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Abstract

The attractor conjecture for Calabi–Yau moduli spaces predicts the algebraicity of the moduli values of certain isolated points picked out by Hodge-theoretic conditions. Using tools from transcendence theory, we provide a family of counterexamples to the attractor conjecture in almost all odd dimensions conditional on a specific case of the Zilber–Pink conjecture in unlikely intersection theory; these Calabi–Yau manifolds were first studied by Dolgachev. We also give constructions of new families of Calabi–Yau varieties, analogous to the mirror quintic family, with all middle Hodge numbers equal to one, which would also give counterexamples to the attractor conjecture.

1. Introduction

1.1 Statement of results

In this paper, we study the following remarkable conjecture due to string theorists.

Conjecture 1.1.1 (Moore). If X is an attractor Calabi–Yau 3-fold, then it is defined over $\bar{\mathbf{Q}}$.

We recall the definition of an attractor Calabi–Yau variety.

Definition 1.1.2. For X a Calabi–Yau d-fold, we say that it is an *attractor variety* if there is a nonzero integral cohomology class $\gamma \in H^d(X, \mathbf{Z})$ satisfying

$$\gamma \perp H^{d-1,1},$$

where $H^{d-1,1} \subset H^d(X, \mathbf{C})$ denotes the $(d-1, 1)$ piece of the Hodge decomposition.

These varieties were originally introduced and studied by Ferrara, Kallosh and Strominger for Calabi–Yau 3-folds [FKS95] as the case most directly of interest in string theory; Calabi–Yau 4-folds were also considered shortly thereafter [Moo07, Section 3.8]. The above definition in general dimension was then given in [BR11], as we discuss further in § 2. To build intuition, note that the above condition should impose $h^{d-1,1}$ conditions, where we denote by $h^{d-1,1} = \dim H^1(X, TX)$ the dimension of Calabi–Yau moduli space; as such, one typically expects attractor Calabi–Yaus (for some fixed γ) to be isolated in moduli space, which is indeed the case for the examples we consider below. Hence, it is certainly of interest, irrespective of the physical origin of the question, to investigate the arithmetic structure of the points picked out by this natural Hodge-theoretic condition. Finally, other than for reasons coming from string theory, we

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see no particular reason for restricting the statement of the attractor conjecture to the case of 3-folds.

Our main result, then, is the following theorem.

Theorem 1.1.3. *Under the Zilber–Pink conjecture, the analogue of Conjecture 1.1.1 for Calabi–Yau varieties of arbitrary dimension is false. More precisely, there exist attractor Calabi–Yau varieties in all odd dimensions except 1, 3, 5 and 9 which are not defined over* $\bar{\mathbf{Q}}$.

Although the Zilber–Pink conjecture (ZPC), whose statement we recall in § 5, is currently far from proven, it is widely believed to hold,[1] and is, moreover, a conjecture of a rather different flavor than the attractor conjecture; we therefore believe that our result constitutes significant progress on the latter. Moreover, we only require a very special case of the ZPC where much more is known [Orr20], in that we are intersecting the moduli space of Calabi–Yau varieties with *Hecke translates of finitely many Shimura subvarieties*, rather than simply arbitrary Shimura subvarieties. This seems to be one of the simplest cases of the conjecture beyond the André–Oort conjecture. Finally, assuming that our construction in § 6 works, we can even reduce the problem to a case where the moduli space of Calabi–Yau varieties is a curve, bringing us even closer to the main result of [Orr20].[2]

Indeed, for the family of counterexamples we consider, we show the much stronger statement that the set of attractor points defined over $\bar{\mathbf{Q}}$ must be non-Zariski dense in the Calabi–Yau moduli space. As we will check that the set of attractor points is indeed dense (even in the analytic topology), these families give extremely strong counterexamples in the sense that almost all attractor points fail to be defined over $\bar{\mathbf{Q}}$.

Amusingly, the Calabi–Yau examples we consider are decidedly *not* counterexamples in dimensions 1, 3, 5 or 9. These examples have already been well-understood in the context of flat surfaces (in the theory of Teichmüller dynamics). Indeed, in these cases the attractors are indeed algebraic and are, moreover, examples of complex multiplication (CM) points on Shimura varieties.

While we give the specific counterexamples above due to the particular techniques we bring to bear, we expect a much more general transcendence property for these attractor points.

Conjecture 1.1.4. The algebraic attractor points in the moduli space of a Calabi–Yau X are Zariski dense if and only if the said moduli space is a Shimura variety.

We pause to explain the nomenclature and history of our examples, as well as to point to some related examples. The Calabi–Yau construction we use is that of a crepant resolution of an n-fold cyclic cover of $\mathbf{P}^{2n-3}$ branched at a suitable hyperplane collection, following [SXZ13]. We follow these authors in citing Dolgachev for his study of the moduli spaces thereof (as attempting to answer the famous question of B. Gross on realizing ball quotients as geometric moduli spaces) as in [DGK05, DK07], terming these *Dolgachev Calabi–Yaus*.

In particular, one could certainly consider variants of the construction we investigate here, such as a family of double covers of projective space now branched at some other suitable hyperplane arrangement. The latter family contains a Calabi–Yau 3-fold example with non-Shimura moduli [SXZ15]. Following our conjecture above, we therefore suggest the following case as a particularly attractive next area for investigation.

[1] See for example [DR18] where it is deduced from certain arithmetic conjectures about point-counting.
[2] Indeed, Orr [Orr20] proves this special case of ZPC that is required for our argument conditional on some arithmetic conjectures.

QUESTION 1.1.5. Does the attractor conjecture hold for the family of double-cover Dolgachev Calabi–Yau 3-folds?

1.2 History of the problem and related works

Attractor varieties in the context of Calabi–Yau 3-folds were originally discovered by Ferrara, Kallosh and Strominger in the context of Calabi–Yau 3-fold compactifications of string theory. They have been the subject of focused study since; mathematically, for example, they are conjectured to govern the behavior of the enumerative geometry of Calabi–Yau 3-folds [KS14]. Moore in [Moo98] performed an in-depth study and made various conjectures about their possible arithmetic properties, including Conjecture 1.1.1 above. In particular, Moore investigated various examples such as $S \times E$ for S a K3 surface and E an elliptic curve, a quotient thereof known as the FHSV model, and abelian 3-folds. In all these cases, the attractor points are defined over $\mathbf{Q}$; however, note that all these examples have Shimura moduli (and indeed, the attractor points are special points in the said Shimura variety). In particular, Moore in his lectures [Moo07] notes a distinction in the plausibility of these conjectures for the *rank-one* and *rank-two* cases, where the rank refers to the rank of the integral sublattice of H^3 that lies within the subspace $H^{3,0} \oplus H^{0,3}$. Indeed, in the latter case where one has an ensuing splitting of Hodge structures, the standard conjectures predict that $H^{3,0} \oplus H^{0,3}$ splits off as a motive and the arithmeticity of such loci would immediately follow. See [KOU20] for recent spectacular work in this direction unconditional on any conjectures, although the specific case at hand is not covered in their paper. By contrast, as essentially remarked above, one should expect arithmeticity almost never to hold in the former case, in the technical sense of (Zariski) nondensity.

These attractor points have many other becoming properties analogous to those of special points of Shimura varieties. Douglas and his coauthors studied the distribution of these points in their moduli space in [Dou03, DD04, DSZ04, DSZ06a, DSZ06b]; the last series of papers by Douglas, Shiffman and Zelditch in particular developed strong heuristics to suggest that attractor points equidistribute in moduli space with its natural Weil–Petersson metric (together with strong numerical evidence in myriad cases to support the said claim). We in fact use such distributional results in our proof of Theorem 1.1.3, although we only need the much weaker statement that attractor points are Zariski dense, which we can verify directly in § 3.5.

1.2.1 *Quantum corrections.* It turns out that the class of Calabi–Yau 3-folds with moduli space a Shimura variety arises naturally in another context: they are precisely those with *no quantum correction* [LY14], which is equivalent to saying that their mirrors have vanishing Gromov-Witten invariants. Since we expect that such Calabi–Yau varieties are precisely those for which the attractor conjecture holds, one could speculate that in the general situation there is some 'quantum correction' to the attractor condition which picks out arithmetically interesting points.

1.2.2 *Flux vacua.* Since the beginning of the mathematical study of attractor points, the underlying theme has been whether Hodge-theoretic conditions other than the existence of extra Hodge classes can lead to loci analogous to Hodge loci. On the other hand, there are other natural loci arising from string theory which are closely related to attractor points, known as flux vacua, in the moduli of Calabi–Yau manifolds; in this setup one again fixes some integral cohomology classes known as the *flux background* and seeks to minimize a function over the moduli space. For details of these loci we refer the reader to [DD04] and the references therein. It turns out that this again picks out loci where the fixed cohomology classes satisfies some Hodge-theoretic

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condition but which, crucially, is *not* equivalent to having extra Hodge classes. Nevertheless, such loci seem to share similarity with Hodge loci: for example, very recent work by Bakker, Grimm, Schnell and Tsimerman [BGST21] proves the surprising analogue of Cattani, Deligne and Kaplan for such loci.

1.2.3 *Donaldson–Thomas invariants.* We would like to briefly point out another appearance of the attractor mechanism. In the case of 3-folds, Kontsevich and Soibelman have used attractor points and attractor flows to define Donaldson–Thomas invariants,[3] which are expected to agree with counts of special Lagrangian manifolds and satisfy the Kontsevich–Soibelman wall-crossing formula. The arithmeticity of such points plays no role in this theory. Nevertheless, in the case of the Dolgachev 3-folds studied in this paper where the attractor points are CM points, one could hope that such explicit control of the attractor points would help in the study of DT invariants, which are still very mysterious in the case of compact Calabi–Yau 3-folds.

1.3 Outline of proof of Theorem 1.1.3

We sketch the proof of Theorem 1.1.3. We proceed by contradiction, so we assume that all attractor Calabi–Yau varieties are in fact defined over $\bar{\mathbf{Q}}$. We consider the Dolgachev Calabi–Yau varieties as constructed in § 3.1; these give examples of Calabi–Yau varieties defined in all odd dimensions and will provide our counterexamples in almost all cases. These Dolgachev Calabi–Yau varieties X are constructed from an associated curve C, and the middle Hodge structures of X and C are closely related as reviewed in § 3.1. Next, it is not difficult to check that the attractors are Zariski (in fact, even analytically) dense in the moduli space $\mathcal{M}$. Then we show that, for X a Dolgachev Calabi–Yau variety, if it is an attractor and defined over $\bar{\mathbf{Q}}$, then its Jacobian $\mathrm{Jac}(C)$ splits in the isogeny category as $A_1 \times A_2$ where the abelian variety A_1 has complex multiplication (CM) by a fixed cyclotomic field. The crucial ingredient here is a theorem of Shiga and Wolfart (following Wüstholtz) in transcendence theory, which, informally, implies that an abelian variety defined over $\bar{\mathbf{Q}}$ is CM as soon as it has sufficiently many algebraic period ratios, which in our case follows from the attractor condition and prior Hodge-theoretic analysis.

This splitting of $\mathrm{Jac}(C)$ up to isogeny then may now be thought of as a problem in the intersection theory of Shimura varieties: *a priori*, $\mathrm{Jac}(C)$ naturally defines a point of an ambient Shimura variety Sh and the isogeny splitting condition above implies that $\mathcal{M}$ intersects the Hecke translates of a sub-Shimura variety Sh_A of Sh in a dense set of points. The attractor condition has thus reduced to a problem in *unlikely intersection theory*. In particular, when the codimensions of $\mathcal{M}$ and Sh_A in Sh sum to less than the dimension of Sh, the ZPC implies that $\mathcal{M}$ is contained in some proper Shimura subvariety. This is the point where the argument fails for small values of the dimension of the Calabi–Yau varieties; in fact, in these cases the moduli space $\mathcal{M}$ turns out to be a Shimura variety, the attractor points are CM points, and therefore the attractor conjecture holds. In the general case, we instead use a result of Deligne and Mostow on the monodromy groups of these varieties to show that, for almost all dimensions, $\mathcal{M}$ is not contained in any proper Shimura subvariety of Sh, and hence we have a contradiction as desired.

