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ON FIXED POINTS OF GENERALIZED SET-VALUED CONTRACTIONS

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Abstract

Using a variational method introduced in [D. Azé and J.-N. Corvellec, 'A variational method in fixed point results with inwardness conditions', *Proc. Amer. Math. Soc.* **134**(12) (2006), 3577–3583], deriving directly from the Ekeland principle, we give a general result on the existence of a fixed point for a very general class of multifunctions, generalizing the recent results of [Y. Feng and S. Liu, 'Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings', *J. Math. Anal. Appl.* **317**(1) (2006), 103–112; D. Klim and D. Wardowski, 'Fixed point theorems for set-valued contractions in complete metric spaces', *J. Math. Anal. Appl.* **334**(1) (2007), 132–139]. Moreover, we give a sharp estimate for the distance to the fixed-points set.

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1. Introduction

Recently some interesting extensions of Nadler's theorem (see [10]) were given in [6, 8]; this was the first generalization to multifunctions of the classical Banach–Picard theorem. In the papers quoted, the authors observed that the assumption that the multifunction is a contraction with respect to the Hausdorff metric could be slightly weakened by requiring only local information on the approximate projection of a point to its image. This observation was anticipated in [1, Example 1.6]. In this work we give a general result on the existence of a fixed point for a large class of multifunctions satisfying a very weak contraction assumption. Moreover, a sharp estimate for the distance to the fixed-points set is given. As a by-product, we derive a very light version of the Banach–Picard theorem.

2. A basic lemma

DEFINITION 2.1. Let (X, d) be a metric space, and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. A point $x \in X$ is said to be a *d-point* of f if

$$f(x) < f(z) + d(z, x) \quad \forall z \in X \setminus \{x\}.$$

Here is the well-known Ekeland variational principle in its simpler form (see [4, 5, 11, 15]).

THEOREM 2.2. *The following are equivalent:*

(a) *the metric space* (X, d) *is complete;*
(b) *every proper (not identically equal to* $+\infty$*), lower semicontinuous, and lower-bounded function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *has a* d*-point.*

Given $x \in X$, let us set $M(x) = \{z \in X \mid f(z) + d(x, z) \leq f(x)\}$. It is an immediate consequence of the triangle inequality that a d-point of the restriction of f to the subset $M(x)$ is a d-point of f on the whole of X. Thus we have the following result.

COROLLARY 2.3. *Let* (X, d) *be a complete metric space. Assume that the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is proper lower semicontinuous and bounded from below. Then, for all* $x \in X$*, there exists a* d*-point of* f *belonging to* $M(x)$.

For $\lambda \in \mathbb{R}$, we denote by $[f \leq \lambda]$ the sublevel set $f^{-1}((-\infty, \lambda])$ and we define $[f < \lambda]$, $[f > \lambda]$, and similar notation analogously. The following simple lemma along the lines of [2, 7, 9, 14] is our basic tool in what follows.

LEMMA 2.4. *Let* $f : X \to [0, +\infty]$ *be a proper lower semicontinuous function defined on a complete metric space* (X, d)*, and let* $0 < \mu \leq +\infty$ *be such that* $[f < \mu] \neq \emptyset$*. Assume that*

$$\forall x \in [0 < f < \mu]\ \exists z \neq x \text{ such that } f(z) + d(x, z) \leq f(x).$$

Then $[f \leq 0] \neq \emptyset$*, and, for all* $x \in [f < \mu]$*, we can find* $z \in [f \leq 0]$ *such that* $d(x, z) \leq f(x)$.

PROOF. Given $x \in [f < \mu]$, then $M(x) \subset [f < \mu]$. Then, a d-point z of f which belongs to $M(x)$, whose existence is guaranteed by Corollary 2.3, is in $[f \leq 0]$ since, from our assumption, an element of $[0 < f < \mu]$ is not a d-point, and z satisfies

$$d(x, z) \leq f(z) + d(x, z) \leq f(x). \qquad \square$$

3. Generalized contractions

A multifunction T from a set X into a set Y is a subset $T \subset X \times Y$. For any $x \in X$, we define $T(x) = \{y \in Y \mid (x, y) \in T\}$. The domain of the multifunction T is the set of those $x \in X$ such that $T(x) \neq \emptyset$. We shall always assume that the domain of T is nonempty. A fixed point of a multifunction $T \subset X \times X$ is a point $x \in X$ such that $x \in T(x)$. We shall denote by $\mathcal{F}_T$ the set of fixed points of T. Given a subset $C \subset X$ of a metric space and given $x \in X$, we set $d(x, C) = \inf_{z \in C} d(x, z)$ with the convention $d(x, C) = +\infty$ if $C = \emptyset$. For $C, D \subset X$, we shall also use $e(C, D) = \sup_{x \in C} d(x, D)$ with the conventions $e(C, D) = 0$ if $C = \emptyset$ and $e(C, \emptyset) = +\infty$ if $C \neq \emptyset$. As is well known, $e(C, D) = \sup_{z \in X}(d(z, D) - d(z, C))$.

