HOMOTOPY GROUPS OF TRANSFORMATION GROUPS

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1. Introduction. In a previous paper (**2**) I defined the fundamental group $\sigma(X, x_0, G)$ of a group G of homeomorphisms of a space X, and showed that if the transformation group admits a family of preferred paths, then $\sigma(X, x_0, G)$ can be represented as a group extension of $\pi_1(X, x_0)$ by G. In this paper the homotopy groups of a transformation group are defined. The nth absolute homotopy group of a transformation group which admits a family of preferred paths is shown to be representable as a split extension of the nth absolute torus homotopy group $\tau_n(X, x_0)$ by G.

In § 6 it is shown that the action of G on X induces a homomorphism of G into a quotient group of a subgroup of the group of automorphisms of $\tau_n(X, x_0)$. This homomorphism is used to obtain a necessary condition for the embedding of one transformation group in another, in particular, for the embedding of a discrete flow in a continuous flow with the same phase space.

The relative homotopy groups of a transformation group are defined only in relation to an invariant subspace Y of X and a particular family of preferred paths. They are shown to be representable as split extensions of the relative torus homotopy groups by G. The homotopy sequence of a transformation group which admits a family of preferred paths is an extension of the torus homotopy sequence by G. Its structure is complex, partly because the torus homotopy sequence has a spiral character and is only exact within each twist of the spiral. However, the sequence contains as a subsequence the extension of the ordinary homotopy sequence of X by G, and this subsequence is at every point "under-exact" relative to G in the sense defined in § 8.

2. Notation. The notation for the n-dimensional cube will be

$$I^n = \{(t_1, \ldots, t_n)\},$$

where $0 \leqq t_i \leqq 1$ for $1 \leqq i \leqq n$. By identifying corresponding points of pairs of opposite faces of the cube we obtain as a quotient space the n-dimensional torus T^n. It will be convenient to use the notation $\hat{t}_i = t_i \pmod 1$, $T^n = \{(\hat{t}_1, \ldots, \hat{t}_n)\}$. The quotient spaces obtained by identifying corresponding points on some pairs of opposite faces of the cube are n-dimensional cylinders. The cylinders $C^n = \{(t_1, \hat{t}_2, \ldots, \hat{t}_n)\}$ obtained by identifying corresponding points on all but the first pair of opposite faces of the cube will be used in the definition of the absolute homotopy groups. The cylinders $D^n = \{(t_1, \hat{t}_2, \ldots, \hat{t}_{n-1}, t_n)\}$

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obtained by identifying corresponding points on all but the first and last pairs of opposite faces of the cube will be used in the definition of the relative homotopy groups.

Let X be a topological space with base-point x_0. The maps $C^n \to x_0$ and $D^n \to x_0$ will both be denoted by ι^n. If $r < n$, a map $f: C^r \to X$ gives rise to a map $f^n: C^n \to X$ defined by $f^n(t_1, \hat{t}_2, \ldots, \hat{t}_n) = f(t_1, \hat{t}_2, \ldots, \hat{t}_r)$. The same notation will be used for the map $f^n: D^n \to X, f^n(t_1, \hat{t}_2, \ldots, \hat{t}_{n-1}, t_n) = f(t_1, \hat{t}_2, \ldots, \hat{t}_r)$. The inverse maps $C^n \to C^n$, $(t_1, \hat{t}_2, \ldots, \hat{t}_n) \to (1 - t_1, \hat{t}_2, \ldots, \hat{t}_n)$, and $D^n \to D^n$, $(t_1, \hat{t}_2, \ldots, t_n) \to (1 - t_1, \hat{t}_2, \ldots, t_n)$, will both be denoted by ρ^n.

3. Transformation groups. Whereas in (**2**) it was convenient to consider a transformation group to be a pair (X, G), where G was a group of homeomorphisms of a space X, in this paper it is necessary to take the more general standpoint that a transformation group is a triple (X, G, π), where X is a topological space, G is a topological group with identity element e, and π is a continuous map $\pi: X \times G \to X$, $\pi: (x, g) \to g(x)$, such that $\pi(x, e) = x$ and $\pi(x, g_1 g_2) = g_1(g_2(x))$. In the study of the absolute homotopy groups of transformation groups, use is made of base-points in the space X. Thus the objects of the category under consideration are of the form (X, x_0, G, π), while a category mapping

$$(\phi, \psi): (X, x_0, G, \pi) \to (X', x_0', G', \pi')$$

consists of a continuous map $\phi: X \to X'$ such that $\phi x_0 = x_0'$ and a homomorphism ψ such that the following diagram is commutative:

$$\begin{tikzcd}
X \times G \arrow[r, "\pi"] \arrow[d, "{(\phi, \psi)}"'] & X \arrow[d, "\phi"] \\
X' \times G' & X' \arrow[l, "\pi'"']
\end{tikzcd}$$

A category mapping (ϕ, ψ) will be called an embedding if ϕ is a homeomorphism of X into X', and ψ is a monomorphism of G into G'. The transformation groups will be said to be of the same homotopy type if there exist category mappings $(\phi, \psi): (X, x_0, G, \pi) \to (X', x_0', G', \pi')$ and $(\phi', \psi'): (X', x_0', G', \pi') \to (X, x_0, G, \pi)$ such that ψ and ψ' are isomorphisms, and $\phi'\phi$ and $\phi\phi'$ are homotopic to the identity maps of X and X', respectively.

A transformation group (X, G, π) is said to be free if the abstract group G is free. With each transformation group (X, G, π) there is associated a free transformation group in the following way. The group G can be expressed as a quotient group F/F_0, where F is an abstract free group. Let $\epsilon: F \to G$ denote the natural epimorphism. Then the topology of G is lifted by ϵ to a topology for F with respect to which F is a topological group. Moreover, if J denotes the identity map of X and if $\pi' = \pi \cdot (J, \epsilon)\colon X \times F \rightarrow X$, then (X, F, π') is a transformation group, and $(J, \epsilon)\colon (X, x_0, F, \pi') \rightarrow (X, x_0, G, \pi)$ is a category mapping. If $g' \in F$ and ${}_\epsilon g' = g \in G$, then for all $x \in X$, $g'(x) = g(x)$. In particular, if $g_0 \in F_0$, then for all $x \in X$, $g_0(x) = x$, so that every element of F_0 acts as the identity mapping of X.

