Canad. Math. Bull. Vol. 61 (4), 2018 pp. 787–801
http://dx.doi.org/10.4153/CMB-2017-080-0

Endpoint Estimates of Riesz Transforms Associated with Generalized Schrödinger Operators

Yu Liu and Shuai Qi

Abstract. In this paper we establish the endpoint estimates and Hardy type estimates for the Riesz transform associated with the generalized Schrödinger operator.

1 Introduction

The Riesz transform is a singular integral operator in harmonic analysis and has been investigated by many scholars. In [8,9], Shen studied L^p estimates for the Riesz transform related to the Schrödinger operator and the generalized Schrödinger operator, respectively. It should be noted that these operators might not be Calderon–Zygmund operators if the potential satisfies some weaker conditions. Recently, Wu and Yan [11] studied the Hardy space by means of a maximal function associated with the heat semigroup generated by the generalized Schrödinger operator and obtained characterizations via atomic decomposition and Riesz transform. Following their works, the goal of our paper is to obtain the weak type estimates and Hardy type estimates for the Riesz transform associated with the generalized Schrödinger operator.

In order to state our main results, we recall some basic facts about the generalized Schrödinger operator which, in this paper, is defined as follows:

$$\mathcal{L} = -\Delta + \mu \text{ on } \mathbb{R}^n, \quad n \geq 3,$$

where μ is a nonnegative Radon measure on $\mathbb{R}^n$ and $\mu \not\equiv 0$ satisfies the following conditions: there exist positive constants C_0, C_1 and δ such that

$$\mu\big(B(x,r)\big) \leq C_0\Big(\frac{r}{R}\Big)^{n-2+\delta}\mu\big(B(x,R)\big), \tag{1.1}$$

$$\mu\big(B(x,2r)\big) \leq C_1\mu\big(B(x,r) + r^{n-2}\big) \tag{1.2}$$

for all $x \in \mathbb{R}^n$ and $0 < r < R$, where $B(x,r)$ denotes the open ball centered at x with radius r. As in [9], the measure μ satisfies conditions (1.1) and (1.2) for some $\delta > 0$ if

Received by the editors July 18, 2017; revised December 7, 2017.
Published electronically March 14, 2018.
Supported by the National Natural Science Foundation of China (No. 11671031, 11471018), the Fundamental Research Funds for the Central Universities (No. FRF-BR-17-004B), and Beijing Municipal Science and Technology Project (No. Z17111000220000).
AMS subject classification: **35J10**, 42B20, 42B30.
Keywords: Schrödinger operator, fundamental solution, Riesz transform.

$d\mu = V(x)dx$ and $V(x) \geq 0$ satisfies

$$\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)^{n/2}\,dy\Big)^{\frac{2}{n}} \leq C\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} V(y)\,dy\Big);$$

in other words, $V(x)$ is in the reverse Hölder class $(RH)_{n/2}$. Moreover, by virtue of [9], the auxiliary function $m(x,\mu)$ is defined by

$$\frac{1}{m(x,\mu)} = \sup_{r>0}\Big\{r : \frac{\mu(B(x,r))}{r^{n-2}} \leq C_1\Big\},$$

where C_1 is the constant in (1.2) and the distance function is defined by

$$d(x,y,\mu) = \inf_{\gamma}\int_0^1 m(\gamma(t),\mu)|\gamma'(t)|dt$$

with the modified Agmon metric

$$ds^2 = m(x,\mu)\{dx_1^2 + \cdots + dx_1^n\},$$

where $\gamma\colon[0,1] \to \mathbb{R}^n$ is absolutely continuous satisfying $\gamma(0) = x, \gamma(1) = y$.

Let $\mathcal{R} = \nabla(-\Delta+\mu)^{\frac{1}{2}}$ be the Riesz transform associated with the generalized Schrödinger operator. Using functional calculus, we can write

$$(-\Delta+\mu)^{-\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(-\Delta+\mu+\lambda)^{-1}d\lambda.$$

For $f \in C_0^\infty(\mathbb{R}^n)$,

$$\mathcal{R}f(x) = \int_{\mathbb{R}^n}\mathcal{K}(x,y)f(y)dy,$$

where

$$\mathcal{K}(x,y) = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\nabla_x\Gamma_{\mu+\lambda}(x,y)d\lambda$$

and $\Gamma_{\mu+\lambda}(x,y)$ denotes the fundamental solution of $-\Delta+\mu+\lambda$.

The following is the first main result of the paper.

Theorem 1.1 *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \geq 3$. Assume that μ satisfies conditions (1.1) and (1.2) for some $\delta \in (0,1)$. Then*

$$\big|\{x \in \mathbb{R}^n : |\mathcal{R}f(x)| > \alpha\}\big| \leq \frac{C}{\alpha}\,\|\,f\,\|_{L^1}, \qquad \textit{for every } \alpha > 0.$$

Remark 1.2 Theorem 1.1 combined with the L^2-boundedness given in [9] implies the L^p-boundedness of the generalized Riesz transforms by the Marcinkiewicz interpolation theorem for $1 < p < 2$.

As we know, the classical Hardy space is a good substitute for the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0,1]$ in the study for the boundedness of some singular integral operators, and it is essentially related to the Laplace operator Δ on $\mathbb{R}^n$. And its generalization, the Hardy space associated with $\mathcal{L}$, which has been studied by Wu and Yan in [11], is the counterpart of the classical Hardy space in the study for the boundedness of some singular integral operators associated with the generalized Schrödinger operator. In particular, this Hardy space was introduced in [3] when $d\mu = V(x)dx$ and $V \in (RH)_{n/2}$.

To state our next result, we need to recall some basic facts on the Hardy space associated with $\mathcal{L}$. We denote by $T_s^{\mathcal{L}}(x,y)$ the kernel of the semigroup $\{T_s^{\mathcal{L}} : s > 0\} = \{e^{-s\mathcal{L}} : s > 0\}$. It follows from [11] that the kernel of the semigroup $\{T_s^{\mathcal{L}} : s > 0\}$ has a Gaussian upper bound, that is,

$$0 \leqslant T_s^{\mathcal{L}}(x,y) \leqslant (4\pi s)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4s}}.$$

The following Hardy space $H^1_{\mathcal{L}}$ has been investigated by Wu and Yan in [11] and is defined as follows.

Definition 1.3 A function $f \in L^1(\mathbb{R}^n)$ is said to be in $H^1_{\mathcal{L}}$ if the maximal function $M^{\mathcal{L}} f$ belongs to $L^1(\mathbb{R}^n)$. The norm of such a function is defined by $\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} = \|M^{\mathcal{L}} f\|_{L^1(\mathbb{R}^n)}$, where $M^{\mathcal{L}} f(x)$ is the maximal function associated with $\{T_s^{\mathcal{L}} : s > 0\}$ defined by $M^{\mathcal{L}} f(x) = \sup_{s>0} |\, T_s^{\mathcal{L}} f(x) \,|$.