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Given functions $\kappa : (0, +\infty) \to [0, 1)$ and $b : (0, +\infty) \to [\underline{b}, 1]$, where $\underline{b} \in (0, 1]$, and given a multifunction $T \subset D \times X$ defined on a subset D of a metric space X, we will say that T satisfies assumption $(\mathcal{H})$ if, setting $d_x = d(x, T(x))$, we have

$$(\mathcal{H}) \begin{cases} \text{for all } x \in D \text{ such that } d(x, T(x)) > 0, \text{ there exists } z \in D \setminus \{x\} \text{ such that} \\ b(d_x)\, d(x, z) \le d(x, T(x)) \le d(x, z) \text{ and } d(z, T(z)) \le \kappa(d(x, z)) d(x, z). \end{cases}$$

It is natural to assume that the function $b(\cdot)$ is nonincreasing since we need less information when $d(x, T(x))$ decreases. Assuming that $D = X$, it is clear that if $T \subset X \times X$ is a multifunction such that $e(T(x), T(z)) \le \kappa(d(x, z))\, d(x, z)$ for all $x, z \in X$, an assumption used for example in [12, 14], then assumption $(\mathcal{H})$ is in force (taking $z \in T(x)$ such that $b(d_x)\, d(x, z) \le d(x, T(x))$, and $d(z, T(z)) \le e(T(x), T(z)) \le \kappa(d(x, z)) d(x, z)$).

EXAMPLE 3.1. Observe that the setting developed by Klim and Wardowski in [8] is contained in our framework. Namely, if $T \subset X \times X$ is a multifunction and if $b : (0, +\infty) \longrightarrow (0, 1)$ is a function, let us define, for all $x \in X$ such that $d(x, T(x)) > 0$, the set

$$I_b^x = \{z \in T(x) : b(d_x)\, d(x, z) \le d(x, T(x))\},$$

so that I_b^x is nonempty. A strengthened version of assumption $(\mathcal{H})$ is then

$$\forall x \in X \text{ such that } d(x, T(x)) > 0, \exists z \in I_b^x$$
$$\text{such that } d(z, T(z)) \le \kappa(d(x, z))\, d(x, z).$$

In the case where $b(t) \equiv b \in (0, 1)$, we recover the setting of [8]. Observe also that our framework allows non-self multifunctions, that is, multifunctions defined on a subset D of X with values in X. We stress the fact that the point z in assumption $(\mathcal{H})$ is not required to belong to $T(x)$, in such a way that there is no Lipschitz property for T in this definition.

The following lemma is a significant extension of a result of Semenov in [13].

LEMMA 3.2. *Let (X, d) be a metric space and let $T \subset D \times X$ be a multifunction defined on a subset D of X. Assume that there exist a function $\kappa : (0, +\infty) \to [0, 1)$ and a nonincreasing function $b : (0, +\infty) \to [\underline{b}, 1]$ where $\underline{b} \in (0, 1]$, such that $\kappa(\cdot) < b(\cdot)$ and:*

(1) *T satisfies assumption $(\mathcal{H})$;*
(2) *$\limsup_{t \downarrow s} b(t)^{-1} \kappa(t) < 1$ for all $s > 0$.*

Then $\inf_{x \in D} d(x, T(x)) = 0$.

PROOF. Let $x_0 \in D$. We may assume that $d(x_0, T(x_0)) > 0$ (otherwise there is nothing to prove). Assume that there are known $x_0, \ldots, x_n \in D$ such that, for all $k \in \{0, \ldots, n\}$, $d(x_k, T(x_k)) > 0$ and for all $k \in \{0, \ldots, n-1\}$,

$$\begin{cases} b(d_{x_k})\, d(x_k, x_{k+1}) \le d(x_k, T(x_k)) \le d(x_k, x_{k+1}), \\ d(x_{k+1}, T(x_{k+1})) \le \kappa(d(x_k, x_{k+1}))\, d(x_k, x_{k+1}). \end{cases} \tag{3.1}$$

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From our assumptions, we can find a point $x_{n+1} \in D$ such that

$$\begin{cases} b(d_{x_n})\, d(x_n, x_{n+1}) \le d(x_n, T(x_n)) \le d(x_n, x_{n+1}), \\ d(x_{n+1}, T(x_{n+1})) \le \kappa(d(x_n, x_{n+1}))\, d(x_n, x_{n+1}). \end{cases} \tag{3.2}$$