1.4 Conjectural one-modulus families of Calabi–Yau varieties

In general, the ZPC is wide open; however, it does simplify greatly when the subvariety in question is a curve. In this case, it is known to hold conditional only on certain arithmetic

[3] For noncompact Calabi–Yaus the attractor flow and attractor points suffice to determine the Donaldson–Thomas invariants, whereas for compact Calabi–Yaus one seems to need some extra input data.

statements by work of Orr [Orr20]. Motivated by this, we construct one-modulus families of Calabi–Yau varieties for which our proof would go through without change; more precisely, we construct one-parameter families of singular Calabi–Yau varieties, and conjecture that they admit crepant resolutions. We believe these families are interesting in their own right as they are analogous to the construction of the mirror quintic[4] family: for example, they are again attached to curves, and have all middle Hodge numbers equal to one.

1.5 Outline of the rest of the paper

In § 2 we introduce the attractor condition on a Calabi–Yau variety as well as the attractor conjecture, along with providing somewhat more context for the interested reader. Section 3 continues with the main thrust of the proof above by defining the Dolgachev Calabi–Yaus X along with their associated curves C before establishing the relation between the Hodge structures thereof. Section 4 applies the theorem of Shiga and Wolfart to reduce to a problem in Shimura theory before setting up the formalism of the ambient Shimura variety Sh and its special Shimura subvariety Sh_A. We conclude the proof of our main result in § 5 with a discussion on the unlikely intersection theory of this Shimura variety problem. Finally, in Section 6 we give the conjectural construction of one-parameter families of Calabi–Yau varieties which would give counterexamples to the attractor conjecture, where the ZPC is more accessible: indeed, there has been significant progress [Orr20] in the case when one considers unlikely intersections between a curve and Hecke translates of a Shimura subvariety.

1.6 Notations and conventions

We now set a few conventions. We work throughout with the Hermitian intersection pairing on the middle-degree complex cohomology of a manifold; under this pairing, distinct Hodge summands are orthogonal. For a vector space V defined over some field K and a field extension $K \subset L$, we often denote by V_L the extension of scalars $V \otimes_K L$. For an algebraic variety X over $\mathbf{C}$, we will sometimes say that X is *algebraic*, meaning that it is defined over $\bar{\mathbf{Q}}$. For a curve C, we denote by $\mathrm{Jac}(C)$ its Jacobian. Starting from § 3.1, we will fix an integer n, such that the Calabi–Yaus we consider have dimension $2n-3$, and we will use the notation $\zeta = e^{2\pi i/n}$ throughout.

2. The attractor conjecture

We now recall the attractor condition for (higher-dimensional) Calabi–Yaus precisely. Note that, in general, we take a Calabi–Yau variety to be a smooth, projective variety X with trivial canonical bundle and (*a priori*) defined over the complex numbers $\mathbf{C}$. We will always assume that X belongs to a family which is versal: that is to say, there is a family of smooth Calabi–Yau varieties $\mathfrak{X} \to \mathcal{M}$ such that for each $s \in S$, the Kodaira–Spencer map[5] $T_s \to \mathrm{Hom}(H^{d,0}(\mathfrak{X}_s), H^{d-1,1}(\mathfrak{X}_s))$ is an isomorphism. The existence of $\mathcal{M}$ follows from Bogomolov [Bog78], Tian [Tia87] and Todorov [Tod89]. In practice, however, we will work with a specific family defined in § 3.1.

Definition 2.0.1. Given a Calabi–Yau d-fold X, then for each nonzero class $\gamma \in H^d(X, \mathbf{Z})$, X is said to be an *attractor* for the class γ if the condition

$$\gamma \perp H^{1,d-1}(X)$$

[4] As well as their higher-dimensional analogues – mirror septic, etc.
[5] Here $\mathfrak{X}_s$ denotes the fiber of $\mathfrak{X}$ at s.

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holds. We will say that X is an *isolated* attractor point if X is an attractor and there is no nontrivial real analytic[6] one-parameter family through X consisting of attractors for the class γ.

Remark 2.0.2. Our reason for emphasizing the *isolated* condition is that in general it is possible to have nonisolated attractors (i.e., positive-dimensional families); it is indeed straightforward to construct examples where this happens, such as the very families of Dolgachev Calabi–Yau's considered in this paper when n is not prime. As such, the higher-dimensional attractor conjecture would be trivially false for such attractor points.

In fact, this attractor condition in high dimensions is essentially due to Brunner and Roggenkamp [BR11, § 4], although they state their condition in a slightly different form. Consider the function $|Z_\gamma|^2$, where Z_γ is the *central charge* function

$$Z_\gamma := \frac{\langle \gamma, \Omega\rangle}{\sqrt{i^d\langle \Omega, \bar{\Omega}\rangle}}.$$

Proposition 2.0.3. *Suppose $\mathfrak{X} \to \mathcal{M}$ is a versal family of smooth Calabi–Yau d-folds. If X is an attractor point, then it is a critical point of $|Z_\gamma|^2$. Conversely, if X corresponds to a critical point of $|Z_\gamma|^2$, and also $\gamma^{d,0} \neq 0$,[7] then it is an attractor point in the sense of Definition 2.0.1.*

Proof. The proof is identical to that of [Moo98, Theorem 2.5.1], though we give the details of the computation here for completeness.

Suppose X corresponds to the point $x \in \mathcal{M}$, and take local holomorphic coordinates z_i which are well defined on an open neighborhood U of x, as well as a holomorphic family of holomorphic volume forms $\Omega(s)$ on $\mathfrak{X}_s$ for $s \in \mathcal{U} \subset \mathcal{M}$.

Write $\partial_{z_i}\Omega(s) = \lambda(s)\Omega(s) + \chi_i$ for some function λ defined on U, and $\chi_i \in H^{d-1,1}(\mathfrak{X}_s)$. Then we have

$$\begin{aligned}
\partial_{z_i}|Z_\gamma|^2 &= \frac{i^d\langle \Omega,\bar{\Omega}\rangle\langle\gamma,\partial_{z_i}\Omega\rangle\langle\gamma,\bar{\Omega}\rangle - \langle\gamma,\Omega\rangle\langle\gamma,\bar{\Omega}\rangle i^d\langle\partial_{z_i}\Omega,\bar{\Omega}\rangle}{(-1)^d\langle\Omega,\bar{\Omega}\rangle^2}\\
&= i^d\frac{\langle \Omega,\bar{\Omega}\rangle\langle\gamma,\lambda\Omega+\chi_i\rangle\langle\gamma,\bar{\Omega}\rangle - \langle\gamma,\Omega\rangle\langle\gamma,\bar{\Omega}\rangle\langle\lambda\Omega+\chi_i,\bar{\Omega}\rangle}{(-1)^d\langle\Omega,\bar{\Omega}\rangle^2}\\
&= \langle\gamma,\chi_i\rangle\frac{\bar{Z}_\gamma}{(i^d\langle\Omega,\bar{\Omega}\rangle)^{1/2}}.
\end{aligned}$$

Now if X corresponds to an attractor point, then $\langle\gamma,\chi_i\rangle = 0$ for all i, and hence $\partial_{z_i}|Z_\gamma|^2 = 0$ for all i, so that X corresponds to a critical point. Conversely, if X corresponds to a critical point, and $\gamma^{d,0} \neq 0$, then $\bar{Z}_\gamma \neq 0$, and therefore $\langle\gamma,\chi_i\rangle = 0$ for all i. Since the χ_i span $H^{d-1,1}$, X corresponds to an attractor point, as required. □

Remark 2.0.4. We expect that, generically, attractor points are isolated as they arise as critical points of the real analytic function $|Z_\gamma|^2$. We believe this is a reasonable generalization of the notion of attractor points from the case of 3-folds, because of the uniform description as critical points of $|Z_\gamma|^2$. Finally, we point out that, for the Calabi–Yau varieties we consider, attractor points are isolated and dense in moduli space can be proved directly: see § 3.5.

[6] That is, a real analytic map $f : [0,1] \to \mathcal{M}$, such that $f(t)$ is an attractor point for all $t \in [0,1]$ and $f(1/2)$ corresponds to X. We have used the real analytic version since we see no reason for the locus of attractor points to be complex analytic in general.

[7] Recall that $\gamma^{d,0} \in H^{d,0}(X)$ is the $(d,0)$ piece of γ according to the Hodge decomposition on X.

Conjecture 2.0.5 ([Moo98, Conjecture 8.2.2] for the case of 3-folds). If X is an attractor variety for a nonzero class $\gamma \in H^d(X, \mathbf{Q})$, which is, moreover, isolated in its moduli space in the sense of Definition 2.0.1, then it has a model over $\bar{\mathbf{Q}}$.

The above conjecture would suggest that these points picked out by the Hodge-theoretic attractor condition are a class of special points analogous in myriad aspects to special points of Shimura varieties. (Indeed, Moore also makes a counterpart conjecture that the periods of these points are algebraic.) And while Moore originally makes the conjecture for Calabi–Yau 3-folds, Brunner and Roggenkamp [BR11, § 4] show that the same considerations apply in all respects to higher-dimensional Calabi–Yaus, and so we find it equally interesting to study the veracity of this conjecture in higher dimensions.

As remarked above, Moore also conjectures [Moo98, Conjecture 8.2.1] not just algebraicity of the attractor points but also algebraicity of certain period ratios (assuming there is a point of maximal unipotent monodromy). This holds in the case of Shimura moduli simply because all period ratios are algebraic, but in the general case would imply very surprising relations between periods. For example, in the recent work [COES19], Candelas, de la Ossa, Elmi and van Straten verified numerically that, in the case of a rank-two attractor point, the periods in question are ratios of special L-values. Note that, morally speaking, the conjecture on algebraicity of attractors is a *mirror* to the conjecture on algebraicity of periods, and hence our main theorem can be thought of as mirror to a transcendence result on special values of L-functions.

3. Dolgachev Calabi–Yau varieties

3.1 Defining the Calabi–Yaus

We will consider a family of Calabi–Yau varieties constructed as crepant resolutions of n-fold cyclic covers of projective space, due to Dolgachev and to Sheng, Xu and Zuo [SXZ13]. Most of the discussion holds for any $n \geq 2$, although the case $n = 2$ is completely classical and returns the Legendre family of elliptic curves; we therefore restrict to $n \geq 3$ simply for convenience. At a crucial point, we find that the cases $n = 3, 4, 6$ are distinguished, giving rise to 'arithmetic' families of Calabi–Yau varieties (e.g., these are exactly the cases for which the resulting $\mathcal{M}_{\mathrm{CY}}$ is Shimura); all three of these therefore display qualitatively different behavior and we note at the appropriate point where the condition $n \neq 3, 4, 6$ is crucial for the 'nonarithmeticity' phenomenon of the statement of our main Theorem 1.1.3.

The definition of the Calabi–Yau varieties goes through that of a curve; this construction has been studied by many authors, and we refer the reader to [Loo07, § 4.1][8] for a more general version.

We first consider $2n$ points $x_1, \ldots, x_{2n}$ in $\mathbf{P}^1$, and the curve C given by the n-fold cyclic cover of $\mathbf{P}^1$ branched at those $2n$ points; more precisely, we mean the cover determined by the same n-cycle monodromy about each branch point in the base, or simply the smooth, projective curve C whose affine model is given by

$$C^\circ = \left\{ y^n = \prod_{i=1}^{2n} (x - x_i) \right\}. \tag{1}$$

By construction, $H^1(C, \mathbf{Q})$ has both a Hodge splitting and a $\mathbf{Z}/n$-action. More precisely, if we let $\zeta = e^{2\pi i/n}$, then $H^1(C, \mathbf{Q}(\zeta))$ splits into eigenspaces for the $\mathbf{Z}/n$-action. Let $\mu \in \mathbf{Z}/n$ be the generator which acts on C by $y \mapsto \zeta y$ in (1).

[8] For the convenience of the reader, one should take n in the notation of [Loo07, § 4.1] to be our $2n - 2$, and the μ_k for each k to be our $1/n$.

