BULLETIN 13 (2), 1970

INVERSION OF A CLASS OF CONVOLUTION TRANSFORMS OF GENERALIZED FUNCTIONS

BY
Z. DITZIAN

1. **Introduction**. The kernels of the transforms in the class that we shall treat satisfy

$$(1.1)\quad G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} [F(s)]^{-1} e^{st}\, ds \equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Big[\prod_{k=1}^{\infty} \{(1-s/a_k)/(1-s/c_k)\} \times \exp(s(a_k^{-1}-c_k^{-1}))\Big]^{-1} e^{st}\, ds$$

where $\mathrm{Re}\, a_k = a_k$, $\mathrm{Re}\, c_k = c_k$, $0 \le a_k/c_k < 1$ and $\sum a_k^{-2} < \infty$ (see also [1], [2], [3], [6], [7] and [8]).

We shall also require that $N_+ + N_- = \infty$, where $N_\pm$ is defined, as in [1], by

$$(1.2)\qquad N_\pm = \liminf_{x\to\pm\infty} (N(\{a_k\}, x) - N(\{c_k\}, x))$$

where $N(\{b_k\}, x)$ is the number of b_k's between 0 and x. All $G(t)$ that shall be mentioned in this paper will satisfy the above conditions. The convolution transform is defined by

$$(1.3)\qquad F(x) = \langle f(t), G(x-t)\rangle$$

where $f(t)$ belongs to a space of generalized functions which is dual to a space of test functions that includes $G(x-t)$. We shall use the space of test functions $L_{c,d}$ defined (as done by A. Zemanian see [9]) by the following sequence of semi norms $\gamma_k[h] \equiv \gamma_{k,c,d}[h] = \sup |K(t)h^{(k)}(t)|$ where

$$(1.4)\qquad \begin{aligned} &K(t) \in C^\infty,\ K(t) \ne 0,\\ &K(t) = e^{ct} \quad \text{for } t > 1 \text{ and}\\ &K(t) = e^{dt} \quad \text{for } t < -1. \end{aligned}$$

Since $\gamma_0(h)$ is a norm so are $\gamma_k'(h) = \max\{\gamma_n(h) : n \le k\}$ which also are monotonic and induce the same topology. $L'_{c,d}$ will be the dual space of $L_{c,d}$.

We shall quote the properties of the spaces $L_{c,d}$ and $L'_{c,d}$ from [9] when needed. In this paper we shall show that the inversion formula proved in [3] will still hold in the sense of weak limit for convolution transform of generalized functions.

2. **The convolution transform on** $L'_{c,d}$. We recall first the definition of α_1 and α_2 (see [1, (2.1)])

$$(2.1)\qquad \alpha_1 = \max(a_k, -\infty \mid a_k < 0),\ \alpha_2 = \min(a_k, \infty \mid a_k > 0)$$

Received by the editors May 5, 1969.

https://doi.org/10.4153/CMB-1970-038-9 Published online by Cambridge University Press

Using asymptotic estimates of $G(t)$ (see [2, Theorem 4.1]) for $N_+ + N_- = \infty$, we have $G(x-t) \in L_{c,d}$ for any x where $c < \alpha_2$ and $d > \alpha_1$ and therefore for $f \in L'_{c,d}$ we can write $F(x) = \langle f(t), G(x-t) \rangle$.

THEOREM 2.1. *Let* $G(t)$ *be defined by* (1.1), $N_+ + N_- = \infty, f \in L'_{c,d}$, $c < \alpha_2$, $d > \alpha_1$ *and* $F(x) = \langle f(t), G(x-t) \rangle$, *then*

$$F^{(n)}(x) = \langle f(t), G^{(n)}(x-t) \rangle \tag{2.2}$$

and $F(x) \in L_{a,b}$ *for any* a *and* b *satisfying* $a < \min(-\alpha_1, -c)$ *and* $b > \max(-\alpha_2, -d)$.

Proof. We follow step by step the proof of Theorem 4.1 in [9, p. 330] using [2, Theorem 4.1] instead of Theorem 2.1 of [4, p. 108].

