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Derivation Algebras of Toric Varieties

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Abstract. We study the Lie algebra of derivations of the coordinate ring of affine toric varieties defined by simplicial affine semigroups and prove the following results:

– Such toric varieties are uniquely determined by their Lie algebra if they are supposed to be Cohen–Macaulay of dimension $\geqslant 2$ or Gorenstein of dimension $= 1$.
– In the Cohen–Macaulay case, every automorphism of the Lie algebra is induced from a unique automorphism of the variety.
– Every derivation of the Lie algebra is inner.

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Key words: toric variety, affine semigroup, derivation, Lie algebra, root space decomposition.

1. Introduction

Normal affine algebraic varieties in characteristic 0 are uniquely determined (up to isomorphism) by the Lie algebra of derivations of their coordinate ring. This was shown by Siebert [Si] and, independently, by Hauser and the third author [HM]. In both papers, the assumption of normality is essential. There are nonisomorphic nonnormal varieties with isomorphic Lie algebras. The third author [M] treated certain nonnormal varieties defined in combinatorial terms by showing that closed simplicial complexes can be reconstructed from the Lie algebra of their Stanley–Reisner ring. Here we study this problem for (in general, nonnormal) toric varieties defined by simplicial affine semigroups.

We show that such toric varieties are uniquely determined by their Lie algebra if they are supposed to be Cohen–Macaulay of dimension $\geqslant 2$. The corresponding statement is false in dimension 1. For toric curves we need the stronger hypothesis that they are Gorenstein. In fact, we can reconstruct from the Lie algebra the semigroup defining the variety. Our result should be compared with a recent one of Gubeladze [G] saying that an affine semigroup is uniquely determined by the toric variety it defines (more precisely, by its coordinate ring as an augmented algebra).

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The main tool in our proofs is a root space decomposition of the Lie algebra of derivations of a Buchsbaum semigroup ring. The set of roots is closely related to the underlying semigroup. This structural description will be used to prove two more results. We show, in the Cohen–Macaulay case, that every automorphism of the Lie algebra is induced from a unique automorphism of the variety. And we establish an infinitesimal analogue of the last statement: Every derivation of the Lie algebra is inner, i.e., the first cohomology of the Lie algebra with coefficients in the adjoint representation vanishes.

2. The Root Space Decomposition

Let S be an affine semigroup, i.e., a finitely generated subsemigroup of some $\mathbb{N}^n$. We stress that, in this paper, semigroup always means semigroup with zero element. Denote by $G = G(S)$ the subgroup of $\mathbb{Z}^n$ generated by S and by $r = \text{rk } S = \text{rk } G(S)$ its rank. Let C_S be the convex polyhedral cone spanned by S in $\mathbb{Q}^n$. We shall suppose throughout that S is *simplicial*, i.e., that the convex cone C_S can be spanned by r elements of S. For an algebraically closed field k of characteristic 0 let $k[S] \subseteq k[t] = k[t_1, \dots, t_n]$ denote the corresponding semigroup ring. We need to recall how the property of $k[S]$ being Cohen–Macaulay or Buchsbaum can be described in terms of S. For this purpose, let $F_1, \dots, F_m$ be the $(r-1)$-dimensional faces of C_S. Set

$$S'_i = \{\lambda \in G,\ \lambda + s \in S \text{ for some } s \in S \cap F_i\}$$

for $i = 1, \dots, m$, and $S' = \bigcap S'_i$.

PROPOSITION 1. *For a simplicial affine semigroup S the semigroup ring $k[S]$ is Cohen–Macaulay (resp. Buchsbaum) if and only if $S' = S$ (resp. $S' + (S \setminus \{0\}) \subseteq S$).*

For the proof see [GSW], [St, Thm. 6.4], [TH, Sect. 4], and [SS, Sect. 6]. The semigroup S' is called the *Cohen–Macaulayfication* of S. Let

$$\bar{S} = \{s \in G,\ ms \in S \text{ for some } m \in \mathbb{N}, m \neq 0\}.$$

It is known [Ho, Sect. 1] that $k[\bar{S}]$ is the normalization of $k[S]$. An affine semigroup S is called *standard* if

(i) $\bar{S} = G(S) \cap \mathbb{N}^n$.
(ii) For all i the image of S under the projection π_i on the ith component is a numerical semigroup, i.e., the complement $\mathbb{N} \setminus \pi_i(S)$ is finite.
(iii) The semigroups $S \cap \ker \pi_i$, $i = 1, \dots, n$, are distinct of rank equal to $\text{rk}\, S - 1$.

It was shown by Hochster [Ho, Sect. 2] that every affine semigroup is isomorphic to a standard one. Hence, we shall assume throughout that S is standard. In that

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case the cone C_S has exactly n faces of dimension $r-1$, namely the convex cones spanned by the $S \cap \ker\ \pi_i$. Hence

$$S'_i = \{\lambda \in \mathbb{Z}^n,\ \lambda + s \in S \text{ for some } s \in S \text{ with } s_i = 0\}$$

for $i = 1, \ldots, n$. A standard affine semigroup S is simplicial if and only if S has elements on every coordinate axis. In fact, the cone of a simplicial affine semigroup of rank r has only r faces of dimension $r-1$. Standardness gives $r = n$. Then the edges of C_S are the intersections of C_S with the coordinate axes, see [SS, Sect. 1]. The reversed implication is obvious. Let $a_i \in \mathbb{N}$, $a_i \neq 0$, be the minimal number such that $\alpha^i = (0, \ldots, 0, a_i, 0, \ldots, 0) \in S$, where the nonzero entry is at the ith place.

PROPOSITION 2. *Every k-linear derivation D of $k[S]$ extends uniquely to a derivation of the polynomial ring $k[t]$.*

Proof. As $S \subseteq \mathbb{N}^n$ is standard and simplicial it has rank n and $k[S]$ has dimension n. Hence, the rational function field $k(t)$ is a separable finite extension of the quotient field $k(S)$ of $k[S]$. Therefore, D extends uniquely to a derivation D of $k(t)$. Write $D = \sum f_i \partial_i$ with $f_i \in k(t)$, say $f_i = g_i/h_i$ with coprime $g_i, h_i \in k[t]$. With the semigroup elements α^i introduced above, we have

$$a_i t_i^{a_i - 1} f_i = D(t^{\alpha^i}) \in k[S] \subseteq k[t]$$

and h_i divides $t_i^{a_i-1}$. As $\pi_i(S)$ is a numerical semigroup there is $s \in G$ with the ith component $s_i = 1$. Using simpliciality, we may assume that $s \in \mathbb{N}^n$, hence $s \in \bar{S}$. It was shown by Seidenberg [Se] that D maps the normalization $k[\bar{S}]$ of $k[S]$ into itself. Then

$$\sum s_j t^s f_j / t_j = D(t^s) \in k[\bar{S}] \subseteq k[t]$$

implies $\prod_{j \neq i} t_j^{a_j - 1} t^s f_i / t_i \in k[t]$. Hence, h_i divides $\prod_{j \neq i} t_j^{a_j-1} t^s / t_i$. But t_i does not divide this product since $s_i = 1$. Thus $h_i \in k$ and $f_i \in k[t]$. This means that D restricts to a derivation of $k[t]$. □