4. Absolute homotopy groups. Given a transformation group (X, x_0, G, π), for each element g of G, let the set $C^n(X, x_0, g)$ consist of all the continuous maps $f\colon C^n \rightarrow X$ such that $f(0, \hat{t}_2, \ldots, \hat{t}_n) = x_0$ and $f(1, \hat{t}_2, \ldots, \hat{t}_n) = g(x_0)$. The elements of the set will be called maps of C^n of order g. Two sets $C^n(X, x_0, g_1)$ and $C^n(X, x_0, g_2)$ are formally distinct even if $g_1(x_0) = g_2(x_0)$.

Two maps f_0 and f_1 of C^n of order g are said to be homotopic if there exists a continuous map $F\colon C^n \times I \rightarrow X$ such that

$$\begin{aligned} F(t_1, \hat{t}_2, \ldots, \hat{t}_n, 0) &= f_0(t_1, \hat{t}_2, \ldots, \hat{t}_n), \\ F(t_1, \hat{t}_2, \ldots, \hat{t}_n, 1) &= f_1(t_1, \hat{t}_2, \ldots, \hat{t}_n), \qquad (t_1, \hat{t}_2, \ldots, \hat{t}_n) \in C^n; \\ F(0, \hat{t}_2, \ldots, \hat{t}_n, t) &= x_0, \\ F(1, \hat{t}_2, \ldots, \hat{t}_n, t) &= g(x_0), \qquad 0 \leqq t \leqq 1, \quad 0 \leqq \hat{t}_i < 1, \quad 2 \leqq i \leqq n. \end{aligned}$$

The homotopy class of a map f of order g will be denoted by $[f; g]$.

If $f_1, f_2\colon C^n \rightarrow X$ are such that

$$f_1\colon (1, \hat{t}_2, \ldots, \hat{t}_n) = f_2(0, \hat{t}_2, \ldots, \hat{t}_n), \qquad 0 \leqq \hat{t}_i < 1, \quad 2 \leqq i \leqq n,$$

then the sum of the two maps, $(f_1 + f_2)\colon C^n \rightarrow X$, can be defined by the equations:

$$(f_1 + f_2)(t_1, \hat{t}_2, \ldots, \hat{t}_n) = \begin{cases} f_1(2t_1, \hat{t}_2, \ldots, \hat{t}_n), & 0 \leqq t_1 \leqq \frac{1}{2}, \\ f_2(2t_1 - 1, \hat{t}_2, \ldots, \hat{t}_n), & \frac{1}{2} \leqq t_1 \leqq 1. \end{cases}$$

Thus if $f_1 \in C^n(X, x_0, g_1)$ and $f_2 \in C^n(X, x_0, g_2)$, $f_1 + g_1 f_2$ is defined and is a map of C^n of order $g_1 g_2$.

The homotopy class of $f_1 + g_1 f_2$ depends only on the homotopy classes of f_1 and f_2. Thus the equation

$$[f_1; g_1] * [f_2; g_2] = [f_1 + g_1 f_2; g_1 g_2]$$

defines a rule of composition for maps of C^n of prescribed order, and the composition is associative. The element $[\iota^n; e]$ is an identity for this rule of composition, and each element $[f; g]$ has an inverse element $[g^{-1} f \rho^n; g^{-1}]$. Thus the set of homotopy classes of maps of C^n of prescribed order with this rule of composition forms a group which will be denoted by $\sigma_n(X, x_0, G, \pi)$.

The results on base-point invariance and naturality of the fundamental group of a transformation group which were proved in (**2**) extend immediately to the absolute homotopy groups. Thus a category mapping

$$(\phi, \psi)\colon (X, x_0, G, \pi) \rightarrow (X', x_0', G', \pi')$$

induces a homomorphism

$$(\phi, \psi)_*\colon \sigma_n(X, x_0, G, \pi) \to \sigma_n(X', x_0', G', \pi'),$$

$$(\phi, \psi)_*\colon [f; g] \to [\phi f; \psi g],$$

such that if

$$(\phi', \psi')\colon (X', x_0', G', \pi') \to (X'', x_0'', G'', \pi'')$$

is another category mapping,

$$(\phi', \psi')_*(\phi, \psi)_* = (\phi'\phi, \psi'\psi)_*;$$

and if (X, x_0, G, π) and (X', x_0', G', π') are of the same homotopy type, then $\sigma_n(X, x_0, G, \pi)$ and $\sigma_n(X', x_0', G', \pi')$ are isomorphic. Also, a path λ from x_0 to x_1 induces a natural isomorphism

$$\lambda_*\colon \sigma_n(X, x_0, G, \pi) \to \sigma_n(X, x_1, G, \pi),$$

$$\lambda_*\colon [f; g] \to [\lambda^n\rho^n + f + g\lambda^n; g];$$

while if G is abelian and $x_1 = gx_0$, then g induces a natural isomorphism

$$g_\flat\colon \sigma_n(X, x_0, G, \pi) \to \sigma_n(X, x_1, G, \pi),$$

$$g_\flat\colon [f_1; g_1] \to [gf_1; g_1].$$

The set of homotopy classes of maps of order e forms a subgroup of $\sigma_n(X, x_0, G, \pi)$ which is isomorphic to the nth torus homotopy group $\tau_n(X, x_0)$ described by Fox (**1**). In the notation introduced in § 2 of this paper, the group $\tau_n(X, x_0)$ is the set of homotopy classes of maps of the cylinder C^n into X such that

$$f(0, \hat{t}_2, \ldots, \hat{t}_n) = f(1, \hat{t}_2, \ldots, \hat{t}_n) = x_0,$$

with the rule of composition induced by the usual rule for addition of maps using the first coordinate.

For $n = 1$, $\tau_1(X, x_0) = \pi_1(X, x_0)$. For $n \geqq 2$, a map $f\colon C^{n-1} \to X$ gives rise to a map $f^\phi\colon C^n \to X$,

$$f^\phi(t_1, \ldots, \hat{t}_n) = f(t_1, \ldots, \hat{t}_{n-1})$$

and a map $f\colon C^n \to X$ gives rise to a map $f^\psi\colon C^{n-1} \to X$,

$$f^\psi(t_1, \ldots, \hat{t}_{n-1}) = f(t_1, \ldots, \hat{t}_{n-1}, 0).$$

These induce a monomorphism $\Phi\colon \tau_{n-1}(X, x_0) \to \tau_n(X, x_0)$ and an epimorphism $\Psi\colon \tau_n(X, x_0) \to \tau_{n-1}(X, x_0)$ such that $\Psi\Phi$ is the identity automorphism. The group $\tau_n(X, x_0)$ is an extension by $\tau_{n-1}(X, x_0)$ of a product of higher homotopy groups; precisely, there is an exact sequence (see **1**, (9.3))

$$0 \to \bigtimes_{2 \leqq i \leqq n} \pi_i^{\binom{n-2}{i-2}} \to \tau_n \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \tau_{n-1} \to 0.$$