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Definition 3.1.1. Let $H^1(C)[i] \subset H^1(C; \mathbf{Q}(\zeta))$ be the sub-$\mathbf{Q}(\zeta)$-vector space such that μ acts by ζ^i.

Then $H^1(C)[i]$ is a $\mathbf{Q}(\zeta)$-sub-Hodge structure of $H^1(C, \mathbf{Q}(\zeta))$ in the sense that we have the decomposition

$$H^1(C)[i] \underset{\mathbf{Q}(\zeta)}{\otimes} \mathbf{C} \simeq H^{1,0}(C)[i] \oplus H^{0,1}(C)[i]$$

given by the Hodge splitting of $H^1(C; \mathbf{C})$.

Proposition 3.1.2. *The Hodge numbers of* $H^1(C)[i]$ *are given by* $(2i-1, 2(n-i)-1)$, *for* $1 \leq i \leq n-1$; *that is,*

$$\dim_{\mathbf{C}} H^{1,0}(C)[i] = 2i - 1, \quad \dim_{\mathbf{C}} H^{0,1}(C)[i] = 2(n-i) - 1.$$

We refer the reader to [Loo07, Lemma 4.2] for this computation. Note that the $i = 0$ eigenspace is trivial as any cohomology class in this space comes from $H^1(\mathbf{P}^1) \simeq 0$ via pullback.

Construction 3.1.3. It remains to introduce the Calabi–Yau $(2n-3)$-folds X. Here we follow the treatment in [SXZ13, Sections 2, 3]. For a collection of $2n$ hyperplanes $(H_1, \ldots, H_{2n})$ of $\mathbf{P}^{2n-3}$, we say that they are in general position if no $2n-2$ of them intersect at any point; that is, there are no unexpected intersections between the H_i. Then we may define an n-fold cyclic cover X' of $\mathbf{P}^{2n-3}$ branched along these hyperplanes.

We give the rigorous construction here. For a line bundle L on an arbitrary variety Y and a positive integer n, consider the rank-n vector bundle

$$\mathcal{E} := \mathcal{O} \oplus L^\vee \oplus \cdots \oplus (L^\vee)^{\otimes n-1}$$

on Y; here $L^\vee$ denotes the dual of L. Now given a section of $L^{\otimes n}$, or equivalently a map

$$(L^\vee)^{\otimes n} \to \mathcal{O},$$

we may define an algebra structure on $\mathcal{E}$ in the obvious way, and therefore we may form the variety

$$X := \mathrm{Spec}(\mathcal{E}),$$

and by construction X admits a map to Y; in fact, this is a cyclic n-fold covering. In other words, a section $\sigma \in \Gamma(L^{\otimes n})$ defines a cyclic n-fold cover $X \to Y$.

Definition 3.1.4. For a collection of $2n$ points $p_1, \ldots p_{2n}$ on $\mathbf{P}^1$, we may consider $2n$ hyperplanes on $\mathbf{P}^{2n-3}$ as follows: recall that there is an isomorphism

$$\mathrm{Sym}^{2n-3}\mathbf{P}^1 \cong \mathbf{P}^{2n-3}, \tag{2}$$

and also that, for each $i = 1, \ldots, 2n$, the set of points of the form $\{p_i\} \times \mathbf{P}^1 \times \cdots \times \mathbf{P}^1$ on the left-hand side of (2) gives a hyperplane on the right-hand side. Therefore, we obtain $2n$ hyperplanes $H_1, \ldots, H_{2n}$ on $\mathbf{P}^{2n-3}$, in general position.

We now apply the above construction to $Y = \mathbf{P}^{2n-3}$, $L = \mathcal{O}(2)$ and $\sigma \in \Gamma(\mathcal{O}(2n))$ such that the zero locus of σ is precisely

$$D := \sum_{i=1}^{2n} H_i \subset \mathbf{P}^{2n-3}$$

to obtain a cyclic n-fold covering of $\mathbf{P}^{2n-3}$, which we denote by X'.

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Note that a pleasant computation shows that the canonical bundle of the cyclic cover X' defined above is trivial: indeed, using the formula $K_X = \pi^* K_Y + R$ for a covering map $\pi : X \to Y$, where $R \subset X$ is the ramification divisor, we deduce that

$$\begin{aligned} K_{X'} &= \pi^*((2-2n)H) + (2n)(n-1)H' \\ &= (n(2-2n) + 2n(n-1))H' \\ &= 0, \end{aligned}$$

and so the canonical class of X' is trivial; here H denotes a hyperplane class in $\mathbf{P}^{2n-3}$ and H' is the class of one of the n components of the pullback of H.

Denote the moduli space of collections $(H_1, \ldots, H_{2n})$ of hyperplanes of $\mathbf{P}^{2n-3}$ in general position by $\mathcal{H}_n$. The following theorem was proved by Sheng, Xu and Zuo [SXZ13, Corollary 2.6].

Theorem 3.1.5. *Denote by $f' : \mathcal{X}' \to \mathcal{H}_n$ the family of $(2n-3)$-folds over the moduli of hyperplane arrangements constructed above. Then the following statements hold.*

(i) *There is a family of smooth Calabi–Yau $(2n-3)$-folds*

$$f : \mathcal{X} \to \mathcal{H}_n$$

as well as a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\sigma"] \arrow[dr, "f"'] & & \mathcal{X}' \arrow[dl, "f'"] \\ & \mathcal{H}_n & \end{tikzcd} \tag{3}$$

where σ is a simultaneous crepant resolution.

(ii) *The middle-degree Hodge structures of the families $\mathcal{X}'$ and $\mathcal{X}$ agree:*

$$R^{2n-3} f'_* \mathbf{Q} \cong R^{2n-3} f_* \mathbf{Q}.$$

(iii) *Furthermore, the family f is maximal in the sense that the Kodaira–Spencer map is an isomorphism at each point $p \in \mathcal{H}_n$.*

Definition 3.1.6. We refer to the varieties constructed in Theorem 3.1.5 as the Dolgachev Calabi–Yau varieties. For $X' = f'^{-1}(p)$ the $(2n-3)$-fold parametrized by some $p \in \mathcal{H}_n$, we denote by $X := f^{-1}(p)$ the corresponding crepant resolution.

Remark 3.1.7. This is a slight abuse of terminology since the crepant resolution $\sigma : \mathcal{X} \to \mathcal{X}'$ is not unique; on the other hand, any two resolutions are of course birational to each other.

3.2 Hodge structures of Dolgachev Calabi–Yaus

Now we relate the curves constructed earlier as branched covers of $\mathbf{P}^1$ to the Calabi–Yau varieties constructed as covers of $\mathbf{P}^{2n-3}$. Recall from Theorem 3.1.5 that the cyclic cover X' has a crepant resolution X which is Calabi–Yau, and we have an isomorphism

$$H^{2n-3}(X; \mathbf{Q}) \simeq H^{2n-3}(X'; \mathbf{Q}).$$

The right-hand side has a natural $\mathbf{Z}/n$-action since it arises as a cyclic cover, and therefore the left-hand side does as well. We may, therefore, decompose $H^{2n-3}(X)$ into eigenspaces as follows. As in § 3.1 we fix a generator $\mu \in \mathbf{Z}/n$.

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Definition 3.2.1. We define

$$H^{2n-3}(X)[i] \subset H^{2n-3}(X; \mathbf{Q}(\zeta))$$

as the sub-vector space over $\mathbf{Q}(\zeta)$ on which μ acts by ζ^i.

Then we have the crucial relationship between the Hodge structures of C and X.

Lemma 3.2.2 [SXZ13, Proposition 3.7]. *We have the following isomorphism of Hodge structures:*

$$H^{2n-3}(X)[i] \simeq \bigwedge^{2n-3} H^1(C)[i].$$

Remark 3.2.3. Perhaps a more intrinsic way of phrasing the above lemma is that there is an isomorphism of $\mathbf{Q}$-Hodge structures with $\mathcal{A} := \mathbf{Q}[X]/(X^n - 1)$-action

$$H^{2n-3}(X, \mathbf{Q}) \cong \bigwedge_{\mathcal{A}}^{2n-3} H^1(C, \mathbf{Q}).$$

In other words, we view $H^1(C, \mathbf{Q})$ as an $\mathcal{A}$-module, where the generator $X \in \mathcal{A}$ acts through $\mu \in \mathbf{Z}/n$, and then we take its $(2n-3)$th wedge power over $\mathcal{A}$.

We have the following statement.

Corollary 3.2.4. *The Hodge structure $H^{2n-3}(X, \mathbf{Q}) \otimes \mathbf{Q}(\zeta)$ decomposes as a sum*

$$H^{2n-3}(X, \mathbf{Q}) \otimes \mathbf{Q}(\zeta) \cong \bigoplus_{i=1}^{n-1} V_i$$

where V_i is a $\mathbf{Q}(\zeta)$-Hodge structure, concentrated only in Hodge degrees

$$(p, q) = (2i-2, 2n-2i-1) \text{ and } (2i-1, 2n-2i-2);$$

furthermore, the dimensions of these pieces of the Hodge decomposition are

$$2i - 1 \text{ and } 2n - 2i - 1,$$

respectively.

Proof. Indeed, we define the Hodge structures V_i to be $\bigwedge^{2n-3} H^1(C)[i]$ from Lemma 3.2.2. By Proposition 3.1.2 we may write the Hodge decomposition of $H^1(C)[i]$ as

$$H^1(C)[i] \otimes \mathbf{C} \cong H^{1,0} \oplus H^{0,1},$$

where we have omitted the dependence on i on the right-hand side, and

$$\dim H^{1,0} = 2i - 1, \quad \dim H^{0,1} = 2(n-i) - 1.$$

For convenience let us pick a basis $\{e_i\}$ (respectively, $\{f_j\}$) for $H^{1,0}$ (respectively, $H^{0,1}$). Since the dimension of $H^1(C)[i]$ is $2n-2$, upon taking the $(2n-3)$th wedge power, the only nonzero elements obtained by wedging together the e_i and f_j must omit precisely one e_i or one f_j. Therefore, the Hodge degrees of such an element are either

$$(p, q) = (2i-2, 2(n-i)-1) \text{ or } (2i-1, 2(n-i)-2);$$

furthermore, there are $2i-1$ (respectively, $2(n-i)-1$) choices of an e_i (respectively, f_j) to omit, and therefore the Hodge numbers are $2i-1$ (respectively, $2n-2i-1$), as claimed. □

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Notation 3.2.5. We will sometimes use the following notation for bookkeeping when dealing with these Hodge numbers. We record the dimensions in a $(n-1)\times 2$ matrix

$$\begin{pmatrix} \dim H^{1,0}(C)[1] & \dim H^{0,1}(C)[1] \\ \dim H^{1,0}(C)[2] & \dim H^{0,1}(C)[2] \\ \vdots & \vdots \\ \dim H^{1,0}(C)[n-1] & \dim H^{0,1}(C)[n-1] \end{pmatrix}.$$

For example, in the case $n=5$, it follows from Proposition 3.1.2 that the above matrix is

$$\begin{pmatrix} 1 & 7 \\ 3 & 5 \\ 5 & 3 \\ 7 & 1 \end{pmatrix}.$$

Remark 3.2.6. We therefore verify from the case of $i=n-1$ that $h^{2n-3,0}=1$, as expected for a Calabi–Yau variety of dimension $2n-3$; moreover, we find the dimension of the deformation space of a Dolgachev Calabi–Yau is $h^{2n-4,1}=2n-3$, which is notably the same as the dimension of the $\mathcal{M}_{0,2n}$, the moduli of $2n$ points in $\mathbf{P}^1$. This implies that the construction exactly accounts for the full moduli space of Calabi–Yau varieties so constructed, that is, the family $\mathcal{X}\to\mathcal{M}_{0.2n}$ is versal. We sometimes write $\mathcal{M}_{\mathrm{CY}}$ for either the space $\mathcal{M}_{0,2n}$ or $\mathcal{H}_n$ to emphasize it as a moduli space of Calabi–Yau varieties.