By Proposition 2, the Lie algebra $\Theta(S) = \text{Der}\ \ k[S]$ of k-linear derivations of the semigroup ring may be viewed as a subalgebra of $\mathbb{D} = \text{Der}\, k[t]$. Let us first describe the latter Lie algebra. The derivations $D_i = t_i \partial_i$ span an Abelian subalgebra H. For a linear form $\lambda \in H^*$ let

$$\mathbb{D}_\lambda = \{D \in \mathbb{D},\ [h, D] = \lambda(h) \cdot D \text{ for all } h \in H\}.$$

Then $\mathbb{D}$ admits a root space decomposition $\mathbb{D} = \bigoplus_{\lambda \in H^*} \mathbb{D}_\lambda$. Given the basis $D_1, \ldots, D_n$ of H one may identify H^* with k^n by identifying the form λ with the vector $(\lambda(D_1), \ldots, \lambda(D_n))$. Then the set of $\lambda \in H^*$ with $\mathbb{D}_\lambda \neq 0$ equals

$$\mathbb{N}^n \cup \{\lambda \in \mathbb{Z}^n,\ \lambda_i = -1 \text{ for exactly one } i \text{ and } \lambda_j \geqslant 0 \text{ for all } j \neq i\}.$$

In fact, for $\lambda \in \mathbb{N}^n$ the root space $\mathbb{D}_\lambda$ is spanned by all $D_{\lambda j} = t^\lambda t_j \partial_j$, $j = 1, \dots, n$. In particular, $\mathbb{D}_0 = H$. And if $\lambda \in \mathbb{Z}^n$ with $\lambda_i = -1$ and $\lambda_j \geqslant 0$ for $j \neq i$, then $\mathbb{D}_\lambda$ is spanned by the single element $D_{\lambda i} = t^\lambda t_i \partial_i$. All these statements follow from the commutator relation $[D_i, D_{\lambda j}] = \lambda_i \cdot D_{\lambda j}$. We need some more notation in order to describe the subalgebra $\Theta(S)$. Let

$$\Lambda_i = \{\lambda \in \mathbb{Z}^n,\ \lambda + s \in S \text{ for all } s \in S \text{ with } s_i \neq 0\}, \quad i = 1, \dots, n$$

$$\Lambda = \Lambda(S) = \bigcup \Lambda_i, \qquad \tilde{S} = \{\lambda \in \mathbb{N}^n,\ \lambda + (S \setminus \{0\}) \subseteq S\}.$$

Remark 1. Let $n = 1$. Then $k[S]$ is always Cohen–Macaulay, and the cardinality of $\Lambda \backslash S$ equals the Cohen–Macaulay type of $k[S]$, see [HK]. For $S = \mathbb{N}$, one has $\tilde{S} = \mathbb{N}$ and $\Lambda = \tilde{S} \cup \{-1\}$. Otherwise $1 \notin S$. Then our assumption that $\mathbb{N} \backslash S$ is finite implies $\Lambda \subseteq \mathbb{N}$ and $\Lambda = \tilde{S}$.

Remark 2. Let $n \geqslant 2$. From $\lambda + \alpha^i \in S$ for $\lambda \in \tilde{S}$ and two indices i one sees $\tilde{S} \subseteq S'$. Hence, $\tilde{S} = S'$ in the Buchsbaum case and $\tilde{S} = S$ in the Cohen–Macaulay case.

PROPOSITION 3. (i) *The Lie algebra* $\Theta(S)$ *admits a root space decomposition* $\Theta(S) = \bigoplus_{\lambda \in H^*} \Theta_\lambda$ *with* $\Theta_\lambda = \Theta(S) \cap \mathbb{D}_\lambda$.

(ii) *Suppose that* $k[S]$ *is Buchsbaum. Then the set of* $\lambda \in H^*$ *with* $\Theta_\lambda \neq 0$ *equals* $\Lambda(S)$. *If* $\lambda \in \tilde{S}$ *then* Θ_λ *is spanned by* $D_{\lambda 1}, \dots, D_{\lambda n}$. *And if* $\lambda \in E_i = \Lambda_i \backslash \tilde{S}$, *then* Θ_λ *is spanned by the single element* $D_{\lambda i}$. *In particular,* $\Lambda(S) = \tilde{S} \cup \bigcup E_i$ *is a disjoint union.*

The elements of $\tilde{S}$ (resp. E_i) will be called *ordinary* (resp. *i-exceptional*) *roots*.

Proof. (i) For $D_\lambda = \sum_i b_{\lambda i} D_{\lambda i} \in \mathbb{D}_\lambda$ one has $D_\lambda t^s = \sum_i b_{\lambda i} s_i \cdot t^{\lambda + s}$. Hence $\sum_\lambda D_\lambda \in \Theta(S)$ if and only if $\lambda + s \in S$ for all $s \in S$ and all occurring λ with $\sum_i b_{\lambda i} s_i \neq 0$ if and only if $D_\lambda \in \Theta(S)$ for all occurring λ.

(ii) Consider $\lambda \in \tilde{S}$. Then $D_{\lambda 1}, \dots, D_{\lambda n}$ are defined and contained in $\Theta(S)$. Next consider $\lambda \in \Lambda_i$. From $\lambda + \alpha^i \in S$ we see $\lambda_j \geqslant 0$ for all $j \neq i$. Moreover, $\lambda_i \in \Lambda(\pi_i(S))$ and Remark 1 yields $\lambda_i \geqslant -1$. Hence, $D_{\lambda i}$ is defined and contained in $\Theta(S)$. Conversely, if $D_{\lambda i} \in \Theta(S)$ then $\lambda \in \Lambda_i$. The proof is completed by the following claim: If Θ_λ contains a linear combination of the $D_{\lambda i}$ with at least two nonvanishing coefficients then $\lambda \in \tilde{S}$. In fact, if $\sum_i b_i D_{\lambda i} \in \Theta(S)$ with $b_1, b_2 \neq 0$ then $\lambda + \alpha^1$ and $\lambda + \alpha^2$ are contained in S. This gives $\lambda \in S' \subseteq \tilde{S}$ as $k[S]$ is Buchsbaum. □

EXAMPLE 1 ([MT, Remark 1.3]). Let $S \subseteq \mathbb{N}^2$ be generated by (0,10), (3,7), (7,3), (8,2), (10,0) and let $\lambda = (9, 11)$. Then $\lambda + (3, 7) \notin S$ but $\lambda + s \in S$ for the remaining generators s. Hence, $\lambda \in S' \backslash \tilde{S}$ and $k[S]$ is not Buchsbaum. Moreover, $\lambda \notin \Lambda(S)$ but $\Theta_\lambda \neq 0$. In fact, $7D_{\lambda 1} - 3D_{\lambda 2} \in \Theta_\lambda$.