The group $\pi_n(X, x_0)$ can be embedded isomorphically in $\tau_n(X, x_0)$ in a number of ways depending on the permutations of the coordinates used (see **1**, § 7). However, in this paper $\pi_n(X, x_0)$ will be regarded as that subgroup of $\tau_n(X, x_0)$ which consists of homotopy classes of maps of $C^n \to X$ subject to the additional restriction that

$$f(t_1, \ldots, \hat{t}_n) = x_0, \qquad \prod_{i=2}^{n} \hat{t}_i = 0.$$

Now the map

$$i_G{}^n\colon \tau_n(X, x_0) \to \sigma_n(X, x_0, G, \pi),$$
$$i_G{}^n\colon [f] \to [f; e],$$

is a monomorphism. Moreover, if X is path-connected, the map

$$j_G{}^n\colon \sigma_n(X, x_0, G, \pi) \to G,$$
$$j_G{}^n\colon [f; g] \to g,$$

is an epimorphism, and these two homomorphisms give rise to a short exact sequence. It will be assumed in the remainder of the paper that X is path-connected.

If G is represented as a quotient group F/F_0, where F is a free group, then there is a corresponding short exact sequence for $\sigma_n(X, x_0, F, \pi')$ which fits into the following commutative diagram in which the monomorphism of F_0 into $\sigma_n(X, x_0, F, \pi')$ is the map $g_0 \to [\iota^n; g_0]$:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & F_0 \arrow[dl] \arrow[dr] & & \\
 & & \sigma_n(X, x_0, F, \pi') \arrow[rr, "j_F{}^n"] \arrow[dd, "(J, \epsilon)_*"] & & F \arrow[r] \arrow[dd, "\epsilon"] & 0 \\
0 \arrow[r] & \tau_n(X, x_0) \arrow[ur, "i_F{}^n"] \arrow[dr, "i_G{}^n"'] & & & & \\
 & & \sigma_n(X, x_0, G, \pi) \arrow[rr, "j_G{}^n"] \arrow[d] & & G \arrow[r] \arrow[d] & 0 \\
 & & 0 & & 0 &
\end{tikzcd}$$

It is easy to see that $i_G{}^n\tau_n(X, x_0)$ is a normal subgroup of $\sigma_n(X, x_0, G, \pi)$. The same is true of $i_G{}^n\pi_n(X, x_0)$.

THEOREM 4.1. *The group* $i_G{}^n\pi_n(X, x_0)$ *is a normal subgroup of* $\sigma_n(X, x_0, G, \pi)$.

Proof. Suppose that $[f_1] \in \pi_n(X, x_0)$ and $[f; g] \in \sigma_n(X, x_0, G, \pi)$. Then

$$[f; g] * [f_1; e] * [g^{-1}f\rho^n; g^{-1}] = [f + gf_1 + f\rho^n; e] = [f_2; e], \quad \text{say.}$$

We have to prove that $[f_2; e] \in i_G{}^n\pi_n(X, x_0)$, i.e. that there exists $[f'] \in \pi_n(X, x_0)$ such that $[f'; e] = [f_2; e] \in \sigma_n(X, x_0, G, \pi)$. The result follows immediately for

$n = 1$. I sketch the geometrical idea for $n = 2$ which gives rise to the required homotopy for $n \geqq 2$. It will be convenient to consider the element f_1 of $C^2(X, x_0, g)$ as a map of I^2 into X subject to the appropriate conditions on the boundary, and to consider a representative of an element of $\pi_2(X)$ as a map of I^2 into X or a map of E^2 into X. Introduce the following notation for the boundary of I^2,

$$\begin{aligned} a_i &= \{t_0 = i\}, \\ b_i &= \{0 \leqq t_0 \leqq \tfrac{1}{2}, t_1 = i\}, \\ c_i &= \{\tfrac{1}{2} \leqq t_0 \leqq 1, t_1 = i\}, \qquad i = 0, 1. \end{aligned}$$

There is a map of I^2 onto E^2 which maps $a_0 \cup a_1$ onto the boundary of E^2 and identifies b_0 with c_0 and b_1 with c_1 in E^2, and a homotopy of E^2 which expands a central disc so as to push the images of b_0 and b_1 onto the boundary of E^2. The map f_2, regarded as a map from E^2 into X, is modified by this homotopy to a map f' of the required type.

It was proved in (**2**) that if G is a group of simplicial transformations of a polyhedron X, then the fundamental group of the orbit space X/G is isomorphic to a quotient group of $\sigma(X, x_0, G)$. The corresponding quotient group can be defined for $\sigma_n(X, x_0, G, \pi)$. Unfortunately, the example discussed in (**2**, § 9) shows that there is no connection between $\pi_2(X/G)$ and the quotient group of $\sigma_2(X, x_0, G, \pi)$.

5. Groups of operators of σ_n. In this section it is shown that if the transformation group (X, x_0, G, π) admits a family of preferred paths, then G acts as a group of operators on $\sigma_n(X, x_0, G, \pi)$, on $\tau_n(X, x_0)$, and on $\pi_n(X, x_0)$. Thus $\sigma_n(X, x_0, G, \pi)$ can be represented as a split extension of $\tau_n(X, x_0)$ by G and contains a subgroup isomorphic to the split extension of $\pi_n(X, x_0)$ by G.

Let the groups of automorphisms of $\sigma_n(X, x_0, G, \pi)$, $\tau_n(X, x_0)$, and $\pi_n(X, x_0)$ be denoted by $A_n(X, x_0, G, \pi)$, $A_n(X, x_0)$, and $B_n(X, x_0)$, respectively. If $r \leqq n$, the mapping

$$\begin{aligned} &\Omega_r^n\colon \sigma_r(X, x_0, G, \pi) \rightarrow \sigma_n(X, x_0, G, \pi), \\ &\Omega_r^n\colon [f; g] \rightarrow [f^n; g], \end{aligned}$$

is a monomorphism. Let $[f^n; g]_*$ denote the inner automorphism

$$\begin{aligned} [f^n; g]_*[f_1; g_1] &= [f^n; g] * [f_1; g_1] * [g^{-1}f^n\rho^n; g^{-1}] \\ &= [f^n + gf_1 + gg_1g^{-1}f^n\rho^n; gg_1g^{-1}]. \end{aligned}$$

Then the map

$$\begin{aligned} &O_r^n\colon \sigma_r(X, x_0, G, \pi) \rightarrow A_n(X, x_0, G, \pi), \\ &O_r^n\colon [f; g] \rightarrow [f^n; g]_*, \end{aligned}$$

is a homomorphism. Since for every category mapping (ϕ, ψ),

$$[\phi f; \psi g]_*(\phi, \psi)_* = (\phi, \psi)_*[f; g]_*,$$

the homomorphism is natural.