Definition 1.4 Let $1 \le q \le \infty$. A function $a \in L^q(\mathbb{R}^n)$ is called an $H^{1,q}_{\mathcal{L}}$*-atom* if $r < \frac{1}{m(x_0,\mu)}$ and the following conditions hold:

(i) $\operatorname{supp} a \subset B(x_0,r)$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \le |B(x_0,r)|^{\frac{1}{q}-1}$;

(iii) if $r < \frac{1}{4m(x_0,\mu)}$, then $\int_{B(x_0,r)} a(x)\,dx = 0$.

In [11], Wu and Yan gave the following atomic decomposition for $H^1_{\mathcal{L}}(\mathbb{R}^n)$.

Proposition 1.5 *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \ge 3$. Assume that μ satisfies conditions (1.1) and (1.2) for some $\delta > 0$. Then $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H^{1,\infty}_{\mathcal{L}}(\mathbb{R}^n)$-atoms, $\sum_j |\lambda_j| < \infty$, and the sum converges in the $H^1_{\mathcal{L}}(\mathbb{R}^n)$ quasi-norm. Moreover,*

$$\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} \sim \inf\Big\{ \sum_j |\lambda_j| \Big\},$$

where the infimum is taken over all atomic decompositions of f into $H^{1,\infty}_{\mathcal{L}}$-atoms.

By Proposition 1.5, we can conclude that the classical Hardy space $H^1(\mathbb{R}^n)$ is a subspace of the Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$. Furthermore, it is really easy to check that an $H^{1,\infty}_{\mathcal{L}}$-atom is also an $H^{1,q}_{\mathcal{L}}$-atom for $1 \le q < \infty$. Then we immediately have another equivalent characterization using the atomic decomposition.

Proposition 1.6 *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \ge 3$. Assume that μ satisfies conditions (1.1) and (1.2) for some $\delta > 0$. Then $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$ if and only if f can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H^{1,q}_{\mathcal{L}}$-atoms with $1 \le q < \infty$, $\sum_j |\lambda_j| < \infty$, and the sum converges in the $H^1_{\mathcal{L}}(\mathbb{R}^n)$ quasi-norm. Moreover,*

$$\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)} \sim \inf\Big\{ \sum_j |\lambda_j| \Big\},$$

where the infimum is taken over all atomic decompositions of f into $H^{1,q}_{\mathcal{L}}$-atoms.

Next, we state our second result.

Theorem 1.7 *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \geq 3$. Assume that μ satisfies conditions (1.1) and (1.2) for some $\delta \in (0,1)$. The Riesz transform $\mathcal{R}$ is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into the classical Hardy space $H^1(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f \in H^1_{\mathcal{L}}(\mathbb{R}^n)$,*

$$\left\|\mathcal{R}(f)\right\|_{H^1(\mathbb{R}^n)} \leq C\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n)}.$$

Remark 1.8 If $\delta > 1$, it follows from [9] that the Riesz transform $\mathcal{R}$ is a Calderón–Zygmund operator. So the weak-type estimate and the boundedness in (classical) Hardy space for $\mathcal{R}$ are therefore already known. If $0 < \delta < 1$, however, $\mathcal{R}$ is not a Calderón–Zygmund operator. Hence, the weak-type estimate for $\mathcal{R}$ is not obvious. Moreover, since the classical Hardy space $H^1(\mathbb{R}^n)$ is a subspace of the Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$, Theorem 1.7 implies that $\mathcal{R}$ is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into $H^1_{\mathcal{L}}(\mathbb{R}^n)$. We also conclude that $\mathcal{R}$ is bounded on $H^1(\mathbb{R}^n)$.

Based on the previous argument, the Schrödinger operator $-\Delta + V$ can be regarded as a special case of generalized Schrödinger operators, where $V \in (RH)_q$ with $1 < q < \infty$. As we know, the boundedness of Riesz transform associated with the Schrödinger operator has been studied by several scholars (*cf.* [1,5,7,8,13]). The endpoint estimates and Hardy type estimates have been investigated in [2,6,7,12], respectively. The ideas of proofs in [7] and [2] provided us with the inspiration to prove our main results in this paper. During the proof of the first main result, we need some estimates for the Riesz transform that can easily be obtained from [9], such as Lemmas 3.1 and 3.2. We also need to apply some new methods and techniques to deal with the proof of the main results. Moreover, since the Laplace operator adds a potential, the atom has no vanishing condition when $r \geq 1/(4m(x_0,\mu))$, which is the important difference between $H^1(\mathbb{R}^n)$ and $H^1_{\mathcal{L}}(\mathbb{R}^n)$. Therefore, the proof of the second main result will be more complicated than the classical case, where the classical Riesz transform $\nabla(-\Delta)^{-\frac{1}{2}}$ is bounded from $H^1(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$.

Throughout the paper, the letters c and C will denote (possibly different) constants that are independent of the essential variables. By $A \sim B$ we mean that there exists a positive constant C such that $\frac{1}{C} \leq \frac{A}{B} \leq C$. By $\mathrm{U} \lesssim \mathrm{V}$ we mean that there is a constant $C > 0$ such that $\mathrm{U} \leq C\mathrm{V}$.

2 Estimates for Kernels

In this section we recall some basic properties of the function $m(x,\mu)$ proved in [9]. In the sequel, C_0, C_1 and δ are positive constants in (1.1) and (1.2).

Lemma 2.1 *Assume that μ satisfies conditions (1.1) and (1.2). Then*

(i) $0 < m(x,\mu) < \infty$ *for any* $x \in \mathbb{R}^n$.

(ii) *If* $r = m(x,\mu)^{-1}$, *then* $r^{n-2} \leq \mu(B(x,r)) \leq C_1 r^{n-2}$.

(iii) $m(x,\mu) \sim m(y,\mu)$ *if* $|x-y| \leq \frac{C}{m(x,\mu)}$.

https://doi.org/10.4153/CMB-2017-080-0 Published online by Cambridge University Press

(iv) *There exist constants $C, c > 0$ such that*

$$m(x,\mu) \le C(1+|x-y|m(y,\mu))^{k_0} m(y,\mu),$$
$$m(x,\mu) \ge c m(y,\mu)\big(1+|x-y|m(y,\mu)\big)^{-k_0/(1+k_0)},$$

where $k_0 = \frac{C_2}{\delta}$ and $C_2 = \log_2(C_1 + 2^{n-2})$.

Remark 2.2 Remark 0.13 in [9] implies that (1.1) is equivalent to the condition

$$\int_{B(x,R)} \frac{d\mu(y)}{|x-y|^{n-2}} \le C \frac{\mu(B(x,R))}{R^{n-2}}.$$

Moreover, there exist two positive constants C and k_1 such that

$$\frac{\mu(B(x,R))}{R^{n-2}} \le C\{1 + Rm(x,\mu)\}^{k_1}$$

for all $x \in \mathbb{R}^n$ and $R > 0$.

Denote by $\Gamma_\mu(x,y)$ the fundamental solution of $-\Delta+\mu$. Then we have the following estimate of the fundamental solution (*cf.* [9]).