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EXAMPLE 2. Let $S \subseteq \mathbb{N}^2$ correspond to the affine cone over the d-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^d$, $d \geqslant 2$, i.e., S is generated by $(0,d), (1,d-1), \ldots, (d-1,1), (d,0)$. Then $k[S]$ is normal and Cohen–Macaulay. The exceptional roots are $(-1,1) + m(0,d)$ and $(1,-1) + m(d,0)$ with $m \in \mathbb{N}$.

EXAMPLE 3. Let $S \subseteq \mathbb{N}^2$ correspond to the product of a cusp with a line, i.e., S is generated by $(2,0)$, $(3,0)$ and $(0,1)$. Then $k[S]$ is Cohen–Macaulay. The 1-exceptional roots are $(1,0) + m(0,1)$ with $m \in \mathbb{N}$. The 2-exceptional roots are $(0,-1) + m(2,0)$ and $(3,-1) + m(2,0)$ with $m \in \mathbb{N}$.

Examples 2 and 3 illustrate the second part of the next result.

PROPOSITION 4. (i) *$\tilde{S}$ is a finitely generated subsemigroup of $\mathbb{N}^n$.*

(ii) *Suppose that $k[S]$ is Buchsbaum and $n \geqslant 2$. For fixed i let A_i be the semigroup generated by all α^j with $j \neq i$. Then the set E_i of i-exceptional roots is a finitely generated A_i-module.*

Proof. (i) Clearly $\tilde{S}$ is a subsemigroup of $\mathbb{N}^n$. Let A be the semigroup generated by $\alpha^1, \ldots, \alpha^n$. We show more generally that every subsemigroup $T \subseteq \mathbb{N}^n$ containing A is finitely generated. Let a_i be the nonzero entry of α^i. For $\beta \in \mathbb{N}^n$ with $\beta_i < a_i$ for all i let $T_\beta = (\beta + A) \cap T$. By Dickson's Lemma each T_β is a finitely generated A-module (or empty). Since $T = \bigcup T_\beta$ is a finite union, T is finitely generated as an A-module and hence as a semigroup.

(ii) We may assume $i = 1$. If $\lambda \in E_1 = \Lambda_1 \backslash \tilde{S}$ then clearly $\lambda + \alpha^2 \in \Lambda_1$. Moreover, $\lambda + \alpha^1 \in S$ so that $\lambda \in S'_i$ for $i \geqslant 2$. If $\lambda + \alpha^2 \in \tilde{S}$ then $\lambda + 2\alpha^2 \in S$, hence $\lambda \in S'_1$ and $\lambda \in S' = \tilde{S}$, contradiction. Thus $\lambda + \alpha^2 \in E_1$. This proves that E_1 is an A_1-module. It remains to show that it is finitely generated. For $\gamma \in \mathbb{N} \times \{0\} \subseteq \mathbb{N}^n$ and $\beta \in \{0\} \times \mathbb{N}^{n-1} \subseteq \mathbb{N}^n$ with $\beta_i < a_i$ for all i let $E_{\gamma\beta} = (\gamma + \beta + A_1) \cap E_1$. As above this is a finitely generated A_1-module (or empty). If $E_{\gamma\beta} \neq \emptyset$ and $\gamma' = \gamma + m\alpha^1$ for some $m \in \mathbb{N}$, $m \neq 0$ then $E_{\gamma'\beta} = \emptyset$. Otherwise, there is $\lambda \in A_1$ with $\gamma + \beta + \lambda, \gamma' + \beta + \lambda \in E_1$, contradicting $\gamma' + \beta + \lambda = \gamma + \beta + \lambda + m\alpha^1 \in S \subseteq \tilde{S}$. Since there are only finitely many congruence classes of $\mathbb{N}$ modulo α^1 the Proposition is proven. □

3. Reconstruction of the Semigroup

Before we explain how to reconstruct the semigroup S from its Lie algebra $\Theta(S)$ we make a remark concerning the reconstruction of S from its semigroup ring $k[S]$ discussed by Gubeladze [G]. Consider the augmentation $k[S] \to k$ defined by $t^s \mapsto 0$ for all $s \in S \setminus \{0\}$. Gubeladze [G, Thm. 2.1] proved that affine semigroups S_1 and S_2 are isomorphic if $k[S_1]$ and $k[S_2]$ are isomorphic as augmented algebras. Moreover [G, Lem. 2.8], if $k[S_1]$ and $k[S_2]$ are normal and isomorphic just as algebras then they are isomorphic as augmented algebras. We shall extend this result (for simplicial semigroups) to the Buchsbaum case.

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For any $\lambda \in \mathbb{Z}^n$ we denote by $|\lambda|$ the sum of its components. Let us say that S corresponds to a *product along a line* if, after permutation of coordinates, $S = \mathbb{N} \oplus M$ for some semigroup $M \subseteq \mathbb{N}^{n-1}$. We shall see that this property only depends on the algebra $k[S]$ and even on the Lie algebra $\Theta(S)$. Let $L = [\Theta(S), \Theta(S)]$ be the derived algebra.

PROPOSITION 5. *Suppose that $k[S]$ is Buchsbaum. Then the following are equivalent:*

(a) *The semigroup S corresponds to a product along a line.*
(b) *There is $\lambda \in \Lambda(S)$ with $|\lambda| < 0$.*
(c) $L = \Theta(S)$.

Proof. (a) $\Leftrightarrow$ (b) If $(-1, 0, \ldots, 0)$ is a root then $(1, 0, \ldots, 0) \in S$ and $S = \mathbb{N} \oplus M$ with $M = S \cap \ker \pi_1$. The converse is clear.