Now $i_G{}^n\tau_n(X, x_0)$ is a normal subgroup of $\sigma_n(X, x_0, G, \pi)$. Thus an element $[f; g] \in \sigma_r(X, x_0, G, \pi)$ induces a map

$$[f; g]_\natural = (i_G{}^n)^{-1} \cdot [f; g]_* \cdot i_G{}^n,$$

$$[f; g]_\natural \colon [f_1] \to [f^n + gf_1 + f^n\rho^n],$$

which is an automorphism of $\tau_n(X, x_0)$. Moreover, the map

$$P_r{}^n \colon \sigma_r(X, x_0, G, \pi) \to A_n(X, x_0),$$

$$P_r{}^n \colon [f; g] \to [f; g]_\natural,$$

is a natural homomorphism.

Again, in view of Theorem 4.1, $i_G{}^n\pi_n(X, x_0)$ is a normal subgroup of $\sigma_n(X, x_0, G, \pi)$, so that $[f; g]_\natural$ restricted to $\pi_n(X, x_0)$ is also an automorphism; and the map

$$Q_r{}^n \colon \sigma_r(X, x_0, G, \pi) \to B_n(X, x_0),$$

$$Q_r{}^n \colon [f; g] \to [f^n; g]_\natural | \pi_n(X, x_0),$$

is a natural homomorphism.

It was proved in (**2**) that if a transformation group (X, x_0, G, π) admits a family of preferred paths at x_0, then the fundamental group $\sigma_1(X, x_0, G, \pi)$ can be represented in terms of $\pi_1(X, x_0)$.

Recall that a family $\mathfrak{k} = \{k_g; g \in G\}$ of paths in X is said to be a *family of preferred paths at* x_0 if for each element g of G, k_g is a path from gx_0 to x_0, if $k_e \sim \iota$, and if for every pair of elements g_1, g_2 of G, $k_{g_1g_2} \sim g_1k_{g_2} + k_{g_1}$. Recall also that every free transformation group admits a family of preferred paths.

PROPOSITION 5.1. *A transformation group* (X, x_0, G, π) *admits a family of preferred paths at* x_0 *if and only if* $\sigma_1(X, x_0, G, \pi)$ *is a split extension of* $\tau_1(X, x_0)$ *by* G.

Proof. An extension B of A by C is said to be split if C can be regarded as a subgroup of B, i.e. if in the short exact sequence there is a monomorphism from C to B which commutes with the epimorphism from B to C. Thus the proposition follows from the observation that if $\mathfrak{k}$ is a family of preferred paths at x_0, then

$$\mathfrak{k}_* \colon G \to \sigma_1(X, x_0, G, \pi),$$

$$\mathfrak{k}_* \colon g \to [k_g\rho; g],$$

is a monomorphism such that $j_G{}^1\mathfrak{k}_* \colon G \to G$ is the identity map.

THEOREM 5.2. *If the transformation group* (X, x_0, G, π) *admits a family* $\mathfrak{k}$ *of preferred paths at* x_0, *then* G *acts as a group of operators of the groups* $\sigma_n(X, x_0, G, \pi)$, $\tau_n(X, x_0)$, *and* $\pi_n(X, x_0)$.

Proof. The mappings $O_1{}^n\mathfrak{k}_*$, $P_1{}^n\mathfrak{k}_*$, and $Q_1{}^n\mathfrak{k}_*$ are homomorphisms of G into appropriate groups of automorphisms.

COROLLARY 5.3. *For each element g of G, the following diagram is commutative:*

$$\begin{tikzcd}
\pi_n(X, x_0) \subset \tau_n(X, x_0) \arrow[r, "\dot{i}_G^{\,n}"] \arrow[d, "Q_1^n\mathfrak{k}_*"'] \arrow[d, shift left=20ex, "P_1^n\mathfrak{k}_*g"'] & \sigma_n(X, x_0, G, \pi) \arrow[d, "O_1^n\mathfrak{k}_*g"'] \\
\pi_n(X, x_0) \subset \tau_n(X, x_0) \arrow[r, "\dot{i}_G^{\,n}"] & \sigma_n(X, x_0, G, \pi)
\end{tikzcd}$$

The proof follows immediately from the definitions.

In conformity with the notation of (**2**), the homomorphism $P_1^n\mathfrak{k}_*$ will be denoted by $\mathfrak{K}^n$, and the automorphism $\mathfrak{K}^n g$ will be denoted by K_g^n. No confusion will be caused by denoting the homomorphism $Q_1^n\mathfrak{k}_*$ also by $\mathfrak{K}^n$. The explicit equation for the action of K_g^n on an element $[f] \in \tau_n(X, x_0)$ is

$$K_g^n[f] = [k_g^n\rho^n + gf + k_g^n].$$

The homomorphism $\mathfrak{K}^n$: $G \rightarrow A_n(X, x_0)$ enables one to define a split extension of $\tau_n(X, x_0)$ by G. The set of ordered pairs $\{\alpha; g\}$, where $\alpha \in \tau_n(X, x_0)$ and $g \in G$ with the rule of composition

$$\{\alpha; g\} * \{\alpha'; g'\} = \{\alpha + K_g^n\alpha'; gg'\}$$

is a group which will be denoted by $\{\tau_n(X, x_0); \mathfrak{K}^n\}$. If $\alpha = [f]$, then the notation $\{f; g\}$ may be used in place of $\{\alpha; g\}$. The set of ordered pairs $\{\alpha; g\}$, where $\alpha \in \pi_n(X, x_0)$ and $g \in G$ with the same rule of composition is a group $\{\pi_n(X, x_0); \mathfrak{K}^n\}$, which is a subgroup of $\{\tau_n(X, x_0); \mathfrak{K}^n\}$.