Lemma 2.3 *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \ge 3$. Assume that μ satisfies conditions* (1.1) *and* (1.2) *for some $\delta > 0$. Then*

$$\frac{ce^{-\varepsilon_2 d(x,y,\mu)}}{|x-y|^{n-2}} \le \Gamma_\mu(x,y) \le \frac{Ce^{-\varepsilon_1 d(x,y,\mu)}}{|x-y|^{n-2}},$$

where $\varepsilon_1, \varepsilon_2, C, c$ are positive constants depending only on n and constants C_0, C_1, δ in (1.1) *and* (1.2).

It is easy to check that the measure $\mu + \lambda$ satisfies conditions (1.1) and (1.2) with constants C_0, C_1, δ independent of $\lambda \ge 0$. For the fundamental solution of $-\Delta + \mu + \lambda$, the estimate

$$\frac{ce^{-\varepsilon_2 d(x,y,\mu+\lambda)}}{|x-y|^{n-2}} \le \Gamma_{\mu+\lambda}(x,y) \le \frac{Ce^{-\varepsilon_1 d(x,y,\mu+\lambda)}}{|x-y|^{n-2}}$$

is also valid. Moreover, [11, (3.1)] tells us that

$$0 \le \Gamma_{\mu+\lambda}(x,y) \le \frac{Ce^{-\varepsilon\sqrt{\lambda}|x-y|}e^{-\varepsilon d(x,y,\mu)}}{|x-y|^{n-2}}, \quad \lambda \ge 0. \tag{2.1}$$

For the kernel of the Riesz transform $\mathcal{R}$, we conclude that the following theorem holds true using the proof of [9, Lemma 7.10].

https://doi.org/10.4153/CMB-2017-080-0 Published online by Cambridge University Press

Theorem 2.4 *Let μ be a nonnegative Radon measure in $\mathbb{R}^n$, $n \geq 3$. Assume that μ satisfies conditions (1.1) and (1.2) for some $\delta \in (0,1)$. Then*

$$(2.2)\quad \big|\mathcal{K}(x,y)\big| \leq Ce^{-\varepsilon(1+|x-y|m(y,\mu))^{\frac{1}{k_0+1}}}\Big(\frac{1}{|x-y|^{n-1}}\int_{B(y,|x-y|)}\frac{d\mu(z)}{|z-y|^{n-1}}+\frac{1}{|x-y|^n}\Big),$$

$$(2.3)\quad \big|\mathcal{K}(x,y)-\mathcal{K}_0(x,y)\big| \leq Ce^{-\varepsilon(1+|x-y|m(y,\mu))^{\frac{1}{k_0+1}}} \times\Big(\frac{1}{|x-y|^{n-1}}\int_{B(y,|x-y|)}\frac{d\mu(z)}{|z-y|^{n-1}}+\frac{(|x-y|m(x,\mu))^{\delta}}{|x-y|^n}\Big),$$

where $\mathcal{K}_0(x,y)$ is the kernel for the operator $\nabla(-\Delta)^{-\frac{1}{2}}$.

3 The Weak Type L^1 Estimate and $(H^1_{\mathcal{L}}, H^1)$ Estimate

In this section we will prove the main results in this paper. Our results are based on the following two lemmas about the kernel of $\mathcal{R}$, where we suppose that $\delta \in (0,1)$.

Lemma 3.1 *Let $r = \frac{1}{m(x,\mu)}$. Then*

$$\int_{|x-y|>r}|\mathcal{K}(y,x)|\,dy \leq C.$$

Proof Let $1 \leq q < \frac{2-\delta}{1-\delta}$ and $I(x) = \int_B |y-x|^{1-n}d\mu(y)$, where $B = B(x_0,r)$. We conclude from [9, Lemma 7.9] or [11, Lemma 4.4] that

$$(3.1)\qquad \| I \|_{L^q(B,dx)} \leq C\frac{\mu(3B)}{r^{n(1-\frac{1}{q})-1}}.$$

Now, let $\frac{1}{p_1} = \frac{1}{q} - \frac{2}{n}$. For $j \geq 1$ integer, we use (3.1), Remark 2.2, and (2.2) to obtain

$$\begin{aligned}
\Big\{\int_{2^{j-1}r<|x-y|\leq 2^j r}&|\mathcal{K}(y,x)|^q dy\Big\}^{\frac{1}{q}}\\
&\leq Ce^{-\varepsilon 2^{\frac{j}{k_0+1}}}\Big\{\frac{1}{(2^j r)^{n-1}}\Big(\int_{|x-y|\leq 2^{j+1}r}I(y)^q dy\Big)^{\frac{1}{q}}+(2^j r)^{\frac{n}{q}-n}\Big\}\\
&\leq Ce^{-\varepsilon 2^{\frac{j}{k_0+1}}}\Big\{\frac{(2^j r)^{\frac{n}{q}-n+1}}{(2^j r)^{n-1}}\mu(3B(x,2^{j+1}r))+(2^j r)^{\frac{n}{q}-n}\Big\}\\
&\leq Ce^{-\varepsilon 2^{\frac{j}{k_0+1}}}\Big\{(2^j r)^{\frac{n}{q}-n}(1+2^j)^{k_1}+(2^j r)^{\frac{n}{q}-n}\Big\}\\
&\leq Ce^{-\varepsilon 2^{\frac{j}{k_0+1}}}(1+2^j)^{k_1}(2^j r)^{-\frac{n}{q'}},
\end{aligned}$$

where $r = \frac{1}{m(x,\mu)}$.

https://doi.org/10.4153/CMB-2017-080-0 Published online by Cambridge University Press

By the Hölder inequality,

$$\int_{|x-y|>r} |\mathcal{K}(y,x)|\,dy \le C\sum_{j=1}^{\infty}\Big(\int_{2^{j-1}r<|x-y|\le 2^j r} |\mathcal{K}(y,x)|^q\,dy\Big)^{\frac{1}{q}}(2^j r)^{\frac{n}{q'}}$$
$$\le C\sum_{j=1}^{\infty} e^{-\varepsilon 2^{\frac{j}{k_0+1}}}(1+2^j)^{k_1} = C. \qquad \blacksquare$$

Lemma 3.2 *Let $r = \frac{1}{m(x,\mu)}$. Then*

$$\int_{|x-y|\le r} |\mathcal{K}(y,x) - \mathcal{K}_0(y,x)|\,dy \le C.$$

Proof Let $j \le 0$ be an integer and let $1 \le q < \frac{2-\delta}{1-\delta}$. Via (3.1), Remark 2.2, and (1.1), we have

$$\Big(\int_{2^{j-1}r<|x_0-y|\le 2^j r} |\,\mathcal{K}(y,x) - \mathcal{K}_0(y,x)\,|^q\,dy\Big)^{\frac{1}{q}}$$
$$\le Ce^{-\varepsilon 2^{\frac{j}{k_0+1}}}\Big\{\frac{1}{(2^j r)^{n-1}}\Big(\int_{|x-y|\le 2^{j+1}r} I(y)^q dy\Big)^{\frac{1}{q}} + (2^j r)^{\frac{n}{q}-n}2^{j\delta}\Big\}$$
$$\le Ce^{-\varepsilon 2^{\frac{j}{k_0+1}}}\Big\{\frac{(2^j r)^{\frac{n}{q}-n+1}}{(2^j r)^{n-1}}\mu(3B(x,2^{j+1}r)) + (2^j r)^{\frac{n}{q}-n}2^{j\delta}\Big\}$$
$$\le Ce^{-\varepsilon 2^{\frac{j}{k_0+1}}}\Big\{(2^j r)^{\frac{n}{q}-n}\frac{\mu(3B(x,2^j r))}{r^{n-2}}2^{j\delta} + (2^j r)^{\frac{n}{q}-n}2^{j\delta}\Big\}$$
$$\le Ce^{-\varepsilon 2^{\frac{j}{k_0+1}}}\Big\{(2^j r)^{\frac{n}{q}-n}2^{j\delta} + (2^j r)^{\frac{n}{q}-n}2^{j\delta}\Big\}$$
$$\le C2^{j\delta}(2^j r)^{-\frac{n}{q'}}.$$