(b) $\Rightarrow$ (c) Here and later we use the commutator relation $[D_{\lambda i}, D_{\mu j}] = \mu_i D_{\lambda+\mu,j} - \lambda_j D_{\lambda+\mu,i}$. It shows $\bigoplus_{\lambda \neq 0} \Theta_\lambda \subseteq L$. Let $\lambda = (-1, 0, \ldots, 0) \in \Lambda$ so that $\mu = (1, 0, \ldots, 0) \in S \subseteq \tilde{S}$. Then L contains $2D_1 = [D_{\lambda 1}, D_{\mu 1}]$ and $D_j = [D_{\lambda 1}, D_{\mu j}]$ for $j \geqslant 2$. Thus $\Theta_0 = H \subseteq L$.

(c) $\Rightarrow$ (b) Assume that $|\lambda| \geqslant 0$ for all roots λ. Then $\eta^1 + \eta^2 = 0$ for roots $\eta^1, \eta^2 \neq 0$ is possible only if (after permutation of coordinates) $\eta^1 = (-1, 1, 0, \ldots, 0)$, $\eta^2 = (1, -1, 0, \ldots, 0)$. In this case $[D_{\eta^1,1}, D_{\eta^2,2}] = D_2 - D_1$. Since Θ_0 is Abelian we obtain $L \subseteq \bigoplus_{\lambda \neq 0} \Theta_\lambda \oplus \langle D_n - D_1, \ldots, D_2 - D_1 \rangle$ and $\Theta_0 \not\subseteq L$. $\square$

PROPOSITION 6. *Suppose that $k[S_1]$ and $k[S_2]$ are Buchsbaum.*

(i) *If $k[S_1]$ and $k[S_2]$ are isomorphic as algebras then they are isomorphic as augmented algebras.*

(ii) *If S_1 and S_2 do not correspond to products along a line then every algebra isomorphism $\phi: k[S_1] \to k[S_2]$ is augmented.*

Proof. Let $I \subseteq k[S_2]$ be a proper differential ideal, i.e., $D(I) \subseteq I$ for every $D \in \Theta(S_2)$. We claim that I is generated by some monomials t^s, $s \in S_2$. In particular, I is contained in the augmentation ideal generated by all t^s, $s \in S_2 \setminus \{0\}$. Given $f = \sum b_s t^s \in I$ fix any s with $b_s \neq 0$. Take any of the remaining $\lambda \in S_2$ with $b_\lambda \neq 0$ and choose j with $\lambda_j \neq s_j$. Then $\sum_\mu (\lambda_j - \mu_j) b_\mu t^\mu = \lambda_j f - D_j(f) \in I$ contains less monomials than f but still the monomial t^s. Repeated application yields $t^s \in I$, proving the claim.

Now assume $S_1 = \mathbb{N}^m \oplus M$ for some $M \subseteq \mathbb{N}^{n-m}$ which does not correspond to a product along a line. Let J be the ideal of $k[S_1]$ generated by all t^μ, $\mu \in M \setminus \{0\}$. We claim that J is differential. Consider any $\lambda \in \Lambda_i$, $i = 1, \ldots, n$. In order to show $D_{\lambda i}(t^\mu) = \mu_i t^{\lambda+\mu} \in J$ we may assume $\mu_i \neq 0$. Then $\lambda + \mu \in S_1$. As M does not correspond to a product along a line we have $|\mu| \geqslant 2$ and conclude $\lambda + \mu = \nu + \mu'$ with $\nu \in \mathbb{N}^m$ and $\mu' \in M \setminus \{0\}$. Hence $t^{\lambda+\mu} = t^{\nu+\mu'} \in J$.

Let $\phi: k[S_1] \to k[S_2]$ be an algebra isomorphism. It induces a Lie algebra isomorphism $\phi^\sharp: \Theta(S_1) \to \Theta(S_2)$ by $D \mapsto \phi \circ D \circ \phi^{-1}$. Since J is differential its

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image in $k[S_2]$ is differential and, hence, contained in the augmentation ideal of $k[S_2]$. We have $k[S_1] = k[M][t_1, \dots, t_m]$. For $i = 1, \dots, m$ let c_i be the constant term of $\phi(t_i)$. Define the $k[M]$-automorphism ψ of $k[S_1]$ by $\psi(t_i) = t_i - c_i$, $i = 1, \dots, m$. Then the $\phi \circ \psi(t_i)$ have no constant term. Since the augmentation ideal of $k[S_1]$ is generated by $t_1, \dots, t_m$ and J this means that $\phi \circ \psi$ is augmented. Assertion (ii) now also is clear because in that case J equals the augmentation ideal. □

THEOREM 1. *Let S_1, S_2 be simplicial affine semigroups such that $k[S_1]$, $k[S_2]$ are Buchsbaum. Suppose that the Lie algebras $\Theta(S_1)$, $\Theta(S_2)$ are isomorphic. Then S_1, S_2 have the same rank and the semigroups $\tilde{S}_1$, $\tilde{S}_2$ are isomorphic.*

Proof. If $\Theta(S_1)$ equals its derived algebra then S_1 and S_2 correspond to products along a line. By a result of Skryabin [Sk, Thm. 2] the semigroup rings $k[S_1], k[S_2]$ are isomorphic. Then [G, Thm. 2.1] and Proposition 6 imply that the semigroups S_1, S_2 themselves are isomorphic. Now suppose that the derived algebra is strictly smaller than $\Theta(S_1)$. Then $|\lambda| \geqslant 0$ for all $\lambda \in \Lambda(S_1)$. As $[\Theta_\lambda, \Theta_\mu] \subseteq \Theta_{\lambda+\mu}$ for all roots λ, μ the subspaces $I_d = \bigoplus_{|\lambda| \geqslant d} \Theta_\lambda$ are ideals of $\Theta(S_1)$ with finite-dimensional quotients $\Theta(S_1)/I_d$ and $\bigcap_{d \in \mathbb{N}} I_d = 0$. Given an isomorphism $\Theta(S_1) \simeq \Theta(S_2)$ we obtain an Abelian subalgebra H_2 of $\Theta(S_1)$ and another root space decomposition $\Theta(S_1) = \bigoplus_{\mu \in H_2^*} \Theta'_\mu$. Every finite-dimensional subspace of $\Theta(S_1)$ is mapped isomorphically onto its image in $\Theta(S_1)/I_d$ if d is sufficiently large. Thus, for $d \gg 0$, H_2 embeds into $Q = \Theta(S_1)/I_d$. For $\mu \in H_2^*$ consider the root spaces

$$Q'_\mu = \{D \in Q,\ [h, D] = \mu(h) \cdot D \text{ for all } h \in H_2\}.$$

Their sum is direct. Since each Θ'_μ is mapped into Q'_μ and the images of the Θ'_μ span Q we see $Q = \bigoplus_{\mu \in H_2^*} Q'_\mu$ and that each Θ'_μ is mapped onto Q'_μ. In particular, $Q'_0 = H_2$. It follows that H_2 equals its normalizer in Q and, hence, is a Cartan subalgebra of Q. Using Proposition 3, Remark 1, and Proposition 4 we may assume that the subsemigroup of H_2^* generated by all μ with $\dim Q'_\mu = \dim H_2 = \operatorname{rk} S_2$ equals $\tilde{S}_2$. Analogous statements hold true for H_1 and $d \gg 0$. Since Q is finite dimensional there is an automorphism of Q mapping the Cartan subalgebra H_1 onto the second Cartan subalgebra H_2, [Hu, Sect. 16]. Its dual induces an isomorphism between the semigroups $\tilde{S}_1$ and $\tilde{S}_2$. □

