Apolar Triads on a Cubic Curve.

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Professor W. P. Milne has shown[1] that if a pencil of plane cubic curves cut in two triads of points which are apolar to the members of the pencil, then the other three points of intersection also form an apolar triad to the pencil. I propose *to show how to obtain a simple geometrical construction for the third apolar triad* though the cubics in this case are not perfectly general. The method of approach is by means of Grassmann's construction for a cubic curve and the use of apolar theorems established for a curve described in this manner.[2]

§ 1. The locus of a point (x_1, x_2, x_3), or briefly x_i, whose joins to three points A, B, C meet three lines

$$a_1 x_1 + a_2 x_2 + a_3 x_3 \equiv \alpha_x = 0 \quad \beta_x = 0 \quad \gamma_x = 0$$

in three collinear points is a general cubic curve. Take the points A, B, C as having coordinates a_i, b_i, c_i, respectively, and the three lines as the sides of the triangle of reference. Then the cubic curve has the equation,

$$\begin{aligned} f = x_2^2 x_3 (b_3 c_1 - b_1 c_3) a_1 + x_2 x_3^2 (b_1 c_2 - b_2 c_1) a_1 + x_1^2 x_3 (c_2 a_3 - c_3 a_2) b_2 \\ + x_1 x_3^2 (c_1 a_2 - c_2 a_1) b_2 + x_1^2 x_2 (a_2 b_3 - a_3 b_2) c_3 + x_1 x_2^2 (a_3 b_1 - a_1 b_3) c_3 \\ + x_1 x_2 x_3 (2a_1 b_2 c_3 - a_3 b_1 c_2 - a_2 b_3 c_1) = 0. \quad \ldots\ldots \text{(i)} \end{aligned}$$

The vertices of the triangle of reference form an apolar triad to the cubic curve if the coefficient of $x_1 x_2 x_3$ vanishes, namely if,

$$2 a_1 b_2 c_3 - a_3 b_1 c_2 - a_2 b_3 c_1 = 0. \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Incidentally, if this holds, then ABC also forms an apolar triad.[2]

Hence if the point A, (a_i) lies on the line,

$$2 x_1 b_2 c_3 - x_2 b_3 c_1 - x_3 b_1 c_2 = 0, \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

[1] Prof. W. P. Milne. *Proc. Edin. Math. Soc.*, vol. 30 (1911-12).

[2] W. Saddler. *Proc. Lond. Math. Soc.* (2) 26 (1927), 249-256.

the cubic curve described, in the Grassmann manner, with the points A, B, C and the sides of the triangle of reference as fundamental elements, will have two triads apolar—namely the triad ABC and the vertices of the triangle of reference.

A ruler construction for this locus (iii) has been obtained[1].

Now describe a new cubic in a manner analogous to f but interchange the relative positions of the points B and C. We thus obtain f' where,

$$\begin{aligned} f' = x_2^2 x_3 (b_1 c_3 - b_3 c_1) a_1 + x_2 x_3^2 (b_2 c_1 - b_1 c_2) a_1 + x_1^2 x_3 (b_2 a_3 - b_3 a_2) c_2 \\ + x_1 x_3^2 (b_1 a_2 - b_2 a_1) c_2 + x_1^2 x_2 (a_2 c_3 - a_3 c_2) b_3 + x_1 x_2^2 (a_3 c_1 - a_1 c_3) b_3 \\ + x_1 x_2 x_3 (2a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1) = 0. \end{aligned}$$

Hence if the point A be taken as the point of intersection of the two lines,

$$2x_1 b_2 c_3 - x_2 b_3 c_1 - x_3 b_1 c_2 = 0,$$
$$2x_1 b_3 c_2 - x_2 b_1 c_3 - x_3 b_2 c_1 = 0,$$

the two triads ABC, $A'B'C'$ will be apolar to both f and f', and hence the pencil of cubics will have a third apolar triad in common. Now from the Grassmann figure one common point will be where BC meets $B'C'$—say A''. The other two points S and T will be shown to be two points common to two definite conics whose other two points of intersection are A and A'.

It is easily shown that,

if $(bcx) = \Sigma \pm b_1 c_2 x_3$, and F, F' are the two conics,

$$F = 2a_1 x_2 x_3 - a_3 x_1 x_2 - a_2 x_1 x_3 = 0,$$
$$F' = x_1 x_3 (2b_2 c_2 a_3 - a_2 b_2 c_3 - a_2 b_3 c_2) + \ldots = 0.$$

Then $\quad (bcx) . F \equiv f - f'$,

and $\quad x_1 . F' \equiv f + f'$.

The tangent to F at the point $A'(1, 0, 0)$, being $a_2 x_3 + a_3 x_2 = 0$, is harmonic to $a_2 x_3 - a_3 x_2 = 0$ with respect to the two sides

$$x_2 = 0 \text{ (or } A'B') \quad x_3 = 0 \text{ (or } A'C');$$

and the line $a_2 x_3 - a_3 x_2$ is the line $A'A$. Hence this conic F is determined. Similarly for the conic F', the lines AA' and the

[1] W. Saddler. *loc. cit.*

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tangent at A are harmonic with respect to AB and AC. This conic passes through the points $ABCA'$ and hence is also uniquely determined. These two conics thus intersect in the points A, A', S, T, and hence the third apolar triad to the pencil of cubics will be $A''ST$. *Q.E.D.*

A further point in connection with the cubic curve f (or f') and the locus (iii) seems worthy of remark.

Take the conic passing through the points B, C, A', B', C', and let the line (iii), viz. $2x_1 b_2 c_3 - x_2 b_3 c_1 - x_3 b_1 c_2 = 0$, meet the conic in A_1 and A_2. Describe the Grassmann cubic with the base points $A_1 B, C, A', B', C'$, and through the same points construct the apolar locus,[1] viz. the locus of a point whose joins to $A_1 BC$ apolarly separate $A'B'C'$. For these two cubics both triads are apolar and hence intersect in a third apolar triad. Now the other three points of intersection are collinear; and hence being a collinear apolar triad, their join is tangent to the Cayleyan of the apolar locus and to this Grassmann cubic.[1]

[1] Milne, *loc. cit.*

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