As a consistency check let us see that the dimensions of $\mathcal{M}_{0,2n}$ and $\mathcal{H}_n$ agree. Indeed, each moduli space parametrizes hyperplanes inside a projective space modulo the action of a projective linear group, and the coincidence of the dimensions is the equality

$$(2n)-(3)=(2n-3)(2n)-((2n-2)^2-1);$$

here the first term on each side of the equation is the number of moduli for the hyperplanes, while the second term is the dimension of the projective linear group.

Remark 3.2.7. Note that in the case where n is not a prime, for any Dolgachev Calabi–Yau variety X there exist classes $\gamma\in H^{2n-3}(X,\mathbf{Q})$ with no component in $H^{2n-3}(X)[1]$: indeed, just take any element in $H^{2n-3}(X,\mathbf{Q}(\zeta))[i]$ for some i not coprime to n, and take the sum of all of its Galois conjugates. The parallel transport of such a class γ will continue to have no component in $H^{2n-3}(X)[1]$ for any X. This gives examples of nonisolated attractor points: indeed, every X is an attractor for such a class γ, and we therefore need the isolated condition in Conjecture 2.0.5.

3.3 The attractor condition for Dolgachev Calabi–Yau varieties

We are now finally in a position to study the attractor condition for Calabi–Yau varieties X constructed as above. We first show that the attractor condition is equivalent to a condition on the periods of the associated curve C as follows.

Lemma 3.3.1. *The variety X satisfies the attractor condition if and only if there exists a nonzero $\omega\in H^{1,0}(C)[1]\cap H^1(C)[1]$.*

Note in the above that $H^{1,0}(C)[1]$ is defined as a subspace of $H^1(C,\mathbf{C})$ while $H^1(C)[1]$ is defined as a subspace of $H^1(C,\mathbf{Q}(\zeta))$. In particular, as $\dim_{\mathbf{C}} H^{1,0}(C)[1]=1$, the above attractor condition is also equivalent to the subspace $H^{1,0}(C)[1]\subset H^1(C)\otimes\mathbf{C}$ being defined over $\mathbf{Q}(\zeta)$.

Proof. Suppose X satisfies the attractor condition. Then there exists $\gamma\in H^{2n-3}(X;\mathbf{Z})$ orthogonal to $H^{2n-4,1}(X)$; equivalently, γ is orthogonal to $H^{1,2n-4}(X)$. Recall from the discussion above

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that $H^{1,2n-4}(X)$ is contained within the $H^{2n-3}(X)[1]$ eigenspace. The distinct μ-eigenspaces are certainly orthogonal under the intersection pairing, and if $\gamma_i \in H^{2n-3}(X)[i]$ denotes the summand of γ under the decomposition of $H^{2n-3}(X;\mathbf{Q}(\zeta))$, the above condition is equivalent to γ_1 orthogonal to $H^{1,2n-4}(X)$. As the Hermitian pairing is perfect on $H^{1,2n-4}(X)$, γ cannot have any support within the said Hodge summand of

$$H^{2n-3}(X)[1]_{\mathbf{C}} \simeq H^{1,2n-4}(X) \oplus H^{0,2n-3}(X),$$

and so we must have $\gamma_1 \in H^{0,2n-3}(X)$. But γ_1 was defined as an element of the vector space $H^{2n-3}(X)[1]$, a vector space defined over $\mathbf{Q}(\zeta)$, and so both $H^{0,2n-3}(X)$ and $H^{1,2n-4}(X)$, as the orthogonal complement of $H^{0,2n-3}(X)$ under the intersection pairing restricted to $H^{2n-3}(X)[1]$, are defined over $\mathbf{Q}(\zeta)$. But then

$$\begin{aligned} H^{1,2n-4}(X) &\simeq \bigwedge\nolimits^{2n-2} H^{0,1}(C)[1] \otimes H^{1,0}(C)[1] \\ &\simeq (H^{0,1}(C)[1])^\vee \otimes (\det H^{0,1}(C)[1]) \otimes H^{1,0}(C)[1] \\ &\simeq (H^{0,1}(C)[1])^\vee \otimes \det H^1(C)[1] \end{aligned}$$

as a subspace of

$$H^{2n-3}(X)[1] \simeq \bigwedge\nolimits^{2n-3} H^1(C)[1] \simeq (H^1(C)[1])^\vee \otimes \det H^1(C)[1].$$

Above, we use the notation $\det V = \bigwedge^{\dim V} V$ and the isomorphism

$$\bigwedge\nolimits^{\dim V - 1} V \simeq V^\vee \otimes \det V.$$

In any case, we have that the decomposition

$$H^{2n-3}(X)[1]_{\mathbf{C}} \simeq H^{1,2n-4}(X) \oplus H^{0,2n-3}(X)$$

is isomorphic to the decomposition

$$\begin{aligned} &((H^1(C)[1])^\vee \otimes \det H^1(C)[1])_{\mathbf{C}} \\ &\quad \simeq ((H^{0,1}(C)[1])^\vee \otimes \det H^1(C)[1]) \oplus ((H^{1,0}(C)[1])^\vee \otimes \det H^1(C)[1]) \end{aligned}$$

induced from the Hodge splitting of $H^1(C)[1]_{\mathbf{C}}$. As $H^1(C)[1]$ and hence $\det H^1(C)[1]$ are defined over $\mathbf{Q}(\zeta)$, however, the condition that the first decomposition be defined over $\mathbf{Q}(\zeta)$ is equivalent to the condition that the second decomposition be defined over $\mathbf{Q}(\zeta)$, which in particular implies that there exists some $\omega \in H^{1,0}(C)[1] \cap H^1(C)[1]$.

Conversely, given such an ω, we have that the subspace $H^{1,0}(C)[1] \subset H^1(C)[1]_{\mathbf{C}}$ is in fact defined over $\mathbf{Q}(\zeta)$ and hence so is $H^{0,1}(C)[1]$ as its orthogonal complement; as above, the decomposition $H^{2n-3}(X)[1]_{\mathbf{C}} \simeq H^{1,2n-4}(X) \oplus H^{0,2n-3}(X)$ is then also defined over $\mathbf{Q}(\zeta)$. Then take some $\gamma_1 \in H^{0,2n-3}(X)$ defined over $\mathbf{Q}(\zeta)$ so that, by construction, γ_1 is orthogonal to $H^{1,2n-4}(X)$, and consider the Galois conjugates γ_i under the action of $\mathrm{Gal}(\mathbf{Q}(\zeta)/\mathbf{Q})$ on $H^{2n-3}(X;\mathbf{Q}(\zeta))$. These Galois conjugates will lie within the $H^{2n-3}(X)[i]$ eigenspaces for values of i coprime to n and thus be concentrated in Hodge summands away from the $(1, 2n-4)$ and $(0, 2n-3)$ summands, so that if we now define $\gamma = \sum_i \gamma_i$, the Galois-theoretic construction will give us $\gamma \in H^{2n-3}(X;\mathbf{Q})$ while its summand in the $H^{2n-3}(X)[1]$ eigenspace is still the original γ_1 we started with. As such, scaling γ as necessary so it in fact lies in $H^{2n-3}(X,\mathbf{Z})$, we have produced some integral cohomology class orthogonal to $H^{1,2n-4}(X)$, or equivalently $H^{2n-4,1}(X)$, as desired. $\square$

3.4 Algebraicity of the associated curve

In this section we show that the algebraicity of the Dolagchev Calabi–Yau variety implies that of the curve associated to it. Therefore, to show that Dolgachev Calabi–Yau varieties provide counterexamples to the attractor conjecture it suffices to show that for the attractor varieties, the associated curves are not defined over $\bar{\mathbf{Q}}$.

Proposition 3.4.1. *X is defined over $\bar{\mathbf{Q}}$ if and only if C is defined over $\bar{\mathbf{Q}}$.*

Proof. We first deal with the easier direction, so suppose C is defined over $\bar{\mathbf{Q}}$. Let $\mu : C \to C$ denote a generator of the $\mathbf{Z}/n$ action, which must also be defined over $\bar{\mathbf{Q}}$; we spell out this argument here as we will use its basic idea, namely that of *spreading out* frequently. So, consider μ as a point of the quasiprojective $\bar{\mathbf{Q}}$-scheme $\mathrm{Aut}(C)$. If the field of definition K of μ is larger than $\bar{\mathbf{Q}}$, and in particular contains some pure transcendental extension thereof, we may freely specialize that transcendental variable to produce a family of automorphisms of C, but any curve has only finitely many automorphisms. Hence, μ must have been defined over $\bar{\mathbf{Q}}$. But now the morphism $C \to C/\mu \simeq \mathbf{P}^1$ is defined over $\bar{\mathbf{Q}}$, and so the $2n$ points $x_1, \ldots, x_{2n} \in \mathbf{P}^1$ of ramification are defined over $\bar{\mathbf{Q}}$ (after an appropriate automorphism of $\mathbf{P}^1$). Now the crepant resolution X constructed by Sheng, Xu and Zuo is obtained by blowing up[9] the cyclic cover X' (with notation as in Definition 3.1.6) along intersections of hyperplanes obtained from the x_i, and their intersections, and so on. It is clear that X' and all the blow-up centers within X' are defined over $\bar{\mathbf{Q}}$, and hence so is X.

The more interesting direction is the reverse argument, where we begin by supposing that X is defined over $\bar{\mathbf{Q}}$. The morphism $X \to \mathbf{P}^{2n-3}$ corresponds to some line bundle $\mathcal{L} \in \operatorname{Pic} X$ given by the pullback of $\mathcal{O}(1)$, but note that $\operatorname{Pic} X \simeq H^2(X; \mathbf{Z})$ is simply a discrete set of points as a scheme over $\bar{\mathbf{Q}}$, and hence all points must be defined over $\bar{\mathbf{Q}}$.

Claim 3.4.2. *The complete linear system of $\mathcal{L}$ defines precisely the morphism $X \to \mathbf{P}^{2n-3}$.*

Proof. It suffices to show that the pullback map induces an isomorphism

$$\Gamma(X, \mathcal{L}) \cong \Gamma(\mathbf{P}^{2n-3}, \mathcal{O}(1)).$$

Recall that we have the factorization

$$X \xrightarrow{\sigma} X' \xrightarrow{\alpha} \mathbf{P}^{2n-3},$$

where σ is the crepant resolution from Theorem 3.1.5, and

$$\alpha : X' \to \mathbf{P}^{2n-3}$$

denotes the n-fold covering of $\mathbf{P}^{2n-3}$ from Definition 3.1.4. We first show that

$$\Gamma(X', \mathcal{L}') \cong \Gamma(\mathbf{P}^{2n-3}, \mathcal{O}(1)),$$

where $\mathcal{L}' := \alpha^*\mathcal{O}(1)$. By construction of X' (see Definition 3.1.4 and the paragraph preceding it),

$$\alpha_*\mathcal{O}_{X'} \cong \mathcal{O}_{\mathbf{P}^{2n-3}} \oplus L^\vee \oplus \cdots \oplus (L^\vee)^{\otimes n-1},$$

where $L^\vee \cong \mathcal{O}(-2)$. Therefore,

$$\alpha_*\mathcal{L} \cong \mathcal{O}_{\mathbf{P}^{2n-3}}(1) \oplus \mathcal{O}_{\mathbf{P}^{2n-3}}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^{2n-3}}(-2n+3),$$

and hence $\Gamma(X', \mathcal{L}') = \Gamma(\mathbf{P}^{2n-3}, \mathcal{O}(1))$ as required. On the other hand, X is obtained from X' by blowing up along subvarieties of codimension at least two, and we claim

$$\Gamma(X, \mathcal{L}) \cong \Gamma(X', \mathcal{L}')$$

[9] We refer the reader to [SXZ13, Section 2.2] for the details of the blow-up procedure.