Therefore, by the Hölder inequality,

$$\int_{|x-y|\le r} |\mathcal{K}(y,x) - \mathcal{K}_0(y,x)|\,dy$$
$$\le \sum_{j=-\infty}^{0}\Big(\int_{2^{j-1}r<|x-y|\le 2^j r} |\mathcal{K}(y,x) - \mathcal{K}_0(y,x)|^q dy\Big)^{\frac{1}{q}}(2^j r)^{\frac{n}{q'}}$$
$$\le C\sum_{j=-\infty}^{0}(2^j)^{\delta} = C. \qquad \blacksquare$$

Now we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1 By the Calderón–Zygmund decomposition, given $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$, we have $f = f_1 + f_2$, with $f_2 = \sum_k b_k$, such that

(a) $|f_1(x)| \le C\alpha$, for a. e. $x \in \mathbb{R}^n$.

(b) Each b_k is supported in a ball B_k,

$$\int_{B_k} |b_k(x)|dx \le C\alpha|B_k| \quad\text{and}\quad \int_{B_k} b_k(x)\,dx = 0.$$

(c) $\sum_k |B_k| \le \frac{C}{\alpha}\,\|\,f\,\|_{L^1}$.

https://doi.org/10.4153/CMB-2017-080-0 Published online by Cambridge University Press

Because $\mathcal{R}$ is bounded on $L^2(\mathbb{R}^n)$ (*cf.* [9, Theorem 7.1]), it is easy to see that

$$\big|\{x \in \mathbb{R}^n : |\mathcal{R}f_1(x)| > \frac{\alpha}{2}\}\big| \le \frac{C}{\alpha^2} \parallel f_1 \parallel_2^2 \le \frac{C}{\alpha} \parallel f \parallel_1 . \tag{3.2}$$

Let $B_k = B(x_k, r_k)$ and $\Omega = \bigcup_k B(x_k, 2r_k)$. Then

$$|\Omega| \le C \sum_k |B_k| \le \frac{C}{\alpha} \parallel f \parallel_1 . \tag{3.3}$$

We only need to consider $\mathcal{R}f_2(x)$ for $x \in \Omega^c$. If $r_k \ge \frac{1}{m(x_k,\mu)}$, by Lemma 2.1(iv), we have $\frac{1}{m(x,\mu)} \le C\, r_k$ for any $x \in B_k$. By Lemma 3.1, we get

$$\int_{|x_k - x| \ge 2r_k} \big|\mathcal{R}b_k(x)\big|\, dx \le \int_{|x_k - x| \ge 2r_k} \int_{B_k} \big|\mathcal{K}(x,y)\big|\,\big|b_k(y)\big|\, dy\, dx \le C \big\| b_k \big\|_{L^1}.$$

If $r_k < \frac{1}{m(x_k,\mu)}$, then (iii) of Lemma 2.1 implies that $\frac{1}{m(x_k,\mu)} \sim \frac{1}{m(x,\mu)}$ for any $x \in B_k$. Since $\mathcal{K}_0(x,y)$ is a Calderón–Zygmund kernel, by Lemmas 3.1 and 3.2 we obtain

$$\begin{aligned}
&\int_{|x_k - x| \ge 2r_k} \big|\mathcal{R}b_k(x)\big|\, dx \\
&\quad \le \int_{2r_k \le |x_k - x| < \frac{2}{m(x_k,\mu)}} \big|\mathcal{R}b_k(x)\big|\, dx + \int_{|x_k - x| \ge \frac{2}{m(x_k,\mu)}} \big|\mathcal{R}b_k(x)\big|\, dx \\
&\quad \le \int_{2r_k \le |x_k - x| < \frac{2}{m(x_k,\mu)}} \int_{B_k} \big|\mathcal{K}(x,y) - \mathcal{K}_0(x,y)\big|\,\big|b_k(y)\big|\, dy\, dx \\
&\qquad + \int_{2r_k \le |x_k - x| < \frac{2}{m(x_k,\mu)}} \int_{B_k} \big|\mathcal{K}_0(x,y) - \mathcal{K}_0(x,x_k)\big|\,\big|b_k(y)\big|\, dy\, dx \\
&\qquad + \int_{|x_k - x| \ge \frac{2}{m(x_k,\mu)}} \int_{B_k} \big|\mathcal{K}(x,y)\big|\,\big|b_k(y)\big|\, dy\, dx \\
&\quad \le C \big\| b_k \big\|_{L^1},
\end{aligned}$$

whence,

$$\big\| \mathcal{R}b_k \big\|_{L^1((B_k^*)^c)} \le C \big\| b_k \big\|_{L^1}.$$

Then

$$\int_{\Omega^c} \big|\mathcal{R}f_2(x)\big|\, dx \le \sum_k \big\| \mathcal{R}b_k \big\|_{L^1((B_k^*)^c)} \le C \sum_k \big\| b_k \big\|_{L^1} \le C\lambda \sum_k |B_k| \le C \big\| f \big\|_{L^1}.$$

Therefore,

$$\big|\big\{x \in \Omega^c : |\mathcal{R}f_2(x)| > \frac{\lambda}{2}\big\}\big| \le \frac{C}{\lambda} \big\| f \big\|_{L^1}. \tag{3.4}$$

Theorem 1.1 follows from the combination of (3.2), (3.3), and (3.4). ∎

Proof of Theorem 1.7 To prove this theorem, we need to use the molecular characterization of $H^1(\mathbb{R}^n)$ in [10] (see also [4]).