Using Remark 2 we conclude

COROLLARY 1. *Simplicial affine semigroups S of rank $\geqslant 2$ with $k[S]$ Cohen–Macaulay are uniquely determined by their Lie algebra $\Theta(S)$.*

Look again at Gubeladze's Theorem that S is uniquely determined by the augmented algebra $k[S]$. In the above proof we applied this only in case S does correspond

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to a product along a line. Therefore, using the Lie algebra $\Theta(S)$ as an intermediate step, we have reproved Gubeladze's Theorem in the special case that S is simplicial, does not correspond to a product along a line, and $k[S]$ is Cohen–Macaulay of dimension $\geqslant 2$. But $\Theta(S)$ cannot distinguish between semigroups with the same Cohen–Macaulayfication:

EXAMPLE 4. Fix $d, l \in \mathbb{N}$, both $\geqslant 2$. Let S consist of all $s \in \mathbb{N}^2$ with $|s| = md$, $m \geqslant l$. Then $k[S]$ is Buchsbaum and the Cohen–Macaulayfication S' is generated by $(0,d), (1,d-1), \ldots, (d-1,1), (d,0)$. Both S and S' have the same exceptional roots, see Example 2. Hence, $\Theta(S) = \Theta(S')$, independently of l.

EXAMPLE 5. Let S_1 (resp. S_2) be generated by all $\lambda \in \mathbb{N}^2$ with $|\lambda| = 6$ except $\lambda = (3,3)$ (resp. $\lambda = (2,4)$). They have a Buchsbaum semigroup ring and the same Cohen–Macaulayfication generated by all $\lambda \in \mathbb{N}^2$ with $|\lambda| = 6$. In both cases the exceptional roots are $(-1,7) + m(0,6)$ and $(7,-1) + m(6,0)$ with $m \in \mathbb{N}$. Hence $\Theta(S_1) = \Theta(S_2)$. But S_1, S_2 are not isomorphic. In fact, any isomorphism would map the set of extremal elements $\{(6,0),(0,6)\}$ onto itself, hence $(6,6)$ onto $(6,6)$. This contradicts $(6,6) = 2(3,3)$ in S_2 but $(6,6) \neq 2s$ for all $s \in S_1$. Observe that both semigroups correspond to affine cones over smooth projective curves in $\mathbb{P}^5$.

In the rank 1 case the situation is different. Although the semigroup ring always is Cohen–Macaulay the semigroup is, in general, not determined by the Lie algebra:

EXAMPLE 6. The numerical semigroups generated by 2 and 3 (resp. 3, 4 and 5) have the same $\tilde{S} = \mathbb{N}$, hence the same Lie algebra. Observe that the semigroup ring is Gorenstein in the first case whereas it has Cohen–Macaulay type 2 in the second, see Remark 1.

EXAMPLE 7. The numerical semigroups generated by 3, 7 and 8 (resp. 4, 5 and 7) have the same $\tilde{S}$ generated by 3, 4 and 5, hence the same Lie algebra. Observe that the Cohen–Macaulay type is 2 in both cases.

COROLLARY 2. *Numerical semigroups S with $k[S]$ Gorenstein are uniquely determined by $\Theta(S)$ and even by the finite-dimensional Lie algebra $\Theta(S)/[L,L]$.*

Proof. If $L = \Theta(S)$ then $S = \mathbb{N}$. So suppose $L \neq \Theta(S)$. Then $\tilde{S}$ is the set of roots and $L = \bigoplus_{\lambda \neq 0} \Theta_\lambda$. This implies $\Theta_\lambda \cap [L,L] = 0$ for λ in the minimal generator system of $\tilde{S}$ and $\Theta_\lambda \subseteq [L,L]$ for every λ which can be decomposed as $\lambda = \mu + \nu$ with two different $\mu, \nu \in \tilde{S}$. We see that $\Theta(S)/[L,L]$ is finite dimensional and that we can use the intrinsically defined ideal $[L,L]$ instead of I_d in the proof of Theorem 1. It remains to show that S is uniquely determined by $\tilde{S}$ in the Gorenstein case. By [HK, Satz 1.9, Prop. 2.21] we know $\tilde{S} = S \cup \{c-1\}$ with the conductor c of S. Consider first the case $\tilde{S} = \mathbb{N}$. Then S must be the semigroup $\mathbb{N} \setminus \{1\}$, generated by 2 and 3. Now let $\tilde{S} \neq \mathbb{N}$. Let a be the smallest element of

S different from 0. As S is a symmetric semigroup we see $c-2, \dots, c-a \in S$ but $c-a-1 \notin S$. Thus, $\tilde{S}$ has conductor $c-a$. Then $c-a \in S \setminus \{0\}$ implies $c-1 > c-a \geqslant a$. Hence, a is the smallest element of $\tilde{S}$ different from 0. Therefore, $S = \tilde{S} \setminus \{c-1\}$ is determined via $c-a$ and a by $\tilde{S}$. □

4. Automorphisms of the Lie Algebra

Every automorphism ϕ of $k[S]$ induces a Lie algebra automorphism

$$\phi^\sharp\colon \Theta(S) \to \Theta(S)\colon D \mapsto \phi \circ D \circ \phi^{-1}.$$

The purpose of this section is to show

THEOREM 2. *Let S be a simplicial affine semigroup such that $k[S]$ is Cohen–Macaulay. For every automorphism Φ of $\Theta(S)$ there is a unique automorphism ϕ of $k[S]$ such that $\Phi = \phi^\sharp$.*