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as well. Indeed, as X and X' fail to be isomorphic only in codimension two, this statement would follow from (algebraic) Hartogs' lemma provided X and X' are both normal. That X is normal follows from its smoothness, while X' is normal as it is both R_1 and S_2 [Sta23]. Indeed, its singular set has codimension two, while it is S_2 given its construction as a hypersurface in a smooth ambient variety (the total space of a line bundle over $\mathbf{P}^{2n-3}$). □

Hence, $\mathcal{L}$ and its linear system $X \to \mathbf{P}^{2n-3}$ are defined over $\bar{\mathbf{Q}}$, and so we learn that the ramification locus with irreducible components the $2n$ hyperplanes $H_1, \dots, H_{2n}$ may be taken to be defined over $\bar{\mathbf{Q}}$ – that is, are defined over $\bar{\mathbf{Q}}$ after, possibly, an application of some PGL_{2n-2} projective transformation to their original definition as corresponding to the points x_i. However, this condition is precisely the same as that the original $2n$ points may be taken to be defined over $\bar{\mathbf{Q}}$, that is, possibly after some PGL_2 transform, or equivalently that all their cross-ratios are in $\bar{\mathbf{Q}}$, and so it is then easy to reconstruct C over $\bar{\mathbf{Q}}$. Indeed, the map from the $2n$ points to the $2n$ hyperplanes (or $2n$ points in a dual projective space) may be regarded as a morphism between (open loci of) $\mathrm{Sym}^{2n-3}\mathbf{P}^1/S_3 \to \mathbf{P}^{2n-3}/S_{2n-2}$ (by using the simple 3- or $(2n-1)$-transitivity of the PGL_2- and PGL_{2n-2}-actions, respectively) which is explicitly defined over $\bar{\mathbf{Q}}$. Indeed, one may write down this map in explicit coordinates; we refer the reader to [SXZ13, Claim 3.6]. □

3.5 Attractors are dense

In this section we show that the attractor points are Zariski dense in moduli space. This fact will be used when we apply the ZPC.

We define the auxiliary space

$$\mathcal{M}'_{0,2n} := \{(s, \omega) \mid s \in \mathcal{M}_{0,2n},\ \omega \in H^{1,0}(C_s)[1],\ \omega \neq 0\}$$

where we have denoted by C_s the n-fold cover of $\mathbf{P}^1$ branched at the configuration of $2n$ points given by $s \in \mathcal{M}_{0,2n}$. Recall that $H^{1,0}(C_s)[1]$ is a one-dimensional $\mathbf{C}$-vector space and hence $\mathcal{M}'_{0,2n}$ is a $\mathbf{G}_m$-bundle over moduli space. Also let $\widetilde{\mathcal{M}}'_{0,2n}$ denote the universal cover of $\mathcal{M}'_{0,2n}$; on this universal cover we have a well-defined basis of the cohomology group $H^1(C, \mathbf{Q}(\zeta_n))[-1]$ which we denote by $\gamma_1, \dots, \gamma_{2n-2}$.

We may now consider the so-called Schwarz map defined as follows:

$$\begin{aligned} \pi : \widetilde{\mathcal{M}}_{0,2n} &\to \mathbf{C}^{2n-2} \\ (s, \omega) &\mapsto \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_{2n-2}} \omega \right). \end{aligned}$$

By Lemma 3.3.1 we have that a point $(s, \omega) \in \widetilde{\mathcal{M}}_{0,2n}$ is an attractor point (more precisely, the point s gives rise to an attractor Calabi–Yau and ω witnesses this) if and only if $\pi((s, \omega))$ has coordinates in $\mathbf{Q}(\zeta) \subset \mathbf{C}$. Note that π is a holomorphic local isomorphism; see, for example, [Loo07, Corollary 1.7]. So the image $\pi(\widetilde{\mathcal{M}}_{0,2n})$ contains some open ball inside $\mathbf{C}^{2n-2}$. Since $\mathbf{Q}(\zeta_n) \subset \mathbf{C}$ is dense, we have that the attractors are topologically dense, and hence Zariski dense as well. Note, moreover, that these attractor points are isolated, as each point of $\mathbf{Q}(\zeta)$ in $\mathbf{C}$ is isolated, and therefore we have the following proposition.

PROPOSITION 3.5.1. *The isolated attractor points are Zariski dense in the moduli space* $\mathcal{M}_{\mathrm{CY}}$.

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4. Reduction to Shimura theory

4.1 Algebraic attractors split off CM abelian varieties

In this section we show that if an attractor is algebraic, then the Jacobian of the corresponding curve C must split off CM factors.

We make use of the following theorem of Shiga and Wolfart [SW95, Proposition 3], a consequence of the analytic subgroup theorem of Wüstholz.

Theorem 4.1.1 (Shiga and Wolfart). *Let A be a simple abelian variety defined over $\bar{\mathbf{Q}}$ endowed with a nonzero differential $\Omega \in \Gamma(A, \Omega^1_A)$ which is also defined over $\bar{\mathbf{Q}}$. For $\gamma \neq 0$, the period $\langle \Omega, \gamma \rangle$ is nonzero. Suppose that for any two classes $\gamma_1, \gamma_2 \in H^1(A; \mathbf{Z})$, the period ratios are algebraic:*

$$\frac{\langle \Omega, \gamma_1 \rangle}{\langle \Omega, \gamma_2 \rangle} \in \bar{\mathbf{Q}}. \tag{4}$$

Then A has complex multiplication. Moreover, if K is the number field generated by the period ratios above then the CM field of A is precisely K.

Proof. For such Ω and $\gamma \neq 0$, $\langle \Omega, \gamma \rangle$ is nonzero by an application of Wüstholz's theorem. More precisely, if $\langle \Omega, \gamma \rangle = 0$, then we apply the theorem in the form stated in [SW95, Lemma 1] to the subspace H defined to be the space of tangent vectors which pair to zero with Ω, to deduce that A has a proper algebraic subgroup, contradicting that A is simple.

The claim that A has CM is precisely the statement of [SW95, Proposition 3]. For the final claim about the CM field, the proof in [SW95] includes the claim that K, the field generated by the period ratios in (4), has degree $2\dim(A)$, and is, moreover, isomorphic to a subalgebra C of $\mathrm{End}_0(A) := \mathrm{End}(A) \otimes \mathbf{Q}$. It follows that $K \cong \mathrm{End}_0(A)$ is the CM field of A. □

We now apply Theorem 4.1.1 to the case we are interested in. Let X be a Dolgachev Calabi–Yau $(2n-3)$-fold as defined in § 3.1, and recall that X is constructed from a curve C, which is a cyclic cover of $\mathbf{P}^1$ and is equipped with an automorphism μ; the latter induces an action on $\mathrm{Jac}(C)$ which we denote by the same symbol.

Proposition 4.1.2. *Suppose X as above satisfies the attractor condition and is defined over $\bar{\mathbf{Q}}$, and let $\omega \in H^{1,0}(C) \cap H^1(C)[1]$*[10] *be the nonzero class given by Lemma 3.3.1. Then $\mathrm{Jac}\ C$ has a summand A_1 in the isogeny category such that the following conditions hold.*

- (i) *ω is supported on A_1, that is, for any isogeny of the form $A_1 \times B \to \mathrm{Jac}\ C$, for some other abelian variety B, then ω pulls back to zero on B.*
- (ii) *A_1 has complex multiplication; more precisely, each simple isogeny factor of A_1 has CM by a subfield of $\mathbf{Q}(\zeta)$.*

Proof. First, note that if X is defined over $\bar{\mathbf{Q}}$, then by Proposition 3.4.1 the same is true of C, and therefore of its Jacobian; let $C_{\bar{\mathbf{Q}}}$ be a model of C over $\bar{\mathbf{Q}}$, namely, it is a curve over $\bar{\mathbf{Q}}$ such that $C_{\bar{\mathbf{Q}}} \otimes_{\bar{\mathbf{Q}}} \mathbf{C} \simeq C$, for some embedding $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$. For brevity we write A for the abelian variety $\mathrm{Jac}\ C$.

Now let A_1 denote the abelian variety, which is a summand of A in the isogeny category, on which ω is supported; more precisely, we write $A \sim B_1 \times B_2 \times \cdots \times B_r$, and take A_1 to be the product of all the factors B_i to which ω pulls back nontrivially. Note that since ω satisfies $\mu^*\omega = \zeta\omega$, μ preserves A_1.

On the other hand, since $\omega \in H^1(C, \mathbf{Q}(\zeta))$, all the periods

$$\langle \omega, \gamma \rangle, \quad \text{for } \gamma \in H^1(A_1, \mathbf{Q}),$$

[10] Recall that $H^1(C)[1]$ is the subspace of $H^1(C, \mathbf{Q}(\zeta))$ on which μ acts via ζ.

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are contained in $\mathbf{Q}(\zeta)$. Since $\dim_{\mathbf{C}} H^{1,0}(C)[1] = 1$ by Proposition 3.1.2, and $H^{1,0}(C)[1] \simeq \Gamma(C_{\bar{\mathbf{Q}}}, \Omega^1_{C_{\bar{\mathbf{Q}}}})[1] \otimes_{\bar{\mathbf{Q}}} \mathbf{C}$, there is a nonzero complex number λ such that $\Omega := \lambda\omega$[11] is defined over $\bar{\mathbf{Q}}$ when considered as a differential on C (or equivalently, A). Applying Theorem 4.1.1 to each simple isogeny factor of A_1' of A_1, and the pullback of Ω to A_1', we deduce that A_1' has complex multiplication by a subfield of $\mathbf{Q}(\zeta)$, as required. ☐

We summarize this section with the following theorem.

Theorem 4.1.3. *Assume the attractor conjecture holds. Then the attractor points for the Dolgachev family of Calabi–Yaus considered here lie on the intersection of $\mathcal{M}_{\mathrm{CY}} \simeq \mathcal{M}_{0,2n}$ with certain Shimura subvarieties of $\mathcal{A}_{(n-1)^2}$ under $\mathcal{M}_{0,2n} \to \mathcal{M}_{(n-1)^2} \to \mathcal{A}_{(n-1)^2}$. More precisely, the attractor points correspond to points of $\mathcal{A}_{(n-1)^2}$ admitting an isogeny factor with CM, all of whose simple factors have CM by a subfield of $\mathbf{Q}(\zeta)$.*

4.2 Prym varieties

In fact, while $\mathcal{M}_{0,2n}$ does naturally map to the Shimura variety parametrizing $(n-1)^2$-dimensional (principally polarized) abelian varieties as above, for the application of the ZPC it is necessary to refine this map slightly, especially when n is not a prime. The end result will be a map to a PEL-type Shimura variety instead of simply $\mathcal{A}_{(n-1)^2}$. Therefore, in this section we study the construction of Prym varieties, which are a certain quotient of the Jacobian.

Recall that, for $n \geq 2$, $C \to \mathbf{P}^1$ denotes the cyclic n-fold covering of $\mathbf{P}^1$ branched at $2n$ points, whose affine model is given in (1); moreover, there is an action of $\mathbf{Z}/n$ on C. Now suppose we have a divisor n' of n, and let $\mathbf{Z}/n' \subset \mathbf{Z}/n$ denote the unique order- n' subgroup of $\mathbf{Z}/n$; we fix a generator $\mu \in \mathbf{Z}/n$ as before, and further denote $\mu' := (n/n')\mu$, which is a generator of this $\mathbf{Z}/n'$ subgroup.

Definition 4.2.1. For each n' dividing n, define

$$C' := C/(\mathbf{Z}/n'),$$

where $\mathbf{Z}/n'$ acts on C via the inclusion $\mathbf{Z}/n' \subset \mathbf{Z}/n$. Also let

$$\pi_{n'} : C \to C'$$

denote the natural quotient map. By further quotienting by a $\mathbf{Z}/(n/n')$, we also have a map $C' \to \mathbf{P}^1$, which is a cyclic n/n'-fold covering.

Proposition 4.2.2. *Let $J_{n'}$ denote the cokernel of the pullback map*

$$\pi_{n'}^* : \mathrm{Jac}(C') \to \mathrm{Jac}(C).$$

Then its cohomology is given by

$$H^1(J_{n'}, \mathbf{Q}(\zeta)) \cong \bigoplus_i H^1(C, \mathbf{Q}(\zeta))[in'];$$

equivalently, the above is the sum of all $H^1(C, \mathbf{Q}(\zeta))[j]$ where j satisfies

$$\zeta^{jn/n'} = 1.$$

Remark 4.2.3. As a consistency check, we see that the above sum is over $i = n', \ldots, (n/n'-1)n'$, and so there are $(n/n'-1)$ nontrivial summands, as expected, since

$$C' \to \mathbf{P}^1$$

is now an n/n'-fold cyclic cover.