Let $\epsilon \in (0,\infty)$ and $b \equiv 1 - 1/p_0 + \epsilon$. Recall that in [10] (see also [4, Definition 7.13, p. 328]), a function $M \in L^{p_0}(\mathbb{R}^n)$ is called a $(1, p_0, \epsilon)$-*molecule* centered at $x_0 \in \mathbb{R}^n$ if

$$\|M\|_{L^{p_0}(\mathbb{R}^n)}^{\epsilon/b} \big\| |\cdot - x_0|^{nb} M \big\|_{L^{p_0}(\mathbb{R}^n)}^{1-\epsilon/b} \le 1, \tag{3.5}$$

$$\int_{\mathbb{R}^n} M(x)\, dx = 0. \tag{3.6}$$

https://doi.org/10.4153/CMB-2017-080-0 Published online by Cambridge University Press

Let $p_0 \in (1, \frac{2-\delta}{1-\delta})$ and a be a $H^{1,\infty}_{\mathcal{L}}$-atom associated with the ball $B \equiv B(x_0, r)$ for some $x_0 \in \mathbb{R}^n$ and $r \in (0, \frac{1}{m(x_0,\mu)})$. By Proposition 1.6, we only need to show that $\mathcal{R}(a)$ is a $(1, p_0, \epsilon)$-molecule up to a harmless multiplicative constant. To this end, we now consider two cases.

Case (i) $r \geq \frac{1}{4m(x_0,\mu)}$. In this case, to prove that $\mathcal{R}(a)$ satisfies (3.5), by the $L^{p_0}(\mathbb{R}^n)$-boundedness of $\mathcal{R}$ (see [9, Theorem 7.1]), we have

$$\big\|\mathcal{R}(a)\big\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \|a\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \Big[\frac{1}{m(x_0,\mu)}\Big]^{-n/p_0'}. \tag{3.7}$$

To estimate $\||\cdot - x_0|^{nb}\mathcal{R}(a)\|_{L^{p_0}(\mathbb{R}^n)}$, for $j \in \mathbb{N}$, let $B_j \equiv B(x_0, \frac{2^j}{m(x_0,\mu)})$. Then we have

$$\begin{aligned}\big\||\cdot - x_0|^{nb}\mathcal{R}(a)\big\|_{L^{p_0}(\mathbb{R}^n)} &\leq \big\|\chi_{B_1}|\cdot - x_0|^{nb}\mathcal{R}(a)\big\|_{L^{p_0}(\mathbb{R}^n)} \\ &\quad + \big\|\chi_{B_1^{\complement}}|\cdot - x_0|^{nb}\mathcal{R}(a)\big\|_{L^{p_0}(\mathbb{R}^n)} \\ &\equiv \mathrm{I} + \mathrm{II},\end{aligned}$$

where $B_1^{\complement} = (B_1)^{\complement}$. By (3.7), we have

$$\mathrm{I} \leq C\Big[\frac{1}{m(x_0,\mu)}\Big]^{n\epsilon}.$$

To estimate II, by (2.2) and Minkowski's inequality, we obtain

$$\begin{aligned}\mathrm{II} &\lesssim \int_B |a(y)|\Bigg[\Big\|\chi_{B_1^{\complement}}|\cdot - x_0|^{nb}\frac{e^{-\varepsilon(1+|\cdot-y|m(y,\mu))^{\frac{1}{k_0+1}}}}{|\cdot - y|^{n-1}}\int_{B(\cdot,|\cdot-y|/4)}\frac{d\mu(z)}{|z-\cdot|^{n-1}}\Big\|_{L^{p_0}(\mathbb{R}^n)} \\ &\qquad + \Big\|\chi_{B_1^{\complement}}|\cdot - x_0|^{nb}\frac{e^{-\varepsilon(1+|\cdot-y|m(y,\mu))^{\frac{1}{k_0+1}}}}{|\cdot - y|^{n}}\Big\|_{L^{p_0}(\mathbb{R}^n)}\Bigg]\,dy \\ &\equiv \int_B |a(y)|(\mathrm{II}_1 + \mathrm{II}_2)\,dy.\end{aligned}$$

To estimate II_2, by Lemma 2.1(iii), we have

$$\mathrm{II}_2 \lesssim C\Big[\frac{1}{m(x_0,\mu)}\Big]^{\frac{N}{k_0+1}}\Bigg\|\frac{\chi_{B_1^{\complement}}(\cdot)}{|\cdot - x_0|^{\frac{N}{k_0+1}+n-nb}}\Bigg\|_{L^{p_0}(\mathbb{R}^n)} \lesssim C\Big[\frac{1}{m(x_0,\mu)}\Big]^{n\epsilon}. \tag{3.8}$$

On II_1, by Minkowski's inequality, we further decompose it into

$$\begin{aligned}\mathrm{II}_1 &\lesssim \sum_{j=1}^{\infty}\Bigg\{\int_{B_{j+1}\setminus B_j}\frac{[\frac{1}{m(y,\mu)}]^{\frac{Np_0}{k_0+1}}}{[2^j\frac{1}{m(x_0,\mu)}]^{(\frac{N}{k_0+1}+n-1-nb)p_0}}\Big|\int_{B(x,|x-y|/4)}\frac{d\mu(z)}{|x-z|^{n-1}}\Big|^{p_0}dx\Bigg\}^{1/p_0} \\ &\equiv \sum_{j=1}^{\infty}\mathrm{II}_{1,j}.\end{aligned}$$

Let k_1 be the constant as in Remark 2.2 and let $N \in ((k_0+1)(k_1+n\epsilon), \infty)$. Combining the boundedness from I (see [9, Lemma 7.9]) with Lemma 2.1(iii), we have

$$\begin{aligned}
\mathrm{II}_{1,j} &\lesssim \left\{ \int_{B_{j+1}\setminus B_j} \frac{\left[\frac{1}{m(y,\mu)}\right]^{\frac{Np_0}{k_0+1}}}{\left[2^j \frac{1}{m(x_0,\mu)}\right]^{(\frac{N}{k_0+1}+n-1-nb)p_0}} \left| \int_{B_{j+2}} \frac{d\mu(z)}{|x-z|^{n-1}} \right|^{p_0} dx \right\}^{1/p_0} \\
&\lesssim \left[\frac{1}{m(x_0,\mu)}\right]^{\frac{N}{k_0+1}} \frac{\mu(3B_{j+2})}{\left[2^j \frac{1}{m(x_0,\mu)}\right]^{n-1-\frac{n}{p_0}}} \left[2^j \frac{1}{m(x_0,\mu)}\right]^{nb+1-n-\frac{N}{k_0+1}} \\
&\lesssim \left[\frac{1}{m(x_0,\mu)}\right]^{\frac{N}{k_0+1}} (1+2^j)^{k_1} \left[2^j \frac{1}{m(x_0,\mu)}\right]^{nb-n-\frac{N}{k_0+1}+\frac{n}{p_0}} \\
&\sim 2^{j(k_1+n\epsilon-\frac{N}{k_0+1})} \left[\frac{1}{m(x_0,\mu)}\right]^{n\epsilon}.
\end{aligned}$$

This implies that $\mathrm{II}_1 \lesssim \left[\frac{1}{m(x_0,\mu)}\right]^{n\epsilon}$, which together with (3.8) shows that

$$\mathrm{II} \lesssim \left[\frac{1}{m(x_0,\mu)}\right]^{n\epsilon}. \tag{3.9}$$

Combining the estimates of I and II, we obtain

$$\left\| |\cdot - x_0|^{nb} \mathcal{R}(a) \right\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \left[\frac{1}{m(x_0,\mu)}\right]^{n\epsilon},$$

which together with (3.7) shows that

$$\|\mathcal{R}(a)\|_{L^{p_0}(\mathbb{R}^n)}^{\epsilon/b} \left\| |\cdot - x_0|^{nb} \mathcal{R}(a) \right\|_{L^{p_0}(\mathbb{R}^n)}^{1-\epsilon/b} \lesssim 1.$$

Thus, we obtain (3.5) up to a harmless multiplicative constant.

