BULL. AUSTRAL. MATH. SOC. 46B20, 46B04
VOL. 53 (1996) [101–107]

ON THE SELF-LENGTH OF TWO-DIMENSIONAL BANACH SPACES

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The aim of this paper is to prove the following result: if X is a 2-dimensional symmetric real Banach space, then its self-length is greater than or equal to 2π. Moreover, the minimum value 2π is uniquely attained (up to isometries) by euclidean space.

1. SYMMETRY NOTIONS AND PROJECTION CONSTANTS

An n-dimensional real Banach space X is symmetric if it has a symmetric basis, that is, a basis $\{x_1, x_2, \ldots, x_n\}$ such that:

$$\left\| \sum_{k=1}^{n} |\alpha_k| \, x_k \right\| = \left\| \sum_{k=1}^{n} \alpha_{\pi(k)} x_k \right\|$$

for any scalars $\alpha_1, \ldots, \alpha_n$ and any permutation π of $\{1, 2, \ldots, n\}$. This notion of symmetry is generalised by the following: An n-dimensional real Banach space X is said to have enough symmetries (e.s.) (see [5]) if the only elements of $\mathcal{L}(X, X)$ which commute with every linear isometry of X have the form κI.

The (absolute) projection constant $\lambda(X)$ of X is defined by:

$$\lambda(X) = \sup\{\lambda(X, Y) : X \subset Y\}$$

where $\lambda(X, Y)$ is the (relative) projection constant of X in Y, defined by:

$$\lambda(X, Y) = \inf\{\|P\| : P \text{ projects } Y \text{ onto } X\}.$$

2. 2-DIMENSIONAL SPACES, SELF-LENGTH

Let X be a 2-dimensional real Banach space, S its unit sphere. We recall the definition of the self-length (or perimeter) $p(X)$ of X. Let A be a convex polygon of vertices $\{a_1, a_2, \ldots, a_n\}$ inscribed in S, then (setting $a_{n+1} = a_1$)

$$p(A) = \sum_{k=1}^{n} \|a_{k+1} - a_k\|_X$$

Received 21st March, 1995

is the "length" (with respect to the metric of X) of the polygon. Parallel to the classic definition of length of a curve we have the definition of self-length:

$$p(X) = \sup\{p(A) : A \text{ a convex polygon inscribed in } S\}.$$

It is clear that if X is isometric to Y ($X \simeq Y$) then $p(X) = p(Y)$. We list now some well known facts about self-length. For more detailed information we refer to [4]. $6 \leqslant p(X) \leqslant 8$; $p(X) = 6$ if and only if $X \simeq H$, the affine regular hexagon; $p(X) = 8$ if and only if $X \simeq l_\infty(2)$, the parallelogram. Of course, if $X \simeq l_2(2)$, then $p(X) = 2\pi$. Also $p(X) = p(X^*)$, where X^* is the dual of X. $p(X)$ has been computed for the affine regular polygons and also for other spaces; see [4].

For the projection constants of 2-dimensional spaces we have: $\lambda(X) = \lambda(X^*)$; $1 \leqslant \lambda(X) \leqslant 4/3$; $\lambda(X) = 4/3$ if and only if $X \simeq H$; $\lambda(X) = 1$ if and only if $X \simeq l_\infty(2)$. The upper bound for λ as well as the unicity statement about the hexagon is a difficult and important result recently proved in [7].

When the dimension is 2 the symmetry conditions become very simple. If X is symmetric then there is a convenient basis such that in the representation of X in R^2, the unit half sphere is symmetric with respect to the x-axis and the unit quarter sphere is symmetric with respect to the (y=x)-axis.

If X is a (e.s.) space then the self-length and the projection constant satisfy the equation

$$p(X) = 8/\lambda(X). \tag{1}$$

(See [4]). This equality does not hold, however, for general spaces.

3. Main result

We state here our main result:

Theorem 1. *Assume that X is a 2-dimensional real symmetric Banach space. Then $p(X) \geqslant 2\pi$ and $p(X) = 2\pi$ if and only if $X \simeq l_2(2)$; consequently, $\lambda(X) \leqslant 4/\pi$, and $\lambda(X) = 4/\pi$ if and only if $X \simeq l_2(2)$*

We note that for spaces with (e.s) this result is not true in general since for H, which has (e.s.), we have $p(H) = 6$ and $\lambda(H) = 4/3$. Before proving the theorem we need some preliminary lemmas.

4. Preliminary lemmas

It is well known that every 2-dimensional Banach space X is embeddable (linearly and isometrically) in a L_1 space, say $L_1[-\pi/2, \pi/2]$. A simple standard way of doing it was shown by Yost [10] (see also [8]): let $(x(t), y(t))$, $-\pi/2 \leqslant t \leqslant \pi/2$, be a

https://doi.org/10.1017/S0004972700016762 Published online by Cambridge University Press

parameterisation of half the unit sphere of X (in a representation in R^2); then (the derivatives $x'(t), y'(t)$ exist almost everywhere and are in $L_1[-\pi/2, \pi/2]$) the subspace $[x', y'] \subset L_1[-\pi/2, \pi/2]$ is isometric to X^*, the dual of X.

LEMMA 1. *Every 2-dimensional symmetric space X is isometric to a subspace $V \subset L_1[-\pi/2, \pi/2]$ of the form $V = [r(t)\cos t, r(t)\sin t]$ with $r \geqslant 0$; $r(-t) = r(t)$; $r(\pi/2 - \tau) = r(\tau)$, $0 \leqslant \tau \leqslant \pi/2$.*

PROOF: We can choose a symmetric basis so that in the representation in R^2 we have a parameterisation $P(t) = (x(t), y(t))$ of the unit sphere C such that

$$x(\pm\pi/2) = 0;\ x(0) = 1;\ x(-t) = x(t);\ x'(t) \leqslant 0,\ \ 0 \leqslant t \leqslant \pi/2;$$
$$y(\pm\pi/2) = \pm 1;\ y(0) = 0;\ y(-t) = -y(t);\ y'(t) \geqslant 0,\ \ 0 \leqslant |t| \leqslant \pi/2;$$
$$x(\pi/2 - t) = y(t);\ y(\pi/2 - t) = x(t),\ \ 0 \leqslant t \leqslant \pi/2.$$

If $Q(t) = (y(t), -x(t))$, then Q is also a parameterisation of C, and therefore $[y', -x'] \subset L_1[-\pi/2, \pi/2]$ is isometric to X^*. Now $(y'(t), -x'(t))$ is in the same octant as $(\cos t, \sin t)$; so, by considering for example only the first octant, there is a rearrangement $t \to \phi(t)$ ($\phi(0) = 0$; $\phi(\pi/4) = \pi/4$) and a positive L_1-function $r(t)$ such that almost everywhere in $[-\pi/2, \pi/2]$ we have

$$(y'[\phi(t)], -x'[\phi(t)]) = (r(t)\cos t, r(t)\sin t).$$

Finally recall that if X is symmetric then also X^* is symmetric; therefore the family of duals of symmetric spaces coincides with the family of all symmetric spaces. ∎

EXAMPLE. If X is 2-dimensional real euclidean space, then note that we can take $(x(t), y(t)) = (\cos t, \sin t)$, $\phi(t) = t$, and $r(t) = 1$ in Lemma 1 and its proof.

LEMMA 2. [2, 8] *If V is a 2-dimensional real space and $V \subset L^1$, then $\lambda(V, L^1) = \lambda(V)$.*

REMARK. It is a well known fact that if V is isometric to W then $\lambda(V) = \lambda(W)$. For 2-dimensional real spaces with (e.s.), this fact follows immediately also from (1).

LEMMA 3. *Let r be an element of $L_1[-\pi/2, \pi/2]$ such that:*

$$r(t) \geqslant 0;\ r(-t) = r(t);\ r(\pi/2 - \tau) = r(\tau)\ (\tau \in [0, \pi/2]).$$

Then, if

$$\sigma(t) = \int_{-\pi/2}^{\pi/2} |\cos(\alpha - t)|\, r(\alpha)\, d\alpha\ ,$$

we have

$$\sigma(t) = \sigma(-t);\ \sigma(\pi/2 - t) = \sigma(t). \tag{2}$$

PROOF: The first equality follows from the fact that

$$\sigma(t) = \int_0^{\pi/2} (|\cos(\alpha+t)| + |\cos(\alpha-t)|) r(\alpha) d\alpha.$$

With the change of variable $\alpha = \pi/2 - \beta$ we obtain

$$\sigma(t) = \int_0^{\pi/2} (|\cos(\pi/2-\beta+t)| + |\cos(\pi/2-\beta-t)|) r(\beta) d\beta\,,$$

$$\sigma(\pi/2-t) = \int_0^{\pi/2} (|\cos(\pi-\beta-t)| + |\cos(-\beta+t)|) r(\beta) d\beta = \sigma(t).$$

□

LEMMA 4. *Let r and σ be as in Lemma 3 and set $V = [v_1, v_2] \subset L_1$; $U = [u_1, u_2] \subset L_\infty$; $v_1 = r(t)\cos t$; $v_2 = r(t)\sin t$; $u_1 = s(t)\cos t$; $u_2 = s(t)\sin t$; $s(t) = c/(\sigma(t))$; $1/c = \int_0^{\pi/2} (r(t))/(\sigma(t)) dt$. Then, if we define $P : L_1 \to V$ by $P = u_1 \otimes v_1 + u_2 \otimes v_2$, the operator P is a projection onto V with $\|P\| = c$.*

PROOF: We must show that $< u_i, v_j >= \delta_{ij}$. Note that by (2) we have $s(t) = s(-t)$; $s(\pi/2 - t) = s(t)$. We have

$$< u_1, v_2 >=< u_2, v_1 >= \int_{-\pi/2}^{\pi/2} r(t) s(t) \cos t \sin t \, dt$$

which is 0 since the integrand is an odd function. Moreover

$$< u_1, v_1 >= 2\int_0^{\pi/2} r(t)s(t)\cos^2 t\, dt = 2\int_0^{\pi/2} r(t)s(t)\sin^2 t\, dt =< u_2, v_2 >;$$

thus

$$< u_i, v_i >= \int_0^{\pi/2} r(t)s(t)\, dt = c\int_0^{\pi/2} \frac{r(t)}{\sigma(t)}\, dt = 1.$$

Recall now that the Lebesgue function Λ of the operator P is defined by $\Lambda(\phi) = \int_{-\pi/2}^{\pi/2} |u_1(\phi)v_1(t) + u_2(\phi)v_2(t)|\, dt$ and that the norm of P is given by $\sup\{\Lambda(\phi) : \phi \in [-\pi/2, \pi/2]\}$, see for example, [1] and [3]. As we shall see in our case, the Lebesgue function is constantly equal to c. Indeed we have

$$\Lambda(\phi) = s(\phi)\int_{-\pi/2}^{\pi/2} r(t)\,|\cos(\phi - t)|\, dt = s(\phi)\sigma(\phi) = s(\phi)\frac{c}{s(\phi)} = c.$$ □

We shall prove that $\|P\| \leqslant 4/\pi$. Once this is done, since by Lemma 2 $\lambda(V) \leqslant \|P\|$, recalling (1) we obtain that

$$p(V) = \frac{8}{\lambda(V)} \geqslant 2\pi.$$

Since $1/(\|P\|) = \int_0^{\pi/2} (r(t))/(\sigma(t)) dt = J$, we have to show that $J \geqslant \pi/4$.

LEMMA 5. *J can be written in the form*

$$J = \int_0^{\pi/4} \frac{r(t)\,dt}{(\cos t + \sin t)\int_0^{\pi/4} \cos\alpha\, r(\alpha)d\alpha + \int_t^{\pi/4} \sin(\alpha - t) r(\alpha)d\alpha}.$$

PROOF: First it is clear that $J = 2\int_0^{\pi/4}(r(t))/(\sigma(t))dt$ and it is also easy to see that $\sigma(t) = \int_0^{\pi/4}(|\cos(\alpha+t)| + |\cos(\alpha-t)| + |\sin(\alpha-t)| + |\sin(\alpha+t)|)\, r(\alpha)d\alpha$. Since $0 \leqslant \alpha \leqslant \pi/4$ and $0 \leqslant t \leqslant \pi/4$ we have:

$$\sigma(t) = \int_0^{\pi/4} (2\cos\alpha\cos t + \sin\alpha\cos t + \cos\alpha\sin t) r(\alpha)d\alpha + \int_0^{\pi/4} |\sin(\alpha - t)|\, r(\alpha)d\alpha,$$

where

$$\int_0^{\pi/4} |\sin(\alpha - t)|\, r(\alpha)d\alpha = \int_0^{\pi/4} \sin(t-\alpha) r(\alpha)d\alpha + 2\int_t^{\pi/4} \sin(\alpha - t) r(\alpha)d\alpha.$$

Thus we obtain

$$\sigma(t) = 2\int_0^{\pi/4} (\cos\alpha\cos t + \cos\alpha\sin t) r(\alpha)d\alpha + 2\int_t^{\pi/4} \sin(\alpha - t) r(\alpha)d\alpha,$$

from which the formula for J follows. □

5. PROOF OF THEOREM 1

Pointing out the dependence of J on the function r, we shall write

$$J = J(r) = \int_0^{\pi/4} \frac{r(t)}{\delta_r(t)} dt\,;$$

$$\delta_r(t) = (\cos t + \sin t)\int_0^{\pi/4} \cos\alpha\, r(\alpha)\, d\alpha + \int_t^{\pi/4} \sin(\alpha - t)\, r(\alpha)\, d\alpha.$$

Let $A = \{r \in L_1[0, \pi/4] : r \geqslant 0\}$; we first prove that

$$\inf\{J(r),\ r \in A\} = \pi/4.$$

We have (omitting the index r in the functional δ):

$$\delta'(t) = (\cos t - \sin t)\int_0^{\pi/4} \cos\alpha\, r(\alpha)\, d\alpha - \int_t^{\pi/4} \cos(\alpha - t)\, r(\alpha)\, d\alpha\,;$$

$$\delta''(t) = -(\cos t + \sin t)\int_0^{\pi/4} \cos\alpha\, r(\alpha)\, d\alpha + r(t) - \int_t^{\pi/4} \sin(\alpha - t)\, r(\alpha)\, d\alpha$$

https://doi.org/10.1017/S0004972700016762 Published online by Cambridge University Press

and consequently

$$\delta'' + \delta = r\,;\ \delta(0) > 0\,;\ \delta'(0) = 0\,;\ \delta'(\pi/4) = 0.$$

These imply that

$$J(r) = \frac{\pi}{4} + \int_0^{\pi/4} \frac{\delta''}{\delta} dt.$$

But we have

$$\int_0^{\pi/4} \frac{\delta''}{\delta} dt = \left[\frac{\delta'}{\delta}\right]_0^{\pi/4} + \int_0^{\pi/4} \left(\frac{\delta'}{\delta}\right)^2 dt$$

and hence we get

$$J(r) = \frac{\pi}{4} + \int_0^{\pi/4} \left(\frac{\delta'}{\delta}\right)^2 dt \geqslant \frac{\pi}{4}.$$

It is clear that every r = constant (positive) is a minimum point for J on A; also $J(r) = \pi/4 \Leftrightarrow \delta' = 0$. We show now that constants are the only minimum points. If the function r is a point of minimality, it follows that $\delta' = \delta'' = 0$, and this implies that δ is constant and (since $\delta'' + \delta = r$) that r = constant. □

6. Remark

It is well known (see for example, [**6**]) that the value of the projection constant of n-dimensional euclidean space is

$$\lambda(l_2(n)) = \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n+1}{2})}.$$

In view of the second statement in Theorem 1, one could ask whether it is true, also for $n > 2$, that for n-dimensional symmetric spaces X_n one has $\lambda(X_n) \leqslant \lambda(l_2(n))$. The answer is no even for $n = 3$ as is shown in the example constructed by Positselskii in [**9**]. In fact he has computed for every n the exact value K_n of the absolute projection constant of a special sequence of symmetric spaces (Marcinkiewicz spaces); it turns out that $K_n > \lambda(l_2(n))$ for all $n > 2$ but $n = 4$.

7. Note

After completing this work we were informed that the result $\lambda(X) \leqslant 4/\pi$ has been proved (independently and with totally different method) in: "Projections onto symmetric spaces" by Hermann Koenig, to appear in Quaest. Math.

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