[11] In fact, one can write down Ω explicitly in terms of $x \in \mathcal{M}_{0,2n}$; see, for example, [Loo07, § 1.1, Lemma 4.2] with r in that paper set to 1.

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Proof. Applying the Riemann–Hurwitz formula to the covering map $C' \to \mathbf{P}^1$, we have

$$2 - 2g(C') = \frac{n}{n'}(2) - 2n\left(\frac{n}{n'} - 1\right),$$

and hence

$$g(C') = (n-1)\left(\frac{n}{n'} - 1\right).$$

Here $g(C')$ denotes the genus of C'. On the other hand, recall that the points of the Jacobian of a curve are the degree-zero divisors modulo rational equivalence, and therefore the image of the pullback map

$$\pi_{n'}^* : \mathrm{Jac}(C') \to \mathrm{Jac}(C)$$

is invariant under the $\mathbf{Z}/n'$-action. On the other hand, $\pi_{n'}^*$ is injective, since if D is a degree-zero divisor on C' such that

$$\pi_{n'}^*(D) = (f)$$

for some rational function f on C, then f is invariant under the Galois group $\mathbf{Z}/n'$ of the covering map $C \to C'$, and therefore f descends to C. Therefore, the map on homology induced by $\pi_{n'}^*$ is also injective, and lands inside the invariant subspace $H_1(\mathrm{Jac}(C), \mathbf{Q}(\zeta))^{\mathbf{Z}/n'}$ (using $\mathbf{Q}(\zeta)$-coefficients). By the genus computation above, this gives an isomorphism

$$H_1(\mathrm{Jac}(C'), \mathbf{Q}(\zeta)) \cong H_1(\mathrm{Jac}(C), \mathbf{Q}(\zeta))^{\mathbf{Z}/n'}.$$

Dualizing, we have $H^1(\mathrm{Jac}(C'), \mathbf{Q}(\zeta))$ being the coinvariants of the $\mathbf{Z}/n'$-action on $H^1(\mathrm{Jac}(C), \mathbf{Q}(\zeta))$, which gives the desired result; indeed, since the action of a generator $\mu' := (n/n')\mu \in \mathbf{Z}/n'$ on $H^1(\mathrm{Jac}(C), \mathbf{Q}(\zeta))[j]$ is given by $\zeta^{nj/n'}$, the coinvariants are given by

$$\bigoplus_{i=1}^{n-1} H^1(\mathrm{Jac}(C), \mathbf{Q}(\zeta))[i]/(\zeta^{ni/n'} - 1)H^1(\mathrm{Jac}(C), \mathbf{Q}(\zeta))[i],$$

and we see that the nontrivial summands are indexed by i such that $\zeta^{in/n'} = 1$, as claimed. □

We immediately deduce the following simple corollary.

Corollary 4.2.4. *We denote by $\pi_{n'}^*$ the map on cohomologies induced by $\pi_{n'}^*$. Then the quotient of $H^1(C, \mathbf{Q}(\zeta))$ by the images of $\pi_{n'}^*$ for all proper divisors n' (i.e. $n' \neq 1, n$) is precisely the sum*

$$\bigoplus_{i \in (\mathbf{Z}/n)^\times} H^1(C, \mathbf{Q}(\zeta))[i].$$

The above direct sum decomposition can be refined integrally, or equivalently as a statement about abelian varieties.

Definition 4.2.5. We now define the abelian variety

$$\mathrm{Prym} := \mathrm{Jac}(C)/\sum_{n'} \mathrm{Im}(\pi_{n'}^*),$$

and refer to it as the Prym variety. Here the sum is over proper divisors n' as above.

Corollary 4.2.6. *The abelian variety* Prym *has endomorphisms by $\mathbf{Q}(\zeta)$, and its $\mathbf{Q}(\zeta)$-Hodge structure is given by*

$$\bigoplus_{i \in (\mathbf{Z}/n)^\times} H^1(C, \mathbf{Q}(\zeta))[i].$$

As such, it has dimension $(n-1)\phi(n)$.

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4.3 PEL Shimura varieties

Now that we have the necessary statements on the Prym construction from § 4.2, we can define the refined period map, whose target is a certain PEL-type Shimura variety. Denote by V the $\mathbf{Q}$ subspace of $H^1(C, \mathbf{Q})$ such that

$$V \otimes \mathbf{Q}(\zeta) = \bigoplus_{r \in (\mathbf{Z}/n)^\times} V[r],$$

where $V[r] \subset H^1(C, \mathbf{Q}(\zeta))$ denotes the ζ^r eigenspace for the action of $\mu \in \mathbf{Z}/n$; as before, V has an action of $\mathbf{Q}(\zeta)$. By Corollary 4.2.6, we may identify V with $H^1(\mathrm{Prym}, \mathbf{Q})$.

Definition 4.3.1. The abelian variety Prym from § 4.2 furnishes us with an integral lattice

$$V_{\mathbf{Z}} := H^1(\mathrm{Prym}, \mathbf{Z}) \subset H^1(\mathrm{Prym}, \mathbf{Q}) = V,$$

equipped with a symplectic form Ψ. Let

$$\mathbf{S} := \mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_m$$

denote the Deligne torus, and let $\mathfrak{H}$ denote the space of homomorphisms

$$h : \mathbf{S} \to \mathrm{GSp}(V_{\mathbf{R}}, \Psi),$$

which in turn define Hodge structures of type $(-1, 0) + (0, -1)$ on $V_{\mathbf{Z}}$. The space $\mathfrak{H}$ is isomorphic to the Siegel upper half space of dimension

$$\frac{(n-1)\phi(n)((n-1)\phi(n)+1)}{2},$$

since it parametrizes abelian varieties of dimension $(n-1)\phi(n)$.

The Shimura datum $(\mathrm{GSp}(V, \Psi), \mathfrak{H})$ certainly defines a Shimura variety to which $\mathcal{M}_{0,2n}$ maps; we can describe its $\mathbf{C}$-points as follows. The integral structure on V defines a maximal compact subgroup $K \subset \mathrm{GSp}(V, \Psi)(\mathbf{A})$, and then we have

$$\mathrm{Sh}(\mathrm{GSp}(V, \Psi), \mathfrak{H})(\mathbf{C}) = \mathrm{GSp}(V, \Psi)(\mathbf{Q}) \backslash \mathfrak{H} \times \mathrm{GSp}(V, \Psi)(\mathbf{A}_f)/K;$$

here $\mathbf{A}$ (respectively, $\mathbf{A}_f$) denotes the ring of (respectively, finite) adeles. However, as we shall presently see, $\mathcal{M}_{0,2n}$ lands inside a smaller Shimura subvariety. In the following we essentially follow the treatment of [Moo10, Section 3.3], the only difference being that we work with the Prym whereas Moonen considers the entire Jacobian.

Definition 4.3.2.

(i) For a $\mathbf{Q}$-algebraic subgroup $H \subset \mathrm{GSp}(V, \Psi)$, define

$$\mathfrak{H}_H := \{h : \mathbf{S} \to \mathrm{GSp}(V_{\mathbf{R}}, \Psi) | h \text{ factors through } H_{\mathbf{R}}\}.$$

A special subvariety associated to the group H is the image of $\mathfrak{H}_H^+ \times \eta K$ under the uniformization map

$$\mathfrak{H} \times G(\mathbf{A}_f)/K \to \mathrm{GSp}(V)(\mathbf{Q}) \backslash \mathfrak{H} \times G(\mathbf{A}_f)/K,$$

where $\mathfrak{H}_H^+$ is a connected component of $\mathfrak{H}_H$.

(ii) Recall that there is a $\mathbf{Q}(\zeta)$-action on V. Define the $\mathbf{Q}$-algebraic group

$$G := \mathrm{GL}_{\mathbf{Q}(\zeta)}(V) \cap \mathrm{GSp}(V, \Psi);$$

here $\mathrm{GL}_{\mathbf{Q}(\zeta)}(V)$ denotes the elements of $\mathrm{GL}(V)$ commuting with the action of $\mathbf{Q}(\zeta)$ on V.

(iii) By construction, $P(\mathcal{M}_{0,2n})$ lies in such a special subvariety, which we denote by Sh.

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We introduce notation to describe the real points of G following Moonen. For each $r \in (\mathbf{Z}/n)^\times$, $V[r] \otimes_{\mathbf{Q}(\zeta)} \mathbf{C} \oplus V[-r] \otimes_{\mathbf{Q}(\zeta)} \mathbf{C}$ is canonically defined over $\mathbf{R}$, and we write $V_{\mathbf{R},(\pm r)}$ for its descent to $\mathbf{R}$: namely, $V_{\mathbf{R},(\pm r)}$ is the $\mathbf{R}$-vector space such that $V_{\mathbf{R},(\pm r)} \otimes \mathbf{C} \simeq V[r] \otimes_{\mathbf{Q}(\zeta)} \mathbf{C} \oplus V[-r] \otimes_{\mathbf{Q}(\zeta)} \mathbf{C}$. The polarization defines a skew-hermitian form $\beta_{(\pm r)}$ on $V_{\mathbf{R},(\pm r)}$. We refer the reader to [Moo10, § 4.5] and the references therein for more details.

Proposition 4.3.3.

(i) *The real points of the group G are given by*

$$G(\mathbf{R}) \cong \prod_{r \in (\mathbf{Z}/n)^\times/\pm} U(V_{\mathbf{R},(\pm r)}, \beta_{\pm r}).$$

For example, the $r = 1$ factor of the above product is isomorphic to $U(1, 2n-3)$, since, by Corollary 3.2.4, $(1, 2n-3)$ are the dimensions of the pieces of the Hodge decomposition of $V[1]$, and therefore give the signature of the hermitian form.

(ii) *The dimension of* Sh *is*

$$\frac{1}{2} \sum_{r \in (\mathbf{Z}/n)^\times} (2r-1)(2n-1-2r). \tag{5}$$

Here the sum is over representatives between 1 and n of the elements of $(\mathbf{Z}/n)^\times$.

Proof. The first part follows from [Moo10, Remark 4.6]. For the second part, it suffices to find the signature of the pairing on each of the subspaces $V[r]$, since the hermitian symmetric domain for the unitary group $U(a, b)$ has dimension ab. The signatures, or equivalently the Hodge numbers, are given by Proposition 3.1.2, and (5) follows immediately. □

The Prym construction therefore gives us a map

$$P : \mathcal{M}_{0,2n} \to \mathrm{Sh}.$$

We have the following result, which is analogous to the classical result that the Torelli map is an embedding, although we only require an infinitesimal version of this.

Lemma 4.3.4. *The derivative of P is injective.*

Proof. We will show equivalently that, for each $x \in \mathcal{M}_{0,2n}$, the codifferential map

$$P^* : T^*_{P(x)}\mathrm{Sh} \to T^*_x \mathcal{M}_{0,2n}$$

is surjective. Here for a variety X and a point $x \in X$ we denote by $T^*_x X$ the cotangent space to X at x. First we identify the source and target of P^* in terms of the geometric structures at hand.