To prove that $\mathcal{R}(a)$ satisfies (3.6), we first show that $\mathcal{R}(a)$ and $\mathcal{L}^{-1/2}(a) \in L^1(\mathbb{R}^n)$. To estimate $\mathcal{R}(a)$, by Hölder's inequality and (3.5), we see that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\mathcal{R}(a)(x)|\, dx &= \int_{|x-x_0|\le 1} |\mathcal{R}(a)(x)|\, dx + \int_{|x-x_0|>1} \cdots\, dx \\
&\lesssim \left\| \chi_{B(x_0,1)} \mathcal{R}(a) \right\|_{L^{p_0}(\mathbb{R}^n)} \\
&\quad + \left\| \chi_{B^\complement(x_0,1)} |\cdot - x_0|^{nb} \times \mathcal{R}(a)(\cdot) \right\|_{L^{p_0}(\mathbb{R}^n)} < \infty.
\end{aligned}$$

In what follows, we need to estimate $\|\mathcal{L}^{-1/2}(a)\|_{L^1(\mathbb{R}^n)}$. Since

$$\mathcal{L}^{-1/2} = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (-\Delta + \mu + \lambda)^{-1}\, d\lambda,$$

using (2.1) we show that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\mathcal{L}^{-1/2}(a)(x)|\,dx \\
&\quad\lesssim \int_0^1 \int_B \int_{|x-y|\ge 2\frac{1}{m(x_0,\mu)}} \lambda^{-1/2}\big|\Gamma_{\mu+\lambda}(x,y)\big|\big|a(y)\big|\,dx\,dy\,d\lambda \\
&\qquad + \int_0^1 \int_B \int_{|x-y|<2\frac{1}{m(x_0,\mu)}} \cdots \quad dx\,dy\,d\lambda + \int_1^\infty \int_B \int_{|x-y|\ge 2\frac{1}{m(x_0,\mu)}} \cdots \quad dx\,dy\,d\lambda \\
&\qquad + \int_1^\infty \int_B \int_{|x-y|<\frac{1}{m(x_0,\mu)}} \cdots \quad dx\,dy\,d\lambda \equiv \sum_{i=1}^4 \mathrm{E}_i .
\end{aligned}
$$

To estimate E_1, we obtain, by [9, (3.19)] and Lemma 2.1(iv),

$$
d(x,y,\mu) \ge C\big(1+|x-y|m(y,\mu)\big)^{\frac{1}{(k_0+1)^2}}
$$

for $|x-y|m(y,\mu)\ge 2$. Note that $m(y,\mu)\sim m(x_0,\mu)$ when $y\in B(x_0,r)$. Then by (2.1), we have

$$
\begin{aligned}
\mathrm{E}_1 &\lesssim \int_0^1 \lambda^{-1/2}\int_B \Big\{ \int_{|x-y|\ge 2\frac{1}{m(x_0,\mu)}} \frac{e^{-\varepsilon\sqrt{\lambda}|x-y|}e^{-\varepsilon d(x,y,\mu)}}{|x-y|^{n-2}}\,dx\Big\}|a(y)|\,dy\,d\lambda \\
&\lesssim \int_0^1 \lambda^{-1/2}\int_B \Big\{ \int_{|x-y|\ge 2\frac{1}{m(x_0,\mu)}} \frac{\big(|x-y|m(y,\mu)\big)^{\frac{-N}{(k_0+1)^2}}}{|x-y|^{n-2}}\,dx\Big\}|a(y)|\,dy\,d\lambda \\
&\lesssim m(x_0,\mu)^{\frac{-N}{(k_0+1)^2}} \int_{\frac{2}{m(x_0,\mu)}}^\infty s^{1-\frac{N}{(k_0+1)^2}}\,ds \lesssim \frac{1}{m(x_0,\mu)^2} < \infty,
\end{aligned}
$$

where we have chosen $N > 2(k_0+1)^2$.

For E_2, by (2.1) again, we obtain

$$
\begin{aligned}
\mathrm{E}_2 &\lesssim \int_0^1 \lambda^{-1/2}\int_B \Big[\int_{|x-y|<2\frac{1}{m(x_0,\mu)}} \frac{1}{|x-y|^{n-2}}\,dx\Big]|a(y)|\,dy\,d\lambda \\
&\lesssim \frac{1}{m(x_0,\mu)^2} < \infty.
\end{aligned}
$$

https://doi.org/10.4153/CMB-2017-080-0 Published online by Cambridge University Press

From (2.1) with $N \in (1/2, \infty)$, it follows that E_3 is controlled by

$$\begin{aligned}
\mathrm{E}_3 &\lesssim \int_1^\infty \lambda^{-1/2} \int_B \Bigg[\int_{|x-y|\ge 2\frac{1}{m(x_0,\mu)}} \frac{e^{-\varepsilon\sqrt{\lambda}|x-y|} e^{-\varepsilon d(x,y,\mu)}}{|x-y|^{n-2}}\,dx \Bigg] |a(y)|\,dy\,d\lambda \\
&\lesssim \int_1^\infty \lambda^{-1/2-N} \int_B \Bigg[\int_{|x-y|\ge 2\frac{1}{m(x_0,\mu)}} \frac{1}{|x-y|^{N+n-2}}\,dx \Bigg] |a(y)|\,dy\,d\lambda \\
&\lesssim m(x_0,\mu)^{N-2} < \infty.
\end{aligned}$$

Similarly, there exists $N \in (1/2, 2)$ such that

$$\begin{aligned}
\mathrm{E}_4 &\lesssim \int_1^\infty \lambda^{-1/2} \int_B \sum_{i=-\infty}^{0} \int_{2^{i-1}\frac{1}{m(x_0,\mu)} \le |x-y| < 2^i \frac{1}{m(x_0,\mu)}} \frac{|a(y)|}{\big(\sqrt{\lambda}|x-y|\big)^N |x-y|^{n-2}}\,dx\,dy\,d\lambda \\
&\lesssim \int_1^\infty \lambda^{-(1+2N)/2} \int_B \Bigg\{ \sum_{i=-\infty}^{0} \int_{2^{i}\frac{1}{m(x_0,\mu)} \le |x-y| < 2^{i+1} \frac{1}{m(x_0,\mu)}} \frac{1}{[2^i \frac{1}{m(x_0,\mu)}]^{N+n-2}}\,dx \Bigg\} \\
&\quad \times |a(y)|\,dy\,d\lambda \\
&\lesssim m(x_0,\mu)^{N-2} < \infty.
\end{aligned}$$

Combining the estimates for E_i with $i \in \{1,2,3,4\}$ implies that $\mathcal{L}^{-1/2}(a) \in L^1(\mathbb{R}^n)$.

