–303; English transl. Soviet Math. Dokl. **55**(1997), 362–364.

[12] ———, *Two-weight inequalities for singular integrals defined on homogeneous groups*. Proc. A. Razmadze Math. Inst., Georgian Acad. Sci. **112**(1997), 57–90.

[13] ———, *Two-weight inequalities for Hardy type transforms and singular integrals defined on homogeneous type spaces*. Proc. A. Razmadze Math. Inst., Georgian Acad. Sci. **114**(1997), 119–123.

[14] B. Muckenhoupt, *Hardy's inequalities with weight*. Studia Math. **44**(1972), 31–38.

[15] B. Muckenhoupt and R. L. Wheeden, *Two-weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform*. Studia Math. **55**(1976), 279–294.

[16] Quan-Tao, *Weighted inequalities concerning the Radon measure of the arc length of curves in the complex plane*. J. Systems Sci. Math. Sci. **6**(1986), 146–152.

[17] Y. Rakotondratsimba, *On boundedness of classical operators on weighted Lorentz spaces*. Georgian Math. J. (1998), to appear.

[18] E. T. Sawyer and R. L. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*. Amer. J. Math. **114**(1992), 813–874.

[19] E. M. Stein, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*. Princeton University Press, Princeton, New Jersey, 1993.

[20] J. O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*. Lecture Notes in Math. **1381**, Springer Verlag, Berlin, 1989.

[21] H. Triebel, *Fractals and spectra related to Fourier analysis and function spaces*. Birkhaüser, Basel, 1997.

Centre for Mathematical Analysis
and its Applications
University of Sussex
Brighton BN1 9QH
Sussex
United Kingdom

A. M. Razmadze Mathematical Institute
Z. Rukhadze Str. 1
380093 Tbilisi
Republic of Georgia

A. M. Razmadze Mathematical Institute
Z. Rukhadze Str. 1
380093 Tbilisi
Republic of Georgia