Proof. If $\Phi = \phi^\sharp$ then $\Phi(f \cdot \Phi^{-1}(D)) = \phi(f) \cdot D$ for all $f \in k[S]$ and $D \in \Theta(S)$. This shows uniqueness. Now take an arbitrary automorphism Φ of $\Theta(S)$. If S corresponds to a product along a line the assertion follows from [Sk, Thm. 2]. Hence we may assume that S does not correspond to a product along a line. Let $Y_i = \Phi(D_i)$ and $Y_{\lambda i} = \Phi(D_{\lambda i})$. We have $\Theta(S) = \bigoplus_{\lambda \in \Lambda} \Theta'_\lambda$ with

$$\Theta'_\lambda = \Phi(\Theta_\lambda) = \{Y \in \Theta(S),\ [Y_i, Y] = \lambda_i \cdot Y \text{ for all } i\}.$$

The map $f \mapsto fY_1$ is an embedding of $\Phi(H)$-modules $k[S] \to \Theta(S)$. Hence $R = k[S]$ admits an eigenspace decomposition $R = \bigoplus_{\lambda \in \Lambda} R_\lambda$ with $R_\lambda = \{f \in R,\ Y_i(f) = \lambda_i \cdot f \text{ for all } i\}$. For any nonzero $x_\mu \in R_\mu$ the elements $x_\mu Y_1, \dots, x_\mu Y_n$ of Θ'_μ are linearly independent. Hence $M = \{\mu \in \Lambda,\ R_\mu \neq 0\}$ is a subsemigroup of $\tilde{S}$ and, for $\mu \in M$, the root space Θ'_μ is spanned by the elements above. It follows easily that the corresponding eigenspace R_μ is one-dimensional. By [GW, Chapter III.1] the M-graded rings R and $k[M]$ are isomorphic. Let K be the localization of R with respect to the multiplicative subset $\bigcup_{\mu \in M}(R_\mu \setminus \{0\})$. It is isomorphic to the group ring $k[G]$ where $G \subseteq \mathbb{Z}^n$ is the subgroup generated by M. We have a decomposition $K = \bigoplus_{\nu \in G} K_\nu$ with

$$K_\nu = \{f \in K,\ Y_i(f) = \nu_i \cdot f \text{ for all } i\}$$

and each K_ν is one-dimensional, say spanned by x_ν, $\nu \in G$. Since G is free Abelian of rank n there is a root space decomposition Der $K = \bigoplus_{\nu \in G} \Theta''_\nu$ with

$$\Theta''_\nu = \{Y \in \text{Der } K,\ [Y_i, Y] = \nu_i \cdot Y \text{ for all } i\}$$

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and each Θ''_ν is spanned by $x_\nu Y_1, \ldots, x_\nu Y_n$, $\nu \in G$. Now there is an embedding $\Theta(S) = \text{Der}\, R \subseteq \text{Der}\, K$. This implies $\tilde{S} \subseteq G$ and $\Theta'_\nu = \Theta''_\nu$ for $\nu \in \tilde{S}$. Next we claim

$$Y_{\mu i} = b_{\mu i} x_\mu Y_i \quad \text{for all} \quad \mu \in M \text{ and all } i$$

with suitable constants $b_{\mu i} \neq 0$. To prove this, note that $[D_{\mu i}, D_{\nu j}] = -\mu_j D_{\mu+\nu,i}$ if $\nu_i = 0$ and thus $Y = Y_{\mu i}$ has the following property: For all $\nu \in \tilde{S}$ with $\nu_i = 0$ the image of ad $Y\colon \Theta'_\nu \to \Theta'_{\mu+\nu}$ has dimension $\leqslant 1$. Hence, it is enough to show that, up to multiplication with a constant, $x_\mu Y_i$ is the unique element of Θ'_μ with this property. In fact, for $Y = \sum_k c_k x_\mu Y_k$ the matrix of coefficients of $([Y, x_\nu Y_j])_j$ with respect to the basis $(x_\mu x_\nu Y_k)_k$ has determinant equal to the value at $\sum c_k \nu_k$ of the characteristic polynomial of the matrix $(\mu_j c_k)_{j,k}$. The semigroup of elements $\nu \in \tilde{S}$ with $\nu_i = 0$ has rank $n-1$. Thus, if $c_k \neq 0$ for some $k \neq i$ it is possible to choose $\nu \in \tilde{S}$ with $\nu_i = 0$ such that $\sum c_k \nu_k$ is not a zero of the characteristic polynomial mentioned above. This proves the claim.

For fixed $\mu \in M$ choose $\nu \in M$ with $\nu_1 \neq \mu_1$ and $\nu_i \neq 0$ for all $i \neq 1$. Then the usual commutator relation implies $b_{\mu i} b_{\nu 1} = b_{\mu+\nu,1}$ for all i. Hence, the $b_{\mu i}$ are independent of i. By a suitable choice of the x_μ we obtain

$$Y_{\mu i} = x_\mu Y_i \quad \text{for all } \mu \in M \text{ and all } i.$$

For $\lambda \in \Lambda_i$ and $\mu \in M$ one calculates

$$Y_{\lambda i}(x_\mu) \cdot Y_j - \lambda_j x_\mu Y_{\lambda i} = \mu_i Y_{\lambda+\mu,j} - \lambda_j Y_{\lambda+\mu,i}.$$

Let us use this equation to show $\tilde{S} + (M \setminus \{0\}) \subseteq M$. In fact, for $\lambda \in \tilde{S}$ and $\mu \in M \setminus \{0\}$ one has $Y_{\lambda i}(x_\mu) \in R_{\lambda+\mu}$. If $\lambda + \mu \notin M$ then $Y_{\lambda i}(x_\mu) = 0$ for all i. This clearly is impossible for $n = 1$. Otherwise, look at

$$\lambda_j x_\mu Y_{\lambda i} = \lambda_j Y_{\lambda+\mu,i} - \mu_i Y_{\lambda+\mu,j}.$$

After choosing i such that $\mu_i \neq 0$ one sees $\lambda_j \neq 0$ for all j. Division by λ_j leads to n equations which are contradictory in case $n \geqslant 2$. Our next claim is

$$x_\lambda x_\mu = x_{\lambda+\mu} \quad \text{for all} \quad \lambda, \mu \in M.$$

For such λ, μ we have

$$\mu_i x_\lambda x_\mu Y_j - \lambda_j x_\lambda x_\mu Y_i = \mu_i x_{\lambda+\mu} Y_j - \lambda_j x_{\lambda+\mu} Y_i.$$

In case $n \geqslant 2$, this immediately implies the claim whereas for $n = 1$ one needs $\mu \neq \lambda$. To show $x_\lambda^2 = x_{2\lambda}$ one may proceed similarly as in the last step of the proof of Theorem 3 below.