Now we choose $\{\varphi_j\}_{j=0}^\infty \subset C^\infty(\mathbb{R}^n)$ such that

(a) $\sum_{j=0}^\infty \varphi_j(x) = 1$ for almost every $x \in \mathbb{R}^n$;
(b) there exists a family $\{Q_j\}_{j\in\mathbb{N}}$ of balls such that $\operatorname{supp} \varphi_j \subset 2Q_j$, $\varphi_j = 1$ on Q_j and $0 \le \varphi_j \le 1$;
(c) there exists a positive constant $C(\varphi)$ such that for all $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $\varphi_j(x) + |\nabla\varphi_j(x)| + |\nabla^2\varphi_j(x)| \le C(\varphi)$;
(d) there exists $N_\varphi \in \mathbb{N}$ such that $\sum_{j=0}^\infty \chi_{2Q_j} \le N_\varphi$.

Using the properties of $\{\varphi_j\}_{j=0}^\infty$ and $\mathcal{L}^{-1/2}(a), \mathcal{R}(a) \in L^1(\mathbb{R}^n)$ together with Lebesgue's dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^n} \nabla\big(\mathcal{L}^{-1/2}\big)(a)(x)\,dx = \sum_{j=0}^\infty \int_{\mathbb{R}^n} \nabla(\varphi_j \mathcal{L}^{-1/2})(a)(x)\,dx.$$

For each j, let $\eta_j \in C^\infty(\mathbb{R}^n)$ satisfy $\eta_j = 1$ on $2Q_j$ and $\operatorname{supp}\eta_j \subset 4Q_j$. Then by the divergence formula, for every $k \in \{1,\dots,n\}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\partial}{\partial x_k}(\varphi_j \mathcal{L}^{-1/2})(a)(x)\,dx &= \int_{\mathbb{R}^n} \eta_j(x) \frac{\partial}{\partial x_k}(\varphi_j \mathcal{L}^{-1/2})(a)(x)\,dx \\
&= -\int_{\mathbb{R}^n} \varphi_j(x)\mathcal{L}^{-1/2}(a)(x) \frac{\partial}{\partial x_l}\eta_j(x)\,dx = 0,
\end{aligned}$$

which implies that $\int_{\mathbb{R}^n} \mathcal{R}(a)(x)\,dx = 0$. Hence, $\mathcal{R}(a)$ satisfies (3.6). Thus, in this case, $\mathcal{R}(a)$ is a $(1, p_0, \epsilon)$-molecule up to a harmless multiplicative constant.

https://doi.org/10.4153/CMB-2017-080-0 Published online by Cambridge University Press

Case (ii) $r < 1/(4m(x_0, \mu))$. In this case, a is a classical $(1, \infty)$-atom of $H^1(\mathbb{R}^n)$. It is well known that $\nabla(-\Delta)^{-\frac{1}{2}}$ is a Calderón–Zygmund operator, and hence it is bounded on $H^1(\mathbb{R}^n)$. Moreover, $\nabla((-\Delta)^{-\frac{1}{2}})(a)$ is a $(1, p_0, \epsilon)$-molecule up to a harmless multiplicative constant; see, for example, [4, Theorem 7.18, p. 335]. By this, we see that in order to show that $\mathcal{R}(a)$ is a $(1, p_0, \epsilon)$-molecule up to a harmless multiplicative constant, it suffices to prove that $L(a)$ is a $(1, p_0, \epsilon)$-molecule up to a harmless multiplicative constant, where $L \equiv \mathcal{R} - \nabla(-\Delta)^{-\frac{1}{2}}$.

To prove that $L(a)$ satisfies (3.5), we estimate $\|L(a)\|_{L^{p_0}(\mathbb{R}^n)}$ by

$$\|L(a)\|_{L^{p_0}(\mathbb{R}^n)} \le \big\| \chi_{B_1} L(a) \big\|_{L^{p_0}(\mathbb{R}^n)} + \big\| \chi_{B_1^\complement} L(a) \big\|_{L^{p_0}(\mathbb{R}^n)} \equiv \mathrm{J}_1 + \mathrm{J}_2,$$

where B_1 is the same as in Case (i).

To estimate J_2, from the size estimate of the kernel of $\nabla(-\Delta)^{-\frac{1}{2}}$ and an argument similar to the estimate of II in Case (i) with a suitable choice of N, we have

$$\mathrm{J}_2 \le \Big\| \chi_{B_1^\complement}(\cdot) \int_B \frac{|a(y)|}{|\cdot - y|^n}\, dy \Big\|_{L^{p_0}(\mathbb{R}^n)} + \big\| \chi_{B_1^\complement} \mathcal{R}(a) \big\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \Big[\frac{1}{m(x_0, \mu)} \Big]^{-\frac{n}{p_0'}}.$$

Using (2.3) and Minkowski's integral inequality, we estimate J_1 by

$$\begin{aligned}
\mathrm{J}_1 &\lesssim \int_B |a(y)| \Big\{ \Big(\int_{B_1} \Big[\frac{1}{|x-y|^{n-1}} \Big(\int_{B(x, |x-y|/4)} \frac{d\mu(z)}{|z-x|^{n-1}} \Big) \Big]^{p_0} dx \Big)^{1/p_0} \\
&\quad + \Big(\int_{B_1} \Big[\frac{1}{|x-y|^n} \Big(\frac{|x-y|}{\frac{1}{m(y,\mu)}} \Big)^{\delta} \Big]^{p_0} dx \Big)^{1/p_0} \Big\} \, dy \\
&\equiv \int_B |a(y)| (\mathrm{U}_1 + \mathrm{U}_2) \, dy.
\end{aligned}$$

To estimate U_2, by Lemma 2.1(iii), we have

$$\mathrm{U}_2 \lesssim \Big[\frac{1}{m(x_0, \mu)} \Big]^{\delta} \Big\{ \int_0^{\frac{3}{m(x_0,\mu)}} s^{(-n+\delta) p_0 + n - 1} \, ds \Big\}^{1/p_0} \lesssim \Big[\frac{1}{m(x_0, \mu)} \Big]^{-\frac{n}{p_0'}}. \tag{3.10}$$

Now, for any $y \in B$ and $j \in \mathbb{Z}$, let $T_j \equiv B(y, 2^{j+1} \frac{1}{m(x_0,\mu)})$. Obviously, $B_1 \subset T_1$ and by Minkowski's inequality, we further have

$$\begin{aligned}
\mathrm{U}_1 &\lesssim \sum_{j=-\infty}^{0} \Big\{ \int_{T_{j+1} \setminus T_j} \frac{1}{|x-y|^{(n-1) p_0}} \Big[\int_{B(x, |x-y|/4)} \frac{d\mu(z)}{|z-x|^{n-1}} \Big]^{p_0} dx \Big\}^{1/p_0} \\
&\equiv \sum_{j=-\infty}^{0} \mathrm{U}_{1,j}.
\end{aligned}$$