If $x_{n+1} = x_n$, then $d(x_{n+1}, T(x_{n+1})) = 0$, so that $\inf_{x\in D} d(x, T(x)) = 0$. If $x_{n+1} \neq x_n$, we derive from (3.2), using the facts that $b(\cdot)$ is nonincreasing and $\kappa(\cdot) < b(\cdot)$, that

$$d(x_{n+1}, T(x_{n+1})) \le \frac{\kappa(d(x_n, x_{n+1}))}{b(d_{x_n})}\, b(d_{x_n})\, d(x_n, x_{n+1}) \tag{3.3}$$

$$\le c(d(x_n, x_{n+1}))\, d(x_n, T(x_n)), \tag{3.4}$$

where $c(t) = b(t)^{-1}\kappa(t)$. In particular, $d(x_{n+1}, T(x_{n+1})) \le d(x_n, T(x_n))$. Moreover, by (3.1) and using again the fact that $\kappa(\cdot) < b(\cdot)$, we have

$$d_{x_n} = d(x_n, T(x_n)) \le b(d(x_{n-1}, x_n))\, d(x_{n-1}, x_n) \le d(x_{n-1}, x_n),$$

yielding $b(d(x_{n-1}, x_n)) \le b(d_{x_n})$, thus, using the fact that $d_{x_n} \neq 0$,

$$d_{x_n} \le b(d_{x_n})\, d(x_{n-1}, x_n) \le \frac{d_{x_n}}{d(x_n, x_{n+1})} d(x_{n-1}, x_n),$$

and then

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n).$$

By induction, either the process ends if $d(x_k, T(x_k)) = 0$ for some $k \in \mathbb{N}^*$ or we obtain a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that the sequences $(d(x_n, T(x_n)))_{n\in\mathbb{N}} \subset \mathbb{R}$ and $(d(x_n, x_{n+1}))_{n\in\mathbb{N}} \subset \mathbb{R}$ are decreasing. Denoting respectively by $d \ge 0$ and $s \ge d$ the limits of the decreasing sequences $(d(x_n, T(x_n)))_{n\in\mathbb{N}}$ and $(d(x_n, x_{n+1}))_{n\in\mathbb{N}}$, and assuming that $d > 0$, we get, using (3.3), the contradiction

$$d \le \limsup_{t\downarrow s} c(t)\, d < d.$$

It follows that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and thus $\inf_{x\in D} d(x, T(x)) = 0$. □

Here is our main result.

THEOREM 3.3. *Let (X, d) be a complete metric space and let $T \subset D \times X$ be a closed valued multifunction defined on a closed subset $D \subset X$. Assume that the function $x \mapsto d(x, T(x))$ is lower semicontinuous, and that there exist a function $\kappa : (0, +\infty) \to [0, 1)$ and a nonincreasing function $b : (0, +\infty) \to [\underline{b}, 1]$, where $\underline{b} \in (0, 1]$, such that $\kappa(\cdot) < b(\cdot)$ and:*

(1) *T satisfies assumption $(\mathcal{H})$;*
(2) *$\limsup_{t\downarrow s} b(t)^{-1}\kappa(t) < 1$ for all $s \ge 0$.*

Then, $\mathcal{F}_T \neq \emptyset$ and, if $\beta > 0$ and $\delta > 0$ satisfy $\sup_{t\in(0,\delta)} \kappa(t)\, b(t)^{-1} \le 1 - \beta$, then

$$\underline{b}\beta\, d(x, \mathcal{F}_T) \le d(x, T(x)) \quad \forall x \in D \text{ such that } d(x, T(x)) < \underline{b}\delta.$$

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PROOF. Let us define $f : D \longrightarrow \mathbb{R}$ by $f(x) = d(x, T(x))$. From Lemma 3.2, we derive that $\inf_{x\in D} f(x) = 0$ thus the set $[f < \underline{b}\delta]$ is nonempty. Given $x \in [0 < f < \underline{b}\delta]$, we can find $z \in D \setminus \{x\}$ such that

$$b(d(x, z))d(x, z) \le b(d_x)\, d(x, z) \le d(x, T(x)),$$

along with

$$d(z, T(z)) \le \kappa(d(x, z))\, d(x, z),$$

so that

$$f(z) + (b(d(x, z)) - \kappa(d(x, z)))\, d(x, z) \le f(x),$$

yielding

$$f(z) + (1 - \kappa(d(x, z))\, b(d(x, z))^{-1}) b(d(x, z))\, d(x, z) \le f(x).$$

Now let $\beta > 0$ and $\delta > 0$ be such that $b(t)^{-1}\kappa(t) \le 1 - \beta$, for all $s \in (0, \delta)$. Then we get

$$\underline{b}d(x, z) \le b(d(x, z))d(x, z) \le b(d_x)d(x, z) \le d(x, T(x)) < \underline{b}\delta,$$

so that $d(x, z) < \delta$ and then $f(z) + \underline{b}\beta\, d(x, z) \le f(x)$ leading to the conclusion of the theorem by using Lemma 2.4 applied with $\mu = \underline{b}\delta$. □