THEOREM 5.4. *If the transformation group (X, x_0, G, π) admits a family $\mathfrak{k}$ of preferred paths at x_0, then $\sigma_n(X, x_0, G, \pi)$ is isomorphic to the group $\{\tau_n(X, x_0); \mathfrak{K}^n\}$ and contains a subgroup isomorphic to $\{\pi_n(X, x_0); \mathfrak{K}^n\}$.*

Proof. The map

$$\mathfrak{k}_\flat^n: \sigma_n(X, x_0, G, \pi) \rightarrow \{\tau_n(X, x_0); \mathfrak{K}^n\},$$
$$\mathfrak{k}_\flat^n: [f; g] \rightarrow \{f + k_g; g\},$$

is easily seen to be an isomorphism. The inverse image of $\{\pi_n(X, x_0); \mathfrak{K}^n\}$ under this isomorphism is a subgroup of the required kind.

Note, however, that this representation of $\sigma_n(X, x_0, G, \pi)$ is not categorical for the category of transformation groups.

Now since G is a topological group, (G, G, π_0), $\pi_0(g, g') = g'g$, is a transformation group. If (G, G, π_0) admits a family $\mathfrak{h}$ of preferred paths at e, then (X, G, π) admits a family $\mathfrak{k}$ of preferred paths at x_0 defined by the equation $k_g t = (h_g t)(x_0)$. Certainly, the additive group R of real numbers admits a family of preferred paths, hence a transformation group (X, R, π) admits a family of preferred paths. In this case Theorem 5.4 can be strengthened considerably.

THEOREM 5.5. *If* (G, G, π_0) *admits a family of preferred paths at* e, *then*

$$\sigma_n(X, x_0, G, \pi) \cong \tau_n(X, x_0) \times G.$$

Proof. The proof of this theorem is the same as that of (**2**, Theorem 7) and consists of proving that the homomorphism $\mathfrak{K}^n\colon G \to A_n(X, x_0)$ is trivial.

6. Embedding. One of the outstanding problems of topological dynamics is that of embedding a discrete flow in a continuous flow with the same phase space. The work of the previous section gives rise to a necessary condition for such an embedding to be possible. The group of integers will be denoted by Z, the additive group of real numbers by R, and the natural monomorphism of Z into R will be denoted by ψ_0. Then an embedding of a discrete flow in a continuous flow is a category mapping

$$(J, \psi_0)\colon (X, Z, \pi) \to (X, R, \pi').$$

The necessary condition for such an embedding will arise as a special case of a more general embedding theorem which depends on the following theorem.

THEOREM 6.1. *Let* (X, G, π) *be a transformation group and set*

$$\begin{aligned} A_n'(X, x_0) &= P_1{}^n\sigma_1(X, x_0, G, \pi) \subset A_n(X, x_0), \\ A_n''(X, x_0) &= P_1{}^n i_G{}^1\tau_1(X, x_0) \subset A_n'(X, x_0), \\ \tilde{A}_n(X, x_0) &= A_n'(X, x_0)/A_n''(X, x_0). \end{aligned}$$

Then π *induces a homomorphism*

$$\pi_*{}^n\colon G \to \tilde{A}_n(X, x_0).$$

Proof. For convenience in the proof, the superfix will be omitted from the symbol $\pi_*{}^n$. Since $i_G{}^1\tau_1(X, x_0)$ is a normal subgroup of $\sigma_1(X, x_0, G, \pi)$, the group $A_n''(X, x_0)$ is a normal subgroup of $A_n'(X, x_0)$. Let G be represented as a quotient group F/F_0 of a free group F, and let (X, F, π') be the corresponding free transformation group. Let $\mathfrak{k}$ and $\mathfrak{l}$ be two families of preferred paths at x_0 in (X, F, π'), and let $\mathfrak{K}^n, \mathfrak{L}^n\colon F \to A_n'(X, x_0)$ be the homomorphisms induced by them. Then for each element g of F, $P_1{}^n i_G{}^1[k_g\rho + l_g] = \beta$, say, is an element of $A_n''(X, x_0)$ such that $\mathfrak{K}^n g = \beta \cdot \mathfrak{L}^n g$, since for each $[f]$ of $\tau_n(X, x_0)$, we have:

$$\begin{aligned} \beta \cdot L_g{}^n[f] &= \beta \cdot [l_g{}^n\rho^n + gf + l_g{}^n] \\ &= [k_g{}^n\rho^n + l_g{}^n + (l_g{}^n\rho^n + gf + l_g{}^n) + l_g{}^n\rho^n + k_g{}^n] \\ &= K_g{}^n[f]. \end{aligned}$$

Hence, if $\delta_n\colon A_n'(X, x_0) \to \tilde{A}_n(X, x_0)$ is the natural epimorphism,

$$\delta_n\mathfrak{K}^n = \delta_n\mathfrak{L}^n\colon F \to \tilde{A}_n(X, x_0).$$

Thus this homomorphism depends only on π' and will be denoted by π_*'.

Since every element of F_0 acts as the identity mapping of X, if $g \in F_0$, then

$[f; g]_\natural = [f; e]_\natural$ so that $\mathfrak{K}^n \colon F_0 \to A_n''(X, x_0)$ and $\pi_*' \colon F_0 \to 0$. Thus π_*' induces a homomorphism $\pi_* \colon G \to \tilde{A}_n(X, x_0)$.

To show that this homomorphism depends only on π, suppose that G is also a quotient group H/H_0 of another free group H, and let (X, H, π'') be the corresponding free transformation group. Then π'' induces $\pi_*'' \colon H \to \tilde{A}_n(X, x_0)$, $\pi_*'' \colon H_0 \to 0$, and hence a homomorphism of G into $\tilde{A}_n(X, x_0)$. Let $\epsilon' \colon F \to G$, $\epsilon'' \colon H \to G$ be the natural epimorphisms, and let $\mathfrak{k}$ and $\mathfrak{l}$ be families of preferred paths at x_0 for (X, F, π') and (X, H, π''). Given $g \in G$, let $g' \in F$ and $g'' \in H$ be elements such that $\epsilon' g' = \epsilon'' g'' = g$. Since for all points x of X,

$$g(x) = g'(x) = g''(x),$$

we have

$$\mathfrak{K}^n g' = P_1{}^n i_G{}^1[k_{g'}\rho + l_{g''}] \cdot \mathfrak{L}^n g''.$$

Thus $\delta_n \mathfrak{K}^n g' = \delta_n \mathfrak{L}^n g''$, and this automorphism depends only on the element g of G.

It does not seem to be possible to derive from each category mapping a natural map between the corresponding groups $\tilde{A}_n$ which commutes with the homomorphisms π_*. However, the following theorem gives a result in this direction which covers the important case of embeddings.