We know by [9, p. 329, v] that for $c \le c'$ and $d' \le d L_{c',d'} \subseteq L_{c,d}$ and also $L'_{c',d'} \supseteq L'_{c,d}$ meaning that the restriction of $f(x) \in L'_{c,d}$ to $L_{c',d'}$ belongs to $L'_{c',d'}$. Since $G(x-t) \in L_{c',d'}$ whenever $c' < \alpha_2$ and $d' > \alpha_1$ we have:

COROLLARY 2.2. *Under the assumptions of Theorem 2.1* $F(x) \in L_{a_*,b_*}$, *where*

$$a_* = \begin{cases} -\alpha_2 + \eta & \alpha_2 \neq \infty \\ -p & \alpha_2 = \infty \end{cases} \quad \text{and}$$
$$b_* = \begin{cases} -\alpha_1 - \eta & \alpha_1 \neq -\infty \\ p & \alpha_1 = -\infty \end{cases} \tag{2.3}$$

for η *small enough and* p *big enough.*

Obviously we have, for all k, $-a_k \notin [a_*, b_*]$ and $-c_k \notin [a_*, b_*]$.

3. The inversion operator.

We shall use the sequence of operators $R_m(D)$ defined in [3] as follows:

$$R_m(D) = e^{-b_m D} \prod_{k=1}^{m} \left(1 - \frac{D}{a_n}\right)\left(1 - \frac{D}{c_k}\right)^{-1} \exp((a_k^{-1} + c_k^{-1})D) \tag{3.1}$$

where $e^{kD} f(x) = f(x+k)$, $D = d/dx$, $(1 - D/c)^{-1}$ is defined as in [3, (1.7)] by

$$\left(1 - \frac{D}{c}\right)^{-1} f(x) = \begin{cases} c\, e^{cx} \int_x^\infty e^{-cy} f(y)\, dy & \text{for } c > 0 \\ -c\, e^{cx} \int_{-\infty}^x e^{-cy} f(y)\, dy & \text{for } c < 0 \end{cases} \tag{3.2}$$

and $\lim_{m \to \infty} b_m = 0$.

The following lemmas about the effect of the inversion operator on classes of functions will be useful later.

LEMMA 3.1. *Let* $F(x) \in L_{a_*,b_*}$, $R_m(D)$ *be defined by* (3.1) *and* (3.4) *and* $-c_k \notin [a_*, b_*]$ *for any* k, *then*

$$R_m(D) F(x) \in L_{a_*,b_*}.$$

Proof. Since $e^{kD} F(x)$ and $(1 - D/a) F(x)$ are obviously in L_{a_*,b_*} we have only to show that so is $(1 - D/c)^{-1} F(x)$. We have therefore to estimate $D^n (1 - D/c)^{-1} F(x)$.

Integrating by parts n times and using the asymptotic behaviour of $F(x)$ and it's derivatives yields

$$\left(\frac{d}{dx}\right)^n\left(e^{cx}\int_x^\infty e^{-cy}F(y)\,dy\right) = ce^{cx}\int_x^\infty e^{-cy}F^{(n)}(y)\,dy.$$

Using the inequalities $|F^{(n)}(y)|\le k\,e^{-a_* y}$ for $y>0$ and $|F^{(n)}(y)|\le k\,e^{-b_* y}$ for $y<0$ we can conclude the proof by simple calculations.

The asymptotic properties of $F(x)$ and all its derivatives imply by Fubini's theorem and other classical theorems that the order in which the elements of $R_m(D)$ are taken has no influence on $R_m(D)F(x)$.

LEMMA 3.2. *Let $\psi(x)\in\mathscr{D}$, then $R_m(-D)\psi(x)\in L_{a,b}$ where $a<\gamma_2$, $b>\gamma_1$ and γ_i are defined as in* [1] *by*:

(3.3) $$\gamma_1 = \max\,(c_k, -\infty \mid c_k < 0) \quad \textit{and} \quad \gamma_2 = \min\,(c_k, \infty \mid c_k > 0).$$

Proof. We shall have to show that

(3.4) $$|D^nR_m(-D)\psi(x)| \le K(n,m)K(x)^{-1}$$

where $K(x)\in c^\infty$, $K(x)=e^{ax}$ for $x>1$ and $K(x)=e^{bx}$ for $x<1$.