REMARK 3.4. In Theorem 3.3, the function $x \mapsto d(x, T(x))$ is required to be lower semicontinuous. This is the case if T is Hausdorff upper semicontinuous, that is,

$$\lim_{z\to x} e(T(z), T(x)) = 0$$

for all $x \in X$. Indeed, we have $e(T(z), T(x)) \ge d(z, T(x)) - d(z, T(z))$, so that

$$\liminf_{z\to x} d(z, T(z)) \ge d(x, T(x)).$$

As a consequence of Theorem 3.3, we get an extension of the main result of [8] along three directions: the following corollary holds for nonself mappings, we use a nonconstant function $b(\cdot)$, and an estimate for the distance to the fixed-points set is available.

COROLLARY 3.5. *Let (X, d) be a complete metric space and let $T \subset D \times X$ be a closed valued multifunction defined on a closed subset $D \subset X$. Assume that the function $x \mapsto d(x, T(x))$ is lower semicontinuous, and that there exist a function $\kappa : (0, +\infty) \to [0, 1)$ and a nonincreasing function $b : (0, +\infty) \to [\underline{b}, 1]$ where $\underline{b} \in (0, 1]$, such that $\kappa(\cdot) < b(\cdot)$ and:*

(1) $\begin{cases} \text{for all } x \in D \text{ such that } d(x, T(x)) > 0, \text{ there exists } z \in I_b^x \cap D \\ \text{such that } d(z, T(z)) \le \kappa(d(x, z))d(x, z); \end{cases}$

(2) $\limsup_{t\downarrow s} b(t)^{-1}\kappa(t) < 1$ *for all* $s \ge 0$.

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Then, $\mathcal{F}_T \neq \emptyset$ and, if $\beta > 0$ and $\delta > 0$ satisfy $\sup_{t\in(0,\delta)} \kappa(t)\, b(t)^{-1} \leq 1-\beta$, then

$$\underline{b}\beta\, d(x, \mathcal{F}_T) \leq d(x, T(x)) \quad \forall x \in D \text{ such that } d(x, T(x)) < \underline{b}\delta.$$

In the case where the functions $\kappa(\cdot)$ and $b(\cdot)$ are constant, it is possible to weaken assumption $(\mathcal{H})$ in order to get a generalization of the main result of [6].

THEOREM 3.6. *Let (X, d) be a complete metric space and let $T \subset D \times X$ be a closed valued multifunction defined on a closed subset $D \subset X$. Assume that the function $x \mapsto d(x, T(x))$ is lower semicontinuous, and that there exists $0 \leq \kappa < b$, such that*

$$\begin{cases} \forall x \in D \text{ such that } d(x, T(x)) > 0,\ \exists z \in D \setminus \{x\} \\ \text{such that } b\, d(x, z) \leq d(x, T(x)) \text{ and } d(z, T(z)) \leq \kappa\, d(x, z). \end{cases}$$

Then, $\mathcal{F}_T \neq \emptyset$ and

$$(b-\kappa)\, d(x, \mathcal{F}_T) \leq d(x, T(x)) \quad \forall x \in D.$$

PROOF. Let us define $f : D \longrightarrow \mathbb{R}$ by $f(x) = d(x, T(x))$ and let $x \in [f>0]$, so that we can find $z \in D \setminus \{x\}$ such that $b\, d(x, z) \leq d(x, T(x))$ and $d(z, T(z)) \leq \kappa\, d(x, z)$, yielding $f(z) + (b-\kappa) d(x, z) \leq f(x)$, and then the conclusion of the theorem follows from Lemma 2.4 applied with $\mu = +\infty$. □

REMARK 3.7. Observe that we do not require that $\kappa < 1$ in Theorem 3.6. When applied to mappings, the previous theorem leads to a very light version of the classical Banach–Picard theorem: any continuous mapping $T : X \to X$ defined on a complete metric space for which we can find $0 \leq \kappa < b$ such that for all $x \in X$ with $T(x) \neq x$, there exists $z \in X$ satisfying $b\, d(x, z) \leq d(x, T(x))$ and $d(z, T(z)) \leq \kappa\, d(x, z)$ has a fixed point and $(b-\kappa)\, d(x, \mathcal{F}_T) \leq d(x, T(x))$ for all $x \in X$. A contraction $T : X \to X$ of modulus $\kappa \in [0, 1)$ satisfies the above assumption with $b = 1$ and $z = T(x)$. The mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ with $|\lambda| < 1$ also satisfies the above assumption, but it is not a contraction. The fact that it is enough to require only $d(T(x), T(T(x))) \leq \kappa d(x, T(x))$ for all $x \in X$ instead of $d(T(x), T(z)) \leq \kappa d(x, z)$ for all $x, z \in X$ was implicit in [3].

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