THEOREM 6.2. *A category mapping*

$$(\phi, \psi) \colon (X_1, x_1, G_1, \pi_1) \to (X_2, x_2, G_2, \pi_2),$$

such that the homomorphism

$$\phi_* \colon \tau_n(X_1, x_1) \to \tau_n(X_2, x_2)$$

is epimorphic, induces a homomorphism

$$(\phi, \psi)_\natural \colon \tilde{A}_n(X_1, x_1) \to \tilde{A}_n(X_2, x_2)$$

such that $(\phi, \psi)_\natural \pi_{1*} = \pi_{2*}\psi$.

Proof. Let $[f_1; g_1]_\natural = [f_2; g_2]_\natural \in A_n'(X_1, x_1)$, and $[f'] \in \tau_n(X_2, x_2)$. Then there exists $[f] \in \tau_n(X_1, x_1)$ such that $[\phi f] = [f']$. Hence, the equality $[f_1; g_1]_\natural[f] = [f_2; g_2]_\natural[f]$ guarantees the equality $[\phi f_1; \psi g_1]_\natural[f] = [\phi f_2; \psi g_2]_\natural[f']$. Thus the map $(\phi, \psi)_\flat \colon A_n'(X_1, x_1) \to A_n'(X_2, x_2)$, $(\phi, \psi)_\flat \colon [f; g]_\natural \to [\phi f; \psi g]_\natural$ is a well-defined homomorphism. Since $(\phi, \psi)_\flat A_n''(X_1, x_1) \subset A_n''(X_2, x_2)$, the homomorphism $(\phi, \psi)_\flat$ induces a homomorphism of the quotient groups

$$(\phi, \psi)_\natural \colon \tilde{A}_n(X_1, x_1) \to \tilde{A}_n(X_2, x_2).$$

Let $\{g_i\}$ be a family of generators for G_1 and let $\{g_i'\}$ be a family of generators for G_2 which includes the family $\{\psi g_i\}$. Let F_1 and F_2 be the free groups on these families of generators. Then ψ induces a monomorphism $\psi_1 \colon F_1 \to F_2$ and there is a homomorphism

$$(\phi, \psi_1)_* \colon \sigma_1(X_1, x_1, F_1, \pi_1') \to \sigma_1(X_2, x_2, F_2, \pi_2').$$

If $\mathfrak{k}$ is a family of preferred paths for (X_1, x_1, F_1, π_1'), then the homomorphism

$$\psi_1\colon F_1 \to \sigma_1(X_2, x_2, F_2, \pi_2'),$$
$$\psi_1\colon (g_{i1} \ldots g_{in}) \to (\phi, \psi_1)_* \mathfrak{k}_* (g_{i1} \ldots g_{in}),$$

can be extended to a homomorphism

$$\mathfrak{l}_*\colon F_2 \to \sigma_1(X_2, x_2, F_2, \pi_2')$$

which gives rise to a family $\mathfrak{l}$ of preferred paths for (X_2, x_2, F_2, π_2').

If $\mathfrak{K}^n = P_1{}^n \mathfrak{k}_*\colon F_1 \to A_n'(X_1, x_1)$ and $\mathfrak{L}^n = P_1{}^n \mathfrak{l}_*\colon F_2 \to A_n'(X_2, x_2)$, then

$$(\phi, \psi_1)_* \mathfrak{k}_* = \mathfrak{l}_* \psi_1$$

ensures that

$$(\phi, \psi)_\flat \mathfrak{K}^n = \mathfrak{L}^n \psi_1$$

and also

$$(\phi, \psi)_\natural \pi_{1*} = \pi_{2*} \psi.$$

The condition imposed in the theorem is certainly satisfied in the case of an embedding $(J, \psi)\colon (X, G_1, \pi_1) \to (X, G_2, \pi_2)$. In this case, $(J, \psi)_\natural$ is the identity and thus the theorem shows that an embedding is only possible if for every positive integer n, the homomorphisms $\pi_{1*}{}^n$ and $\pi_{2*}{}^n$ commute with ψ. As a special case of this we have the following theorem.

THEOREM 6.3. *The discrete flow (X, Z, π) can be embedded in a continuous flow (X, R, π') only if for every positive integer n, $\pi_*{}^n\colon Z \to 0$.*

Proof. The proof of Theorem 5.5 contains the result that (X, R, π') admits a preferred family of paths $\mathfrak{k}$ such that, for all n, $\mathfrak{K}^n\colon R \to 0 \in A_n'(X, x_0)$. Thus $\pi_*'\colon R \to 0 \in \tilde{A}_n(X, x_0)$. It follows therefore from the previous theorem that $\pi_*{}^n\colon Z \to 0 \in \tilde{A}_n(X, x_0)$.

7. Relative homotopy groups. The theorems of §§ 4 and 5 have analogues for relative homotopy groups which will now be given. Some of the details of the proofs will be suppressed in order to avoid undue repetition of arguments and complexity of notation.

The relative homotopy groups of a transformation group (X, x_0, G, π) are defined relative to an invariant subspace Y of X and a family $\mathfrak{k}$ of preferred paths at x_0 in Y. Thus, the appropriate category for relative homotopy groups is that whose objects are of the form $(X, Y, \mathfrak{k}, G, \pi)$ and whose mappings $(\phi, \psi)\colon (X, Y, \mathfrak{k}, G, \pi) \to (X', Y', \mathfrak{k}', G', \pi')$ consist of a continuous map $\phi\colon X \to X'$ and a homomorphism $\psi\colon G \to G'$ with the following properties: (i) $\phi\colon Y \to Y'$, (ii) $\phi\pi = \pi'(\phi, \psi)$, (iii) for every element g, $\phi k_g \sim k_{\psi g}'$. As always, the homotopy is to keep the end points of the paths fixed.

For each integer $n \geqq 2$ and each element g of G and its associated preferred path k_g, let the set $D^n(X, Y, k_g, g)$ consist of all the continuous maps $f\colon D^n \to X$ such that $f|(t_1 = 0) = x_0$, $f|(t_1 = 1) = gx_0$, $f|(t_n = 0) \in Y$ and $f|(t_n = 1) = f'(t_1) \in Y$, where f' is a path which is homotopic in Y to $k_g p$. Two such maps f_0 and f_1 are said to be homotopic if there exists a continuous map

$F\colon D^n \times I \to X$ such that $F|(t = 0) = f_0$, $F|(t = 1) = f_1$, and for every λ, $F|(t = \lambda) \in D^n(X, Y, k_g, g)$. The homotopy class of such a map f will be denoted by $[f; g]$. With the rule of composition $[f_1; g_1] * [f_2; g_2] = [f_1 + g_1 f_2; g_1 g_2]$, these form the relative homotopy group $\sigma_n(X, Y, \mathfrak{k}, G, \pi)$.