Claim 4.3.5. *We have the following identifications of the cotangent spaces:*

$$T^*_{P(x)}\mathrm{Sh} \cong \bigoplus_{\substack{r \in (\mathbf{Z}/n)^\times \\ r < n/2}} H^0(C, \Omega^1_C)[r] \otimes H^0(C, \Omega^1_C)[n-r], \tag{6}$$

$$T^*_x \mathcal{M}_{0,2n} \cong H^0(C, (\Omega^1_C)^{\otimes 2})_{\mathrm{inv}}. \tag{7}$$

Here $H^0(C, \Omega^1_C)[r]$ denotes the ζ^r eigenspace of $H^0(C, \Omega^1_C)$, and the subscript inv *in (7) denotes the invariant part of the $\mathbf{Z}/n$-action.*

Furthermore, under these identifications, the restriction of P^* on each of the factors is given in (6) by the cup product map

$$\cup : H^0(C, \Omega^1_C)[r] \otimes H^0(C, \Omega^1_C)[n-r] \to H^0(C, (\Omega^1_C)^{\otimes 2})_{\mathrm{inv}}$$

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Let us first see how to conclude the proof of this lemma, assuming this claim. Note that the dimension of $H^0(C,(\Omega_C^1)^{\otimes 2})_{\mathrm{inv}}$ is $2n-3$, since it has to be the dimension of $\mathcal{M}_{0,2n}$. We will now show that the restriction (which we continue to denote by P^*)

$$P^* : H^0(C,\Omega_C^1)[1] \otimes H^0(C,\Omega_C^1)[n-1] \to H^0(C,(\Omega_C^1)^{\otimes 2})_{\mathrm{inv}} \tag{8}$$

is in fact an isomorphism; certainly the dimensions of the source and target agree, and so it suffices to show this map is injective. But this is clear since the space $H^0(C,\Omega_C^1)[1]$ is one-dimensional and spanned by ω, say, and so anything in the kernel of the map (8) takes the form $\omega \otimes \eta$ for some $\eta \in H^0(C,\Omega_C^1)[n-1]$. On the other hand, ω and η are nonzero 1-forms on C, and the quadratic differential obtained by multiplying them together is certainly nonzero. This shows that the map (8) is injective, and therefore it is an isomorphism, as required. Therefore, it suffices to prove Claim 4.3.5.

Proof of Claim. By construction, we have an embedding

$$\mathrm{Sh} \to \mathrm{Sh}(\mathrm{GSp}(V,\Psi),\mathfrak{H}),$$

where the right-hand side denotes the Shimura variety attached to the Shimura datum $(\mathrm{GSp}(V,\Psi),\mathfrak{H})$. The latter is the moduli space of abelian varieties of dimension $(n-1)\phi(n)$ equipped with a polarization of the fixed type specified by the polarization on the Prym variety. Therefore, the tangent space to $\mathrm{Sh}(\mathrm{GSp}(V,\Psi),\mathfrak{H})$ at $P(x)$ is given by

$$\mathrm{Sym}^2(t_{\mathrm{Prym}}) \subset t_{\mathrm{Prym}} \otimes t_{\mathrm{Prym}^\vee}, \tag{9}$$

where for an abelian variety A we denote by t_A the tangent space at the origin, and by $A^\vee$ its dual abelian variety. On the left-hand side of the above we have also made the identification

$$t_{\mathrm{Prym}} \cong t_{\mathrm{Prym}^\vee}$$

using the polarization on Prym. Note that in (9) the right-hand side is the deformation space of Prym with no reference to polarizations. By Corollary 4.2.6, we may make the identification

$$t_{\mathrm{Prym}} \cong \bigoplus_{i\in(\mathbf{Z}/n)^\times} H^1(C,\mathcal{O}_C)[i];$$

for convenience we denote the right-hand side of this identification by $H^1(C,\mathcal{O}_C)_{\mathrm{prim}}$. Similarly, we define $H^0(C,\Omega_C^1)_{\mathrm{prim}}$ to be the sum of the eigenspaces of $H^0(C,\Omega_C^1)$ with eigenvalues primitive nth roots of unity.

On the other hand, as mentioned above, the right-hand side of (9) is the deformation space of the abelian variety Prym (without reference to a polarization), and hence there is a Kodaira–Spencer map

$$\mathrm{KS} : t_{\mathrm{Prym}} \otimes t_{\mathrm{Prym}^\vee} \to \mathrm{Hom}(H^0(C,\Omega_C^1)_{\mathrm{prim}}, H^1(C,\mathcal{O}_C)_{\mathrm{prim}}), \tag{10}$$

which is the natural isomorphism once we make the identifications

$$t_{\mathrm{Prym}} \otimes t_{\mathrm{Prym}^\vee} \cong H^1(C,\mathcal{O}_C)_{\mathrm{prim}}^{\otimes 2}$$

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and

$$H^1(C,\mathcal{O})_{\mathrm{prim}} \cong H^0(C,\Omega^1_C)^\vee_{\mathrm{prim}},$$

the latter of which is induced by Serre duality.

By definition, Sh is contained in the locus of $\mathrm{Sh}(\mathrm{GSp}(V,\Psi),\mathfrak{H})$ where the Hodge structure H^1 admits a $\mathbf{Q}(\zeta)$-action and a splitting

$$H^1 \otimes_{\mathbf{Q}} \mathbf{Q}(\zeta) \cong \bigoplus_{r\in(\mathbf{Z}/n)^\times} H^1[r]$$

with prescribed Hodge numbers. Therefore, the Kodaira–Spencer map (10) restricted to Sh must preserve the different eigenspaces. In other words, we have

$$\mathrm{KS}|_{\mathrm{Sh}} : T_{P(x)}\mathrm{Sh} \to \bigoplus_{r\in(\mathbf{Z}/n)^\times} \mathrm{Hom}(H^0(C,\Omega^1_C)[r], H^1(C,\mathcal{O}_C)[r]); \tag{11}$$

now since KS itself is an isomorphism, $\mathrm{KS}|_{\mathrm{Sh}}$ is injective at least. On the other hand, deformations in Sh are also required to preserve the polarization: concretely this means that each element in the image of $\mathrm{KS}|_{\mathrm{Sh}}$ is invariant under the involution which identifies $\mathrm{Hom}(H^0(C,\Omega^1_C)[r], H^1(C,\mathcal{O}_C)[r])$ and $\mathrm{Hom}(H^0(C,\Omega^1_C)[n-r], H^1(C,\mathcal{O}_C)[n-r])$ for each $r \in (\mathbf{Z}/n)^\times$.[12] Therefore, the projection

$$\mathrm{KS}|_{\mathrm{Sh}} : T_{P(x)}\mathrm{Sh} \to \bigoplus_{\substack{r\in(\mathbf{Z}/n)^\times \\ r<n/2}} \mathrm{Hom}(H^0(C,\Omega^1_C)[r], H^1(C,\mathcal{O}_C)[r]). \tag{12}$$

is also injective. Since the dimensions of the two sides of (12) now agree, and the map is injective, it must in fact be an isomorphism.

Again using Serre duality, for each $r = 1,\ldots,n$, we have

$$H^0(C,\Omega^1_C)[r] \cong H^1(C,\mathcal{O}_C)[n-r]^\vee,$$

and using the remark above we may rewrite (12) as

$$T_{P(x)}\mathrm{Sh} \cong \bigoplus_{\substack{r\in(\mathbf{Z}/n)^\times \\ r<n/2}} (H^0(C,\Omega^1_C)[r])^\vee \otimes H^1(C,\mathcal{O}_C)[r] \tag{13}$$

$$\cong \bigoplus_{\substack{r\in(\mathbf{Z}/n)^\times \\ r<n/2}} H^1(C,\mathcal{O}_C)[n-r] \otimes H^1(C,\mathcal{O}_C)[r]. \tag{14}$$

Dualizing, we therefore deduce

$$T^*_{P(x)}\mathrm{Sh} \cong \bigoplus_{\substack{r\in(\mathbf{Z}/n)^\times \\ r<n/2}} H^0(C,\Omega^1_C)[r] \otimes H^0(C,\Omega^1_C)[n-r],$$

which proves (6), as claimed.

The identification (7) is well known; see, for example, [LO11, Proposition 4.1]. Furthermore, [LO11, Proposition 4.1] also shows that the codifferential of the map

$$\mathcal{M}_{0,2n} \to \mathrm{Sh}(\mathrm{GSp}(V,\Psi),\mathfrak{H}),$$

[12] The identification is to first take the dual of an element of $\mathrm{Hom}(H^0(C,\Omega^1_C)[r], H^1(C,\mathcal{O}_C)[r])$ and then apply Serre duality.

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namely

$$P^* : \mathrm{Sym}^2(H^0(C, \Omega^1_C)_{\mathrm{prim}}) \to H^0(C, (\Omega^1_C)^{\otimes 2})_{\mathrm{inv}},$$

is the cup product map followed by projection onto the invariant factor, and therefore the same is true for the codifferential of the map

$$\mathcal{M}_{0,2n} \to \mathrm{Sh},$$

as claimed. This concludes the proofs of all the statements in Claim 4.3.5. $\square$

This gives immediately the following corollary.

Corollary 4.3.6. *The dimension of the image $P(\mathcal{M}_{0,2n})$ inside* Sh *is $2n-3$.*

4.4 Special subvarieties

Now that we have defined the relevant PEL-type Shimura variety Sh to which $\mathcal{M}_{0,2n}$ maps naturally, we can rephrase Theorem 4.1.3 in terms of special subvarieties of Sh. In this section we assume the validity of the attractor conjecture, so that the conclusion of Theorem 4.1.3 holds.

Let X be an attractor Dolgachev Calabi–Yau variety corresponding to $x \in \mathcal{M}_{0,2n}$, with corresponding curve C_x[13] which comes with an automorphism μ, as in § 3.1. Recall that from Theorem 4.1.3, the attractor conjecture implies that Jac C_x has an isogeny factor $\tilde{A}_x$,[14] which has CM and is invariant under the action μ, and such that the distinguished differential ω is supported on $\tilde{A}_x$. Let A_x denote the image of $\tilde{A}_x$ under the map Jac $C_x \to \mathrm{Prym}_x$, where the latter map is as in § 4.3. Since there are only finitely many choices of $\tilde{A}_x$ up to isogeny,[15] the same is true for A_x. Note that the quotient map Jac $C_x \to \mathrm{Prym}_x$ is compatible with the action of μ, and hence A_x is also acted on by μ.

Let $V(A_x)$ and V_x denote the first cohomology groups $H^1(A_x, \mathbf{Q})$ and $H^1(\mathrm{Prym}_x, \mathbf{Q})$, respectively. We have a splitting of $\mathbf{Q}$-Hodge structures of the form $V_x = V(A_x) \oplus V_x'$, corresponding to the splitting up to isogeny of Prym_x; here V_x' denotes the $\mathbf{Q}$-Hodge structure of the quotient Prym_x/A_x. Note that both $V(A_x)$ and V_x' have actions by $\mathbf{Q}(\zeta)$ induced by the action of μ, and the decomposition is compatible with these actions. Recall from Definition 4.3.2 that G acts naturally on V_x.

Definition 4.4.1. Let G_x denote the $\mathbf{Q}$-algebraic subgroup of G preserving the above decomposition.

Recall that in Definition 4.3.2 we defined special subvarieties associated to subgroups of G through which the homomorphism on the Deligne torus may factor. From Theorem 4.1.3 we immediately deduce the following slight refinement.

Proposition 4.4.2. *For each attractor point x in $\mathcal{M}_{0,2n}$, x lies on a special subvariety* Sh_x, *associated to the subgroup G_x.*

Remark 4.4.3. Every point of Sh_x corresponds to a point of Sh such that the associated abelian variety admits an isogeny factor isomorphic to A_x.

Definition 4.4.4. For $r \in (\mathbf{Z}/n)^\times$, let $d_r := \dim H^{1,0}(A_x)[r], d_{-r} := \dim H^{0,1}(A_x)[r]$.[16]

Proposition 4.4.5. *We have $d_1 = 1$, and $d_r + d_{-r}$ is independent of r.*

[13] In this section we include the subscript x on all relevant objects for clarity.
[14] Note the slight change of notation from Proposition 4.1.2.
[15] Since each simple factor has CM by a subfield of $\mathbf{Q}(\zeta)$.
[16] We have omitted the reference to x in the notation d_r by a slight abuse of notation.

Proof. That $d_1 \geq 1$ follows from Proposition 4.1.2, since the distinguished differential ω is supported on A_x; therefore, $d_1 = 1$ since $\dim H^{1,0}(\mathrm{Prym}_x)[1] = 1$.[17] The second statement follows from the fact that the dimension of $H^1(A_x)[r]$ is independent of r, as these spaces are Galois conjugate to each other. □

Proposition 4.4.6. *We have*

$$\mathrm{codim}_{\mathrm{Sh}}\mathrm{Sh}_x \geq 2n-3, \tag{15}$$

with equality if and only if $\phi(n) = 2$.