Suppose that $n \geqslant 2$. As $R = k[S] \simeq k[M]$ is Cohen–Macaulay we have $\tilde{S} = S$ and $M' = M$ with M' as defined at the beginning of Section 2, see [TH, Cor. 2.2]. But then $\tilde{S} + (M \setminus \{0\}) \subseteq M$ yields $M = S$. Therefore, $Y_{\lambda i}(x_\mu) = \mu_i \cdot x_{\lambda+\mu}$ for all $\lambda, \mu \in S$. We want to show the same equation for $\lambda \in E_i$ and $\mu \in S$. This is clear if $\lambda + \mu \notin S$ because then $Y_{\lambda i}(x_\mu) = 0$ and $\mu_i = 0$. Otherwise, given $\lambda \in E_i$ we may choose $s \in S \setminus \{0\}$ with $s_i = 0$, $\lambda + s \notin S$ and then j with $s_j \neq 0$. The claim follows by applying

$$Y_{\lambda i}(x_\mu) \cdot Y_j - \lambda_j x_\mu Y_{\lambda i} = \mu_i x_{\lambda+\mu} Y_j - \lambda_j x_{\lambda+\mu} Y_i$$

to x_s. We can define an automorphism ϕ of $k[S]$ by $\phi(t^s) = x_s$ and obtain $\Phi(D) = \phi \circ D \circ \phi^{-1}$ for all $D \in \Theta(S)$.

Finally, consider the case $n = 1$. Then $\tilde{S} + (M \setminus \{0\}) \subseteq M$ implies that M is a numerical semigroup. Let c be the conductor of M and $x = x_{c+1}/x_c \in k(t)$. For $\mu \in M$ one calculates $x_c^\mu x^\mu = x_c^\mu x_\mu$ and $x^\mu = x_\mu$. In particular, x is integral over $k[t]$ and hence contained in $k[t]$. Write for short $Y = \Phi(t\partial_t)$ and $Y_\lambda = \Phi(t^\lambda t\partial_t)$. From $Y(x_\mu) = \mu \cdot x_\mu$ one deduces $Y(x) = x$. This implies $Y = f\partial_t$ with a polynomial f of degree 1. Because $S \neq \mathbb{N}$, the constant term of f vanishes. Then x must be a monomial, say of degree r. Now $k[S] = \bigoplus_{\mu \in M} R_\mu$ with R_μ spanned by $t^{r\mu}$. Since S is a numerical semigroup we obtain $r = 1$ and $M = S$. For $\lambda \in \tilde{S}$ one has $x^\lambda Y \in \Theta'_\lambda$ and $x^\lambda Y$ is a scalar multiple of Y_λ. Using $Y_s = x^s Y$ for $s \in S$ and $[Y_\lambda, Y_s] = (s - \lambda) Y_{\lambda+s}$ one can deduce $Y_\lambda = x^\lambda Y$. Then $Y_\lambda(x^s) = s \cdot x^{\lambda+s}$ for all $\lambda \in \tilde{S}$ and $s \in S$. Therefore, the automorphism ϕ of $k[S]$ defined by $\phi(t^s) = x^s$ satisfies $\Phi = \phi^\sharp$. $\square$

5. Derivations of the Lie Algebra

In this section we show

THEOREM 3. *Let $S \subseteq \mathbb{N}^n$ be a simplicial affine semigroup such that $k[S]$ is Buchsbaum. Then every derivation Δ of $\Theta(S)$ is inner: $\Delta = \operatorname{ad} D$ for some $D \in \Theta(S)$.*

Proof. The cochain complex of the Lie algebra $\Theta(S)$ with coefficients in the adjoint representation has a $\mathbb{Z}^n$-grading given by the root space decomposition. By [F, Thm. 1.5.2b] it is acyclic in degrees different from zero. Hence, we may assume that the given Δ has degree 0, i.e. $\Delta(\Theta_\lambda) \subseteq \Theta_\lambda$ for all λ. For each root λ denote by $M(\lambda)$ the set of i such that $D_{\lambda i} \in \Theta(S)$. Thus $M(\lambda) = \{1, \ldots, n\}$ for ordinary roots and $M(\lambda) = \{i\}$ for i-exceptional roots. We have

$$\Delta(D_{\lambda i}) = \sum_{m \in M(\lambda)} b_{\lambda i m} D_{\lambda m} \quad \text{for} \quad i \in M(\lambda) \tag{1}$$

with suitable constants $b_{\lambda i m} \in k$. The brackets of the generators are given by

$$[D_{\lambda i}, D_{\mu j}] = \mu_i D_{\lambda+\mu,j} - \lambda_j D_{\lambda+\mu,i} \tag{2}$$

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Inserting (1) and (2) into the cocycle condition

$$\Delta([D_{\lambda i}, D_{\mu j}]) = [\Delta(D_{\lambda i}), D_{\mu j}] + [D_{\lambda i}, \Delta(D_{\mu j})]$$

gives

$$\sum_m (\mu_i \cdot b_{\lambda+\mu,j,m} - \lambda_j \cdot b_{\lambda+\mu,i,m}) D_{\lambda+\mu,m}$$
$$= \sum_m (\mu_i \cdot b_{\mu jm} - \lambda_j \cdot b_{\lambda im}) D_{\lambda+\mu,m} +$$
$$+ \left(\sum_m \mu_m \cdot b_{\lambda im}\right) D_{\lambda+\mu,j} - \left(\sum_m \lambda_m \cdot b_{\mu jm}\right) D_{\lambda+\mu,i}.$$

By comparing the coefficients one obtains

$$\mu_i \cdot b_{\lambda+\mu,j,m} - \lambda_j \cdot b_{\lambda+\mu,i,m} = \mu_i \cdot b_{\mu jm} - \lambda_j \cdot b_{\lambda im} \quad \text{for} \quad m \neq i, j, \tag{3}$$

$$\mu_i \cdot b_{\lambda+\mu,j,j} - \lambda_j \cdot b_{\lambda+\mu,i,j} = \mu_i \cdot b_{\mu jj} - \lambda_j \cdot b_{\lambda ij} + \sum_m \mu_m \cdot b_{\lambda im} \quad \text{for} \quad j \neq i, \tag{4}$$

$$(\mu_i - \lambda_i) b_{\lambda+\mu,i,i} = \mu_i \cdot b_{\mu ii} - \lambda_i \cdot b_{\lambda ii} + \sum_m \mu_m \cdot b_{\lambda im} - \sum_m \lambda_m \cdot b_{\mu im}. \tag{5}$$

Equation (4) with $\lambda = \mu = \alpha^j$ yields

$$b_{2\alpha^j,i,j} = 0 \quad \text{for} \quad i \neq j. \tag{6}$$

Let us show that $b_{\lambda ij} = 0$ for all $\lambda \in \tilde{S}$ and all $i, j \in M(\lambda)$ with $i \neq j$. Set $\mu = 2\alpha^j$. In case $\lambda_i = 0$ the claim follows from (5) and (6). If $\lambda_i \neq 0$ use (3) with $j = i$ and m replaced by j to show $b_{\lambda+\mu,i,j} = b_{\lambda ij}$. Then (4) gives the claim.

Now we have $\Delta(D_{\lambda i}) = b_{\lambda i} D_{\lambda i}$ for $i \in M(\lambda)$, with suitable $b_{\lambda i} \in k$. Equations (4) and (5) reduce to

$$\mu_i \cdot b_{\lambda+\mu,j} = \mu_i \cdot b_{\mu j} + \mu_i \cdot b_{\lambda i} \quad \text{for} \quad j \neq i, \tag{7}$$

$$(\mu_j - \lambda_j) b_{\lambda+\mu,j} = (\mu_j - \lambda_j)(b_{\lambda j} + b_{\mu j}). \tag{8}$$

For fixed $\lambda \in \tilde{S}$ the coefficients $b_{\lambda i}$ are independent of $i \in M(\lambda)$. In fact, for $j \neq i$ apply (7) and (8) where μ is any element of $\tilde{S}$ with $\mu_i \neq 0$ and $\mu_j \neq \lambda_j$. Thus we may write b_λ instead of $b_{\lambda i}$.

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Consider first the case $n \geqslant 2$. Then (7) implies $b_{\lambda+\mu} = b_\lambda + b_\mu$ for $\lambda, \mu \in \tilde{S}$. Let $c_i = b_{\alpha^i}/a_i$ where a_i denotes the nonzero entry of α^i. Using the fact that $\tilde{S}$ is torsion modulo the semigroup generated by the α^i one shows $b_\lambda = \sum_i c_i \lambda_i$ for all $\lambda \in \tilde{S}$. The same is seen to hold for $\lambda \in \Lambda_i$ by applying (7) with some $\mu \in S$, $\mu_i \neq 0$. We have proven

$$\left[\sum_i c_i D_i, D_{\lambda j}\right] = \sum_i c_i \lambda_i D_{\lambda j} = b_\lambda D_{\lambda j} = \Delta(D_{\lambda j})$$

for all $\lambda \in \Lambda$ and $j \in M(\lambda)$. This means $\Delta = \operatorname{ad} D$ for $D = \sum_i c_i D_i$.

In the case $n = 1$ only Equation (8) is available. Then $b_{5\lambda} = b_{3\lambda} + b_{2\lambda} = 2b_{2\lambda} + b_\lambda$ and $b_{5\lambda} = b_{4\lambda} + b_\lambda = b_{3\lambda} + 2b_\lambda = b_{2\lambda} + 3b_\lambda$, hence $b_{2\lambda} = 2b_\lambda$ and then $b_{m\lambda} = mb_\lambda$ for all $m \in \mathbb{N}, \lambda \in \tilde{S}$ with $m, \lambda > 0$. This shows that the ratio b_λ/λ is independent of λ, say $b_\lambda/\lambda = c$. Hence $b_\lambda = c\lambda$ for all positive roots. Since the same clearly holds for $\lambda = 0$ (and $\lambda = -1$ in the special case $S = \mathbb{N}$) we have again shown that Δ is inner. □

Remark 3. In the special case $S = \mathbb{N}^n$ Theorem 3 was proven by Heinze [He, Kap. II, Satz 2.8]. More generally, for semigroups corresponding to a product along a line it follows from work of Skryabin [Sk, Thm. 3].

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References

[F] Fuks, D. B.: *Cohomology of Infinite-Dimensional Lie Algebras*, Consultants Bureau, New York, 1986.

[GSW] Goto, S., Suzuki, N. and Watanabe, K.: On affine semigroup rings, *Japan. J. Math.* **2** (1976), 1–12.

[GW] Goto, S. and Watanabe, K.: On graded rings, II ($\mathbb{Z}^n$-graded rings), *Tokyo J. Math.* **1** (1978), 237–261.

[G] Gubeladze, J.: The isomorphism problem for commutative monoid rings, Preprint.

[HM] Hauser, H. and Müller, G.: On the Lie algebra $\Theta(X)$ of vector fields on a singularity, *J. Math. Sci. Univ. Tokyo* 1 (1994), 239–250.

[He] Heinze, J.: Über die rechts-symmetrische Algebra und die Lie-Algebra der konvergenten Potenzreihen eines Banachraumes, *Schriftenreihe Math. Inst. Univ. Münster 2. Ser.* 16 (1979).

[HK] Herzog, J. and Kunz, E.: Die Wertehalbgruppe eines lokalen Rings der Dimension 1, *Sitzungsber. Heidelb. Akad. Wiss. Math.-Natur. Kl.* **2** (1971), 27–67.

[Ho] Hochster, M.: Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, *Ann. of Math. (2)* **96** (1972), 318–337.

[Hu] Humphreys, J. E.: *Introduction to Lie Algebras and Representation Theory*, Springer, New York, 1972.

[MT] Molinelli, S and Tamone, G.: On the derivations of the homogeneous coordinate ring of a monomial curve in $\mathbb{P}^d_k$, *Comm. Algebra* **20** (1992), 3279–3300.

[M] Müller, G.: Lie algebras attached to simplicial complexes, *J. Algebra* **177** (1995), 132–141.

[SS] Schäfer, U. and Schenzel, P.: Dualizing complexes and affine semigroup rings, *Trans. Amer. Math. Soc.* **322** (1990), 561–582.

[Se] Seidenberg, A.: Derivations and integral closure, *Pacific J. Math.* **16** (1966), 167–173.

[Si] Siebert, T.: Lie algebras of derivations and affine algebraic geometry over fields of characteristic 0, *Math. Ann.* **305** (1996), 271–286.

[Sk] Skryabin, S. M.: Regular Lie rings of derivations, *Moscow Univ. Math. Bull.* **43**(3) (1988), 56–58.

[St] Stanley, R. P.: Hilbert functions of graded algebras, *Adv. in Math.* **28** (1978), 57–83.

[TH] Trung, N. V. and Hoa, L. T.: Affine semigroups and Cohen–Macaulay rings generated by monomials, *Trans. Amer. Math. Soc.* **298** (1986), 145–167.