The map $i_G{}^n\colon [f] \to [f; e]$ is a monomorphism of the relative torus homotopy group $\tau_n(X, Y, x_0)$ into $\sigma_n(X, Y, \mathfrak{k}, G, \pi)$. Here, $\pi_n(X, Y, x_0)$ will be regarded as a subgroup of $\tau_n(X, Y, x_0)$, just as in § 4, $\pi_n(X, x_0)$ was regarded as a subgroup of $\tau_n(X, x_0)$. By a proof similar to that of Theorem 4.1, it can be shown that $i_G{}^n \tau_n(X, Y, x_0)$ and $i_G{}^n \pi_n(X, Y, x_0)$ are normal subgroups of $\sigma_n(X, Y, \mathfrak{k}, G, \pi)$.

THEOREM 7.1. *The group G acts as a group of operators of the groups* $\sigma_n(X, Y, \mathfrak{k}, G, \pi)$, $\tau_n(X, Y, x_0)$ *and* $\pi_n(X, Y, x_0)$.

Proof. Given an element g of G, $\mathfrak{k}_* g = [k_g \rho; g]$ is an element of $\sigma_1(X, x_0, G, \pi)$, and $[k_g{}^n \rho^n; g]$ can be regarded as an element of $\sigma_n(X, Y, \mathfrak{k}, G, \pi)$. The inner automorphism $[k_g{}^n \rho^n; g]_*$ of $\sigma_n(X, Y, \mathfrak{k}, G, \pi)$ leaves invariant the normal subgroups $i_G{}^n \tau_n(X, Y, x_0)$ and $i_G{}^n \pi_n(X, Y, x_0)$, and thus induces automorphisms of $\tau_n(X, Y, x_0)$ and $\pi_n(X, Y, x_0)$, both of which will be denoted by the same symbol $K_g{}^n$ which was used in § 5 to denote automorphisms of $\tau_n(X, x_0)$ and $\pi_n(X, x_0)$. The maps $\mathfrak{K}^n\colon g \to K_g{}^n$ are homomorphisms of G into the groups of automorphisms of $\tau_n(X, Y, x_0)$ or $\pi_n(X, Y, x_0)$.

The homomorphism $\mathfrak{K}^n$ gives rise to the group extension $\{\tau_n(X, Y, x_0); \mathfrak{K}^n\}$ and its subgroup $\{\pi_n(X, Y, x_0); \mathfrak{K}^n\}$, as in § 5.

THEOREM 7.2. *The group* $\sigma_n(X, Y, \mathfrak{k}, G, \pi)$ *is isomorphic to the group* $\{\tau_n(X, Y, x_0); \mathfrak{K}^n\}$.

Proof. The map $\mathfrak{k}_\flat{}^n\colon [f; g] \to \{f + k_g; g\}$ is an isomorphism.

A category mapping (ϕ, ψ) induces homomorphisms $(\phi, \psi)_*$ of the relative homotopy groups. Moreover, we have the following result.

THEOREM 7.3. *A category mapping* (ϕ, ψ) *induces homomorphisms*

$$(\phi, \psi)_\natural\colon \{\tau_n(X, Y, x_0); \mathfrak{K}\} \to \{\tau_n(X', Y', x_0'); \mathfrak{K}'\}$$

such that

$$(\phi, \psi)_\natural\colon \{\pi_n(X, Y, x_0); \mathfrak{K}\} \to \{\pi_n(X', Y', x_0'); \mathfrak{K}'\},$$

and corresponding homomorphisms for the extensions of the absolute homotopy groups.

Proof. For each element g of G, $K_{\psi g}{}' \phi_* = \phi_* K_g$, where $\phi_*\colon [f] \to [\phi f]$. One can now check that $(\phi, \psi)_\natural\colon \{f; g\} \to \{\phi f; \psi g\}$ is a homomorphism.

Within the category of transformation groups with families of preferred paths, the representations of the σ_n as split extensions of the τ_n are natural in that $(\phi, \psi)_\natural \mathfrak{k}_\flat = \mathfrak{k}_\flat{}'(\phi, \psi)_*$.

8. The homotopy sequence. The inclusion map $i\colon Y \to X$ gives rise to homomorphisms

$$i_n\colon \sigma_n(Y, x_0, G, \pi) \to \sigma_n(X, x_0, G, \pi),$$

and

$$i_n\colon \{\tau_n(Y, x_0);\ \mathfrak{K}^n\} \to \{\tau_n(X, x_0);\ \mathfrak{K}^n\}$$

such that $i_n\colon \{\pi_n(Y, x_0);\ \mathfrak{K}^n\} \to \{\pi_n(X, x_0);\ \mathfrak{K}^n\}$.

The inclusion homomorphism $j_n'\colon \pi_n(X, x_0) \to \pi_n(X, Y, x_0)$ commutes with K_g^n for each element g of G, and hence induces a homomorphism

$$j_n\colon \{\pi_n(X, x_0);\ \mathfrak{K}^n\} \to \{\pi_n(X, Y, x_0);\ \mathfrak{K}^n\}.$$

On the other hand, there is in general no homomorphism of $\tau_n(X, x_0)$ into $\tau_n(X, Y, x_0)$, or of $\sigma_n(X, x_0, G, \pi)$ into $\sigma_n(X, Y, \mathfrak{f}, G, \pi)$. However, $\tau_n(X, x_0)$ has a subgroup $\tau_n(X, x_0, x_0)$ (see **1**, § 11) which admits a natural homomorphism into $\tau_n(X, Y, x_0)$. Similarly, $\sigma_n(X, x_0, G, \pi)$ contains as a subgroup the group $\sigma_n(X, \mathfrak{f}, G, \pi)$ of homotopy classes of continuous maps $f\colon C^n \to X$ of prescribed order which satisfy the conditions $f|(t_1 = 0) = x_0, f|(t_1 = 1) = gx_0$, $f|(t_n = 0) = f'(t_1) \in Y$, where f' is a path which is homotopic in Y to $k_g\rho$. This group contains $\tau_n(X, x_0, x_0)$ as a subgroup and is isomorphic to a split extension $\{\tau_n(X, x_0, x_0);\ \mathfrak{K}^n\}$ defined in the same way as the previous extensions. Since $\pi_n(X, x_0)$ is a subgroup of $\tau_n(X, x_0, x_0)$, $\{\pi_n(X, x_0);\ \mathfrak{K}^n\}$ is a subgroup of $\{\tau_n(X, x_0, x_0);\ \mathfrak{K}^n\}$.