To estimate $\mathrm{U}_{1,j}$, by (1.1) and the boundedness from I (see [9, Lemma 7.9]) again, we obtain

$$\mathrm{U}_{1,j} \lesssim \frac{1}{[2^j \frac{1}{m(x_0,\mu)}]^{n-1}} \frac{\mu(3T_{j+2})}{[2^j \frac{1}{m(x_0,\mu)}]^{n-1-\frac{n}{p_0}}} \lesssim 2^{j\delta} \Big[2^j \frac{1}{m(x_0, \mu)} \Big]^{-\frac{n}{p_0'}}.$$

https://doi.org/10.4153/CMB-2017-080-0 Published online by Cambridge University Press

Thus, we have

$$\mathrm{U}_1 \lesssim \sum_{j=-\infty}^{0} \mathrm{U}_{1,j} \lesssim \Big[\frac{1}{m(x_0,\mu)}\Big]^{-\frac{n}{p_0'}},$$

which together with (3.10) and the estimate for J_2 imply that

$$\|L(a)\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \Big[\frac{1}{m(x_0,\mu)}\Big]^{-\frac{n}{p_0'}}. \tag{3.11}$$

To estimate $\||\cdot - x_0|^{nb}L(a)\|_{L^{p_0}(\mathbb{R}^n)}$, we write it as

$$\begin{aligned}\big\||\cdot - x_0|^{nb}L(a)\big\|_{L^{p_0}(\mathbb{R}^n)} &\le \big\|\chi_{B_1}|\cdot - x_0|^{nb}L(a)\big\|_{L^{p_0}(\mathbb{R}^n)}\\ &\quad + \big\|\chi_{B_1^\complement}\big|\cdot - x_0\big|^{nb}L(a)\big\|_{L^{p_0}(\mathbb{R}^n)}\\ &\equiv \mathrm{S}_1 + \mathrm{S}_2.\end{aligned}$$

To estimate S_2, by the size estimate of the kernel of $\nabla(-\Delta)^{-\frac{1}{2}}$ and (3.9), we have

$$\mathrm{S}_2 \le \Big\|\chi_{B_1^\complement}\int_B \frac{|a(y)|}{|\cdot - x_0|^{n(1-b)}}\,dy\Big\|_{L^{p_0}(\mathbb{R}^n)} + \mathrm{II} \lesssim \Big[\frac{1}{m(x_0,\mu)}\Big]^{n\epsilon},$$

where II is the same as in Case (i). From (3.11), it follows that

$$\mathrm{S}_1 \lesssim \Big[\frac{1}{m(x_0,\mu)}\Big]^{nb}\|L(a)\|_{L^{p_0}(\mathbb{R}^n)} \lesssim \Big[\frac{1}{m(x_0,\mu)}\Big]^{n\epsilon}.$$

Thus, $\big\||\cdot - x_0|L(a)\big\|_{L^{p_0}(\mathbb{R}^n)} \lesssim [\frac{1}{m(x_0,\mu)}]^{n\epsilon}$, which together with (3.11) implies (3.5).

To prove that $L(a)$ satisfies (3.6), we make use of the fact that $\nabla(-\Delta)^{-\frac{1}{2}}(a)$ is a $(1,p_0,\epsilon)$-molecule up to a harmless multiplicative constant to deduce that $\int_{\mathbb{R}^n}\nabla(-\Delta)^{-\frac{1}{2}}(a)(x)\,dx = 0$. Thus, we only need to show that $\mathcal{R}(a)$ satisfies (3.6). Notice that in Case (i), when proving $\int_{\mathbb{R}^n}\mathcal{R}(a)(x)\,dx = 0$, we did not use the condition $r \ge 1/(4m(x_0,\mu))$. Thus, the same argument also shows that $\int_{\mathbb{R}^n}\mathcal{R}(a)(x)\,dx = 0$ when $r < 1/(4m(x_0,\mu))$, which further implies that $L(a)$ satisfies (3.6).

Thus, in both cases, $\mathcal{R}(a)$ is a $(1,p_0,\epsilon)$-molecule up to a harmless multiplicative constant, which completes the proof of Theorem 1.7. ∎

References

[1] P. Auscher and B. Ben Ali, *Maximal inequalities and Riesz transform estimates on L^p spaces for Schrödinger operators with nonnegative potentials.* Ann. Inst. Fourier (Grenoble) **57**(2007), no. 6, 1975–2013. http://dx.doi.org/10.5802/aif.2320

[2] J. Cao, Y. Liu, and Da. Yang, *Hardy spaces $H^1_{\mathcal{L}}(\mathbb{R}^n)$ associated to Schrödinger type operators* $(-\Delta)^2 + V^2$. Houston J. Math. **36**(2010), no. 4, 1067–1095.

[3] J. Dziubański and J. Zienkiewicz, *Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality.* Rev. Mat. Iberoamericana **15**(1999), no. 2, 279–296. http://dx.doi.org/10.4171/RMI/257

[4] J. García-Cuerva and J. Rubio de Francia, *Weighted norm inequalities and related topics.* North-Holland Mathematics Studies, 116, Notas de Matemática, 104, North-Holland Publishing Co., Amsterdam, 1985.

[5] H.-Q. Li, *Estimations L^p des opérateurs de Schrödinger sur les groupes nilpotents.* J. Func. Anal. **161**(1999), no. 1, 152–218. http://dx.doi.org/10.1006/jfan.1998.3347

[6] C. Lin, H. Liu, and Y. Liu, *Hardy spaces associated with Schrödinger operators on the Heisenberg group.* arxiv:1106.4960

[7] Y. Liu and J. Dong, *Some estimates of higher order Riesz transform related to Schrödinger operator.* Potential. Anal. **32**(2010), no. 1, 41–55. http://dx.doi.org/10.1007/s11118-009-9143-7
[8] Z. Shen, L^p *estimates for Schrödinger operators with certain potentials.* Ann. Inst. Fourier (Grenoble) **45**(1995), 513–546. http://dx.doi.org/10.5802/aif.1463
[9] ———, *On fundamental solutions of generalized Schrödinger operators.* J. Funct. Anal. **167**(1999), 521–564. http://dx.doi.org/10.1006/jfan.1999.3455
[10] M. H. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces.* In: Representation theorems for Hardy spaces, Astérisque, 77, Soc. Math. France, Paris, 1980, pp. 67–149.
[11] L. Wu and L. Yan, *Heat kernels, upper bounds and Hardy spaces associated to the generalized Schrödinger operators.* J. Funct. Anal. **270**(2016), no. 10, 3709–3749. http://dx.doi.org/10.1016/j.jfa.2015.12.016
[12] Da. Yang, Do. Yang, and Y. Zhou, *Endpoint properties of localized Riesz transforms and fractional integrals associated to Schrödinger operators.* Potential Anal. **30**(2009), no. 3, 271–300. http://dx.doi.org/10.1007/s11118-009-9116-x
[13] J. Zhong, *Harmonic analysis for some Schrödinger type operators.* Ph.D., Princeton University, 1993.

School of Mathematics and Physics, University of Science and Technology Beijing, 100083, China
e-mail: liuyu75@pku.org.cn 1005777218@qq.com