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CHARACTERIZATION OF A FAMILY OF SIMPLE GROUPS BY THEIR CHARACTER TABLE, II

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Abstract

It is shown that the simple groups $G_2(q)$, $q = 3^f$, are characterized by their character table. This result completes characterization of the simple groups $G_2(q)$, q odd, by their character table.

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The aim of this paper is to prove the following result:

THEOREM 1. *The character table of $G_2(q)$, q odd, determines $G_2(q)$.*

By Theorem 3.2 in Herzog and Wright (1977), it suffices to deal with the case $q = 3^f$. Thus we prove, using the character tables of $G_2(3^f)$ recently computed by Enomoto (1976), that the following theorem holds:

THEOREM 2. *The character table of $G_2(q)$, $q = 3^f$, determines $G_2(q)$.*

PROOF. We shall use the notation of Enomoto (1976) for elements and characters of $G_2(q)$, $q = 3^f$. In addition, we shall denote by $\mathrm{Irr}(G)$ the set of irreducible characters of G and if $x \in G$, $\mathrm{Cl}(x)$ denotes the conjugacy class of x in G.

Suppose that $*G$ is a group with the same character table as $G_2(q)$, $q = 3^f$. Put an asterisk in front of each conjugacy class representative, character, and so on, of $*G$, to distinguish it from the same in $G_2(q)$. Since a character table determines the order of the group and the lattice of normal subgroups, see Feit (1967), $*G$ is simple with $|*G| = q^6(q^2-1)(q^6-1)$. The first step is to establish that $*G$ has a unique conjugacy class of involutions.

LEMMA 3. *The only conjugacy class of involutions in $*G$ is that represented by $*B_1$.*

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PROOF. By Enomoto (1976), p. 239, $*B_1$ is the only class representative with the full 2-power of $|*G|$ dividing the order of its centralizer. Hence $*B_1$ is an involution. The classes of $G_2(q)$ denoted by A_i or A_{ij} in Enomoto (1976) consist of 3-elements, and by Lambert (1972), Property 2.5, also $*A_i$ and $*A_{ij}$ are 3-elements. Let $*F$ be a conjugacy class representative in $*G$, $*F \neq *A_i, *A_{ij}, *B_1$. Then by Enomoto (1976), p. 239,

$$|C_{*G}(*F)| \leqslant q(q+1)(q^2-1)$$

hence

$$|\mathrm{Cl}_{*G}(*F)| \geqslant |G|/q(q+1)(q^2-1) = q^5(q^5-q^4+q^3-q^2+q-1).$$

Consequently, we get

$$(1) \qquad |\mathrm{Cl}_{*G}(*F)| \geqslant q^{10}-q^9.$$

Suppose that $*F$ is an involution. If t is the number of involutions in $*G$, then by Feit (1967), p. 23,

$$(2) \qquad t+1 \leqslant \sum *Y_i(1) = \sum Y_i(1),$$

where $*Y_i(Y_i)$ runs through $\mathrm{Irr}(*G)(\mathrm{Irr}(G_2(q))$. To obtain an upper bound for $\sum Y_i(1)$ the following inequalities were used:

$$q \geqslant 3, \quad (q+1)^2 \leqslant 2q^2, \quad q^i+1 \leqslant 4q^i/3, \quad i = 1,\dots,$$

$$q^i-d \leqslant q^i \quad \text{for } d \geqslant 0, \quad (q^i)^2+q^i+1 \leqslant 3q^{2i}/2, \quad i = 1,\dots,$$

and $\qquad (q^i)^2-q^i+1 \leqslant q^{2i}, \quad i = 1,\dots.$

We get, using the notation of Enomoto (1976):

$$\sum_{i=0}^{12} \theta_i(1) \leqslant 1+\tfrac{3}{2}q^4+6\tfrac{13}{18}q^5+2\tfrac{1}{2}q^6 \leqslant 1+5q^6$$

and

$$\sum_{i=1}^{14} r(X_i)X_i(1) \leqslant 3q^6+3q^7+1\tfrac{35}{108}q^8 \leqslant 3q^8,$$

where $r(X_i)$ is the number of characters of type X_i and degree $X_i(1)$ in $\mathrm{Irr}(G_2(q))$. Thus:

$$(3) \qquad \sum Y_i(1) \leqslant 1+4q^8 \leqslant 1+4q^9/3.$$

If $*F$ were an involution, we would get from (1), (2) and (3) that

$$1+q^{10}-q^9 \leqslant 1+4q^9/3$$

hence $q \leqslant 7/3$, a contradiction. Thus $*F$ is not an involution, proving the lemma.

We need also:

LEMMA 4. 2-rank $*G \leqslant 3$.

PROOF. By Lemma 3 and by Lemma 2.1(b) in Herzog and Wright (1977), it suffices to find an $X \in \operatorname{Irr}(G_2(q))$ such that

$$X(A_1) - X(B_1) \not\equiv 0 \pmod{16}. \tag{4}$$

First suppose that $q = 3^f$, f odd. Then $q \equiv 3$ or $11 \pmod{16}$ and we get, using the tables of Ecomoto (1976):

$$3(\theta_3(A_1) - \theta_3(B_1)) = q(q^4 + q^2 - 2) \equiv 8 \pmod{16}.$$

Hence θ_3 satisfies (4) in this case. For $q = 3^f$, f even, we have: $q \equiv 1$ or $9 \pmod{16}$. Consider $X_1(k)$, $k \in {}^2R_2$. Clearly $1, 2 \in {}^2R_2$, hence:

$$(X_1(1)(A_1) - X_1(1)(B_1)) - (X_1(2)(A_1) - X_1(2)(B_1)) = 2(q+1)^2 \equiv 8 \pmod{16}$$

and either $X_1(1)$ or $X_1(2)$ satisfies (4). The lemma is proved in all cases.

We now complete the proof of Theorem 2. By Lemma 4 2-rank $*G \leqslant 3$. Since $G_2(q)$ has 2-rank 3, by Lemma 2.1.(b) in Herzog and Wright (1977) $X(A_1) - X(B_1) \equiv 0 \pmod 8$ for each $X \in \operatorname{Irr}(G_2(q))$. Consequently, Corollary 2.5 in Herzog and Wright (1977) and the Note following it yield: 2-rank $*G = 3$. As in Herzog and Wright (1977), p. 303, we conclude, using Stroth's (1976) classification of simple groups of 2-rank 3, that $*G = G_2(q)$ unless $q = 3$. In the latter case $|*G| = 3^6 \cdot 8(3^6 - 1)$ and it is easy to check that the only group of that order in Stroth's list is $G_2(3)$. Hence $*G = G_2(q)$ for each $q = 3^f$, thus proving Theorem 2.

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