The natural homomorphism of $\sigma_n(X, \mathfrak{f}, G, \pi)$ into $\sigma_n(X, Y, \mathfrak{f}, G, \pi)$ will be denoted by j_n. The natural homomorphism of $\tau_n(X, x_0, x_0)$ into $\tau_n(X, Y, x_0)$ commutes with the automorphisms K_g for each element g of G, and thus induces a homomorphism of the split extensions which may also be denoted by j_n.

Now a map f in $D^{n+1}(X, Y, k_g, g)$ gives rise to a map $\delta f = f|(t_{n+1} = 0)$ in $C^n(Y, x_0, g)$; and $[f; g] \to [\delta f; g]$ and $\{f; g\} \to \{\delta f; g\}$ are homomorphisms

$$\partial_{n+1}\colon \sigma_{n+1}(X, Y, \mathfrak{f}, G, \pi) \to \sigma_n(Y, x_0, G, \pi),$$

and

$$\partial_{n+1}\colon \{\tau_{n+1}(X, Y, x_0);\ \mathfrak{K}^{n+1}\} \to \{\tau_n(Y, x_0);\ \mathfrak{K}^n\},$$

respectively, such that

$$\partial_{n+1}\colon \{\pi_{n+1}(X, Y, x_0);\ \mathfrak{K}^{n+1}\} \to \{\pi_n(Y, x_0);\ \mathfrak{K}^n\}.$$

The isomorphisms $\mathfrak{f}_\flat$ commute with the homomorphisms i, j, and ∂. Thus, we have the following commutative diagram, in which, to save space, the symbols for the groups have been abbreviated by the suppression of the symbols G, π, and $\mathfrak{K}$.

$$\begin{tikzcd}
\sigma_{n+1}(X, x_0) \arrow[r, phantom, "\supset"] & \sigma_{n+1}(X, \mathfrak{f}) \arrow[r, "j_{n+1}"] \arrow[d, "\mathfrak{f}_\flat"'] & \sigma_{n+1}(X, Y, \mathfrak{f}) \arrow[r, "\partial_{n+1}"] \arrow[d, "\mathfrak{f}_\flat"'] & \sigma_n(Y, x_0) \arrow[r, "i_n"] \arrow[d, "\mathfrak{f}_\flat"'] & \sigma_n(X, x_0) \arrow[d, "\mathfrak{f}_\flat"'] \\
\{\tau_{n+1}(X, x_0)\} \arrow[r, phantom, "\supset"] & \{\tau_{n+1}(X, x_0, x_0)\} \arrow[r, "j_{n+1}"] \arrow[d, phantom, "\cup"] & \{\tau_{n+1}(X, Y, x_0)\} \arrow[r, "\partial_{n+1}"] \arrow[d, phantom, "\cup"] & \{\tau_n(Y, x_0)\} \arrow[r, "i_n"] \arrow[d, phantom, "\cup"] & \{\tau_n(X, x_0)\} \arrow[d, phantom, "\cup"] \\
 & \{\pi_{n+1}(X, x_0)\} \arrow[r, "j_{n+1}"] & \{\pi_{n+1}(X, Y, x_0)\} \arrow[r, "\partial_{n+1}"] & \{\pi_n(Y, x_0)\} \arrow[r, "i_n"] & \{\pi_n(X, x_0)\}
\end{tikzcd}$$

The lower sequence is a split extension by G of the homotopy sequence of (X, Y, x_0), while the middle sequence is a split extension by G of the torus homotopy sequence.

$$\begin{tikzcd}
{} \arrow[r, "i_{n+1}'"] & \tau_{n+1}(X, x_0) & & & & \\
& \cup & & & & \\
& \tau_{n+1}(X, x_0, x_0) \arrow[r, "j_{n+1}'"] & \tau_{n+1}(X, Y, x_0) \arrow[r, "\partial_{n+1}'"] & \tau_n(Y, x_0) \arrow[r, "i_n'"] & \tau_n(X, x_0) & \\
& & & & \cup & \\
& & & & \tau_n(X, x_0, x_0) \arrow[r, "j_n'"] & {}
\end{tikzcd}$$

This sequence has a spiral character, in that $\tau_n(X, x_0) \subset \tau_{n+1}(X, x_0)$ and is exact at each level in that $\operatorname{Ker} \partial_{n+1}' = \operatorname{Im} j_{n+1}'$, $\operatorname{Ker} i_n' = \operatorname{Im} \partial_{n+1}'$, and $\operatorname{Ker} j_n' = \tau_n(X, x_0, x_0) \cap \operatorname{Im} i_n'$.

The equalities here can be regarded as inclusions in both directions. The split extensions of the sequences give inclusion in one direction only, namely $\operatorname{Ker} \partial_{n+1} \subset \operatorname{Im} j_{n+1}$.

Definition 8.1. A sequence

$$A \xrightarrow{i} B \xrightarrow{j} C$$

will be said to be under-exact at B if $\operatorname{Ker} j \subset \operatorname{Im} i$, and under-exact relative to a subgroup G at B if $G \operatorname{Ker} j = \operatorname{Im} i$. The following lemma can be proved by standard arguments.

LEMMA 8.2. *A split extension of an exact sequence by a group G is under-exact relative to G.*

Using these notions, the structure of the homotopy sequence of a transformation group can be described in the following way.

THEOREM 8.3. *If $(X, Y, \mathfrak{k}, G, \pi)$ is a transformation group with a family of preferred paths, then its homotopy sequence is under-exact relative to G at $\sigma_{n+1}(X, Y, \mathfrak{k}, G, \pi)$ and at $\sigma_n(Y, x_0, G, \pi)$, that is to say,*

$$G \operatorname{Ker} \partial_{n+1} = \operatorname{Im} j_{n+1}, \qquad G \operatorname{Ker} i_n = \operatorname{Im} \partial_{n+1}.$$

Moreover, it contains a subsequence which is under-exact relative to G at every point.

REFERENCES

1. R. H. Fox, *Homotopy groups and torus homotopy groups*, Ann. of Math. (2) *49* (1948), 471–510.
2. F. Rhodes, *On the fundamental group of a transformation group*, Proc. London Math. Soc. (3) *16* (1966), 635–650.

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