Remark 4.4.7. In particular, whenever $\phi(n) \neq 2$, any intersection between the subvarieties $P(\mathcal{M}_{0,2n})$ and Sh_x of Sh is unlikely in the sense that $\dim P(\mathcal{M}_{0,2n}) + \mathrm{codim}_{\mathrm{Sh}}\mathrm{Sh}_x < \mathrm{Sh}$. In the next section we will recall the ZPC, which predicts that an abundance of such unlikely intersections, as we have here, implies restrictions on $P(\mathcal{M}_{0,2n})$ itself.

Proof. The subvariety Sh_x is itself a unitary Shimura variety, whose dimension is governed by the signatures of V'_x; see below for an example demonstrating this. We have that

$$\dim \mathrm{Sh}_x = \sum_{r\in(\mathbf{Z}/n)^\times/\pm} (2r-1-d_r)(2n-1-2r-d_{-r}).$$

Comparing to[18]

$$\dim \mathrm{Sh} = \sum_{r\in(\mathbf{Z}/n)^\times/\pm} (2r-1)(2n-1-2r),$$

we see that the difference between the $r=1$ terms of the two above expressions is $2n-3$, which gives us the desired inequality. Equality holds if and only if all the other terms in the two above expressions agree, and since from Proposition 4.4.6 we have $d_r + d_{-r} \geq 1$ for all r, this occurs if and only if there are no other terms, that is, $\phi(n) = 2$. □

Example 4.4.8. We give an example to illustrate the numerology that is at play here, for $n=5$, which is the smallest value for which the Dolgachev Calabi–Yau varieties give counterexamples to the attractor conjecture. In this case, since 5 is a prime number, the Prym variety is simply the Jacobian of C. For example, if we fix the numerical data $d_1 = d_2 = 1$, $d_3 = d_4 = 0$, then we may write schematically the splitting of the Hodge structure as

$$\begin{pmatrix} 1 & 7 \\ 3 & 5 \\ 5 & 3 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 7 \\ 2 & 5 \\ 5 & 2 \\ 7 & 0 \end{pmatrix}. \tag{16}$$

In the equation above we follow Notation 3.2.5, and on the right-hand side we have written the dimension matrices of the summands of this splitting: the last term is the signatures of the summand V'_x. In this case, the special subvariety Sh_x has dimension 10, while the ambient Shimura variety Sh has dimension 22.

4.5 A brief digression: the arithmetic cases of $n = 3, 4, 6$ and connections to tilings of the sphere

Here we observe that the attractor conjecture works remarkably well in the cases when the Calabi–Yau moduli space does happen to be a Shimura variety, and point out a connection to the tilings of the sphere by polygons due to Engel and Smillie [ES18].

[17] Recall that $H^{1,0}(\mathrm{Prym}_x)[1]$ is the subspace of $H^{1,0}(\mathrm{Prym}_x)$ on which μ acts by ζ.
[18] In the above formulas, the representatives for $r \in (\mathbf{Z}/n)^\times$ should be taken in $\{1, \ldots, n-1\}$.

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PROPOSITION 4.5.1. *For $n = 3, 4, 6$, there is a bijection between attractor points and tilings of the sphere by triangles, squares and hexagons, respectively.*

Proof. After quotienting by the appropriate arithmetic group, the Schwarz map π considered in § 3.5 coincides precisely with the map denoted by D in [ES18, Proof of Proposition 2.5, p.7]. Furthermore, the integral points in the image of D correspond to tilings of the sphere, as required. □

In fact, we mention an intriguing question for the interested reader: Engel and Smillie in the above paper study a very precise generating function of the attractor points in the arithmetic $n = 3, 4, 6$ cases in terms of a mock modular form; it is reasonable to ask if there may be any analogous (but presumably more complicated) behavior in the nonarithmetic cases.

5. Unlikely intersection

We now recall the ZPC.

CONJECTURE 5.0.1 [Pin05, Conjecture 1.3]. Given a subvariety $Y \subset \mathcal{X}$ of a Shimura variety and a countable collection of special subvarieties $\{X_\alpha\}$ of codimension greater than $\dim Y$, if

$$\bigcup_\alpha Y \cap \mathcal{X}_\alpha \subset Y$$

is Zariski dense, then Y is contained within some proper special subvariety $\mathcal{X}' \subset \mathcal{X}$.

Here, the definition of a special subvariety is exactly as in part (i) of Definition 4.3.2. We will now argue that $\mathcal{M}_{0,2n}$ is not contained in any special Shimura subvariety of Sh. Once again, these arguments hold for all $n \geq 2$; the only place where the $\phi(n) > 2$ condition enters is to force the dimensional inequality.

As in § 3.1, we have a curve C_x attached to each $x \in \mathcal{M}_{0,2n}$, and we defined the Prym Prym_x in § 4.2. Recall that we denote by V the $\mathbf{Q}$-subspace of $H^1(\mathrm{Prym}_x, \mathbf{Q})$, and we have a decomposition

$$V \otimes \mathbf{Q}(\zeta) = \bigoplus_{r \in (\mathbf{Z}/n)^\times} V[r].$$

Recall also from Definition 4.3.2 that the Shimura variety Sh is defined by the group $G = \mathrm{GL}_{\mathbf{Q}(\zeta)}(V) \cap \mathrm{GSp}(V, \Psi)$, and that $G(\mathbf{R}) = \prod_{r \in (\mathbf{Z}/n\mathbf{Z})^\times/\pm} U(V_{\mathbf{R},(\pm r)}, \beta_{\pm r})$ by Proposition 4.3.3.

PROPOSITION 5.0.2. *Fix a basepoint $x \in \mathcal{M}_{0,2n}$. Let $\mathcal{G} \subset G$ denote the $\mathbf{Q}$-Zariski closure of the fundamental group of $\mathcal{M}_{0,2n}$ acting on $V = H^1(\mathrm{Prym}_x)$. Then $\mathcal{G}(\mathbf{R})$ contains $\prod_{r \in (\mathbf{Z}/n)^\times/\pm} SU(V_{\mathbf{R},(\pm r)}, \beta_{\pm r})$, where the latter is embedded in the natural way into $G(\mathbf{R}) = \prod_{r \in (\mathbf{Z}/n\mathbf{Z})^\times/\pm} U(V_{\mathbf{R},(\pm r)}, \beta_{\pm r})$.*

Before turning to the argument for this proposition, we note that this requirement is exactly the hypothesis necessary to apply the ZPC to conclude Theorem 1.1.3 in the $\phi(n) > 2$ cases of unlikely intersection. Indeed, the special Shimura subvarieties of Sh correspond to subgroups of G, and being contained within some Shimura subvariety would imply a corresponding restriction on the Zariski closure of the monodromy group, namely $\mathcal{G}$; that is, we would have inclusions $\mathcal{G} \subset H \subset G$ with H corresponding to some Shimura subvariety. However, the above proposition implies that $\mathcal{G}(\mathbf{R})$ and $G(\mathbf{R})$ have the same adjoint groups, and therefore H cannot correspond to a proper Shimura subvariety of Sh.

To conclude the proof of Theorem 1.1.3, it hence remains to establish the above proposition.

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Proof. This is essentially due to Deligne and Mostow [DM86], but we will use the version stated by Looijenga in [Loo07], as we now describe.[19]

First, as explained in [Loo07, Proof of Theorem 4.3], we have a decomposition

$$\mathcal{G} \otimes \mathbf{Q}(\zeta) = \prod_{r \in (\mathbf{Z}/n)^\times} \mathcal{G}_r$$

with $\mathcal{G}_r$ defined over $\mathbf{Q}(\zeta)$, and such that $\mathcal{G}_r \subset \mathrm{GL}(V[r])$ under the natural inclusion $\mathcal{G} \otimes \mathbf{Q}(\zeta) \subset \mathrm{GL}(V \otimes \mathbf{Q}(\zeta))$.

Furthermore, again according to [Loo07, Lemma 4.4],[20] for each $r \in (\mathbf{Z}/n\mathbf{Z})^\times$, $\mathcal{G}_r \times \mathcal{G}_{-r}$ is canonically defined over $\mathbf{R}$, and the real points $(\mathcal{G}_r \times \mathcal{G}_{-r})(\mathbf{R})$ contain $SU(V_{\mathbf{R},(\pm r)}, \beta_{\pm r})$. This implies the proposition. □

We now put all the ingredients together to conclude the proof of our main result.

Proof of Theorem 1.1.3. Suppose that $n \neq 3, 4, 6$. We assume for the sake of contradiction that attractor points are defined over $\bar{\mathbf{Q}}$. By § 3.5 the isolated attractor points are Zariski dense in moduli space. Now recall that we have the Prym map

$$P : \mathcal{M}_{0,2n} \to \mathrm{Sh},$$

where Sh denotes the Shimura variety from Definition 4.3.2, and consider its image $P(\mathcal{M}_{0,2n})$, which has dimension $2n-3$, the same dimension as $\mathcal{M}_{0,2n}$, by Corollary 4.3.6.

On the other hand, by Proposition 4.4.2, each attractor point lies in a sub-Shimura variety, whose codimension inside Sh is strictly greater than $2n-3$ by Proposition 4.4.6 Therefore, by the ZPC, the variety $P(\mathcal{M}_{0,2n})$ must be contained in some proper special subvariety of Sh, which is impossible by Proposition 5.0.2, as explained before the proof of this proposition. Therefore, the attractor points cannot be defined over $\bar{\mathbf{Q}}$, as required. □

6. Conjectural one-parameter Calabi–Yau families

6.1 A conjectural construction

We first recall an alternative description of the Dolgachev Calabi–Yau varieties. Recall that we have the curve C whose affine model is

$$C^\circ = \left\{ y^n = \prod_{i=1}^{2n} (x - x_i) \right\}.$$

Then (see [SXZ13]) the cyclic cover X' of $\mathbf{P}^{2n-3}$ in Definition 3.1.4 is given by

$$C^{2n-3}/S_{2n-3} \ltimes N',$$

where N' is the kernel of the sum map

$$(\mu_n)^{2n-3} \to \mu_n.$$

In the above, each μ_n factor acts via

$$x \mapsto x, \quad y \mapsto \zeta y,$$

for ζ a primitive nth root of unity.

[19] In the notation of [Loo07], one should take μ there to be our $1/n$, and n there should be our $2n-2$, to match with the cases considered here.

[20] As well as the Galois conjugate statements; see, for example, [Loo07, Proof of Theorem 4.3].

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We now consider a special one-parameter family $\mathcal{N} \subset \mathcal{M}_{CY}$ given by

$$x_1 = a, x_2 = \zeta a, \ldots, x_n = \zeta^{n-1} a, \ y_1 = 1, \ldots, y_n = \zeta^{n-1},$$

for some parameter a. For each such curve C we can further quotient by the μ_n acting by

$$x \mapsto \zeta x, \quad y \mapsto y;$$

let us denote this quotient by Z, and the total space of this family over $\mathcal{N}$ by $\mathcal{Z}$.

Note that this new action commutes with the action of the group $S_{2n-3} \ltimes N'$ above.

Lemma 6.1.1. *Under the action of $S_{2n-3} \ltimes N' \times \mu$, the invariant part H_{inv} of the Hodge structure on $H^{2n-3}(C^{2n-3}, \mathbf{Q})$ has Hodge numbers all equal to one.*

Proof. This is a straightforward calculation using [Loo07, Lemma 4.2]. □

Conjecture 6.1.2. The family $\mathcal{Z}$ over $\mathcal{N}$ admits a simultaneous crepant resolution, such that the middle Hodge structure is isomorphic to H_{inv}. In particular, we have a one-modulus family of Calabi–Yau varieties parametrized by $\mathcal{N}$.

6.2 Zilber–Pink for one-parameter families

Assuming Conjecture 6.1.2, the same argument as in the case of the Dolgachev Calabi–Yau varieties yields counterexamples to the attractor conjecture, again conjectural on the ZPC. However, in the case of one-parameter families inside Shimura varieties, the ZPC is known to hold conditional on certain purely arithmetic statements as a result of work of Orr [Orr20, Theorem 3.4].

Corollary 6.2.1. *Assuming certain arithmetic conjectures, namely Conjectures 3.2 and 3.3 of [Orr20] in our setting, the families $\mathcal{Z}$ give counterexamples to the attractor conjecture.*

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Conflicts of Interest

None.

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