Since obviously $\psi(x)$ satisfies the conditions on $R_m(-D)\psi(x)$ in (3.4) we can complete the proof if we show that if $\psi_1(x)\in L_{a,b}$ then

(i) $D\psi_1(x)\in L_{a,b}$,

(ii) $e^{kD}\psi_1(x)=\psi_1(x+k)\in L_{a,b}$, and

(iii) $(1+D/c)^{-1}\psi_1(x)\in L_{a,b}$ whenever $c\in\{c_k\}$.

Obviously (i) and (ii) are valid. We shall show (iii) for $c>0$. Since $\psi_1(x)\in L_{a,b}$ the definition of $(1+D/c)^{-1}$ and integration by parts n time yields

$$D^n\left(1+\frac{D}{c}\right)^{-1}\psi_1(x) = D^n\left(c\,e^{-cx}\int_{-\infty}^x e^{cy}\psi_1(y)\,dy\right)$$

$$= c\,e^{-cx}\int_{-\infty}^x e^{cy}\psi_1^{(n)}(y)\,dy = \Phi(x)$$

which yields $\Phi(x)\in L_{a,b}$ since $c\ge\gamma_2>a$. For $c<0$ the proof of (iii) is similar. Q.E.D.

REMARK 3.2a. When the multiplicity of γ_1 and γ_2 (both finite) is one, we can show that $R_m(-D)\psi(x)\in L_{\gamma_2,\gamma_1}$.

4. The inversion result for $L'_{c,d}$.

THEOREM 4.1. *Let c and d satisfy for a given $G(t)$, $c<\alpha_2$ and $d>\alpha_1$ and suppose $f(t)\in L'_{c,d}$ and $F(x)=\langle f(t), G(x-t)\rangle$, then, for all $\psi(x)\in\mathscr{D}$,*

(4.1) $$\lim_{m\to\infty}\langle R_m(D)F(x), \quad \psi(x)\rangle = \langle f(t), \psi(t)\rangle.$$

https://doi.org/10.4153/CMB-1970-038-9 Published online by Cambridge University Press

Proof. We first outline the major steps of the proof which we shall justify later.

(4.2) $\langle R_m(D)F(x), \psi(x)\rangle = \langle F(x), R_m(-D)\psi(x)\rangle$

(4.3) $= \langle\langle f(t), G(x-t)\rangle, R_m(-D)\psi(x)\rangle = \langle f(t), \langle G(x-t), R_m(-D)\psi(x)\rangle\rangle$

(4.4) $= \langle f(t), \langle R_m(D)G(x-t), \psi(x)\rangle\rangle = \langle f(t), \langle G_m(x-t), \psi(x)\rangle\rangle;$

(4.5) $\lim_{m\to\infty} \langle f(t), \langle G_m(x-t), \psi(x)\rangle\rangle = \langle f(t), \psi(t)\rangle.$

To justify (4.2) we first observe, using Lemma 3.2, that the order of applying terms of $R_m(-D)$ to $\psi(x) \in \mathscr{D}$ does not make any difference. Therefore (4.2) can be proved termwise. It will be enough if we show for $F_1(x) \in L_{a_*,b_*}$ and $\psi_1 \in L_{a,b}$, where both a_* and b_* are defined by (2.3), $a<\gamma_2$, $b<\gamma_1$ and $\eta<\min(\alpha_1-\gamma_1, \gamma_2-\alpha_2)$, the following:

(a) $\langle DF_1(x), \psi_1(x)\rangle = \langle F_1(x), -D\psi_1(x)\rangle;$

(b) $\langle e^{kD}F_1(x), \psi_1(x)\rangle = \langle F_1(x), e^{-kD}\psi_1(x)\rangle;$

(c) $\left\langle \left(1-\frac{D}{c}\right)^{-1} F_1(x), \psi_1(x)\right\rangle = \left\langle F_1(x), \left(1+\frac{D}{c}\right)^{-1}\psi_1(x)\right\rangle, \quad c \notin (\gamma_1, \gamma_2).$

Integrating by parts and using the asymptotic estimates of $F_1(x)$ and $\psi_1(x)$ yields (a). Change of variable yields (b). By Fubini Theorem, that can be used because $F_1(x) \in L_{a_*,b_*}$ and $\psi_1(x) \in L_{a,b}$, we show (c) for $c>0$ (similarly it can be shown for $c<0$) as follows:

$$\left\langle \left(1-\frac{D}{c}\right)^{-1} F_1(x), \psi_1(x)\right\rangle = \int_{-\infty}^{\infty} c\, e^{cx}\left(\int_x^{\infty} e^{-cy}F_1(y)\,dy\right)\psi_1(x)\,dx$$

$$= \int_{-\infty}^{\infty} F_1(y)\,dy\left(c\,e^{-cy}\int_{-\infty}^{y} e^{cx}\psi_1(x)\,dx\right) = \left\langle F_1(x), \left(1+\frac{D}{c}\right)^{-1}\psi_1(x)\right\rangle\cdot$$

To prove (4.3), which is a Fubini type Lemma, we recall that for any finite A and B an argument similar to that used by A. Zemanian in [10, Lemma 2.1] yields

(4.6) $$\int_A^B \langle f(t), G(x-t)\rangle R_m(-D)\psi(x)\,dx = \left\langle f(t), \int_A^B G(x-t)R_m(-D)\psi(x)\,dx\right\rangle\cdot$$

Therefore it is left to be shown that

(4.7) $$\left|\left\{\int_{-\infty}^{A} + \int_B^{\infty}\right\} \langle f(t), G(x-t)\rangle R_m(-D)\psi(x)\,dx\right| < \varepsilon \text{ for } A < A(\varepsilon) \text{ and } B > B(\varepsilon)$$

(which is clear by Theorem 2.1 and Lemma 3.2) and

(4.8) $$\left\langle f(t), \left\{\int_{\infty}^{A} + \int_B^{\infty}\right\} G(x-t)R_m(-D)\psi(x)\,dx\right\rangle < \varepsilon \text{ for } A < A(\varepsilon) \text{ and } B > B(\varepsilon).$$

https://doi.org/10.4153/CMB-1970-038-9 Published online by Cambridge University Press

We shall now show that $\int_{-\infty}^{A} G(x-t)R_m(-D)\psi(x)\,dx$ tends to zero in $L_{c,d}$ as A tends to $-\infty$, this combined with the analogous result for $\int_{B}^{\infty}\ldots$ (the proof of which is similar) will prove (4.8). Recalling $R_m(-D)\psi(x) \equiv \psi_1(x) \in L_{a,b}$ by Lemma 3.2, we write

$$D^n \int_{-\infty}^{A} G(x-t)\,\psi_1(x)\,dx = \int_{\infty}^{A} G(x-t)D^n\psi_1(x)\,dx.$$

A calculation using the asymptotic properties of $G(x-t)$ (proved in Theorem 4.1 of [2]) and $\psi_1^{(n)}(x) \in L_{a,b}$ imply $\left|\int_{-\infty}^{A} G(x-t)\psi_1^{(n)}(x)\,dx\right| \le \epsilon[K(t)]^{-1}$, where $K(t)$ is defined by (1.4). This concludes the proof of (4.3). The method used for proving (4.2) can be used to show that

$$\langle G(x-t), R_m(-D)\psi(x)\rangle = \langle R_m(D)G(x-t), \psi(x)\rangle. \tag{4.9}$$

Using the definition (see [3, (5.14)]),

$$R_m(D)G(x-t) = G_m(x-t), \tag{4.10}$$

we prove the validity of (4.4).

To prove (4.5) we have to show that $\langle G_m(x-t), \psi(x)\rangle \to \psi(t)$ in $L_{c,d}$; that is, for $I(n, m)$ defined by

$$I(n, m) = \left|K(t)\left\{D_t^n \int_{-\infty}^{\infty} G_m(x-t)\psi(x)\,dx - \psi^{(n)}(t)\right\}\right|$$

for every ε and n there exists an $m_0 = m_0(\varepsilon, n)$ such that for $m > m_0$ $I(n, m) < \varepsilon$.

$$\begin{aligned} I(n, m) &= \left|K(t)\int_{-\infty}^{\infty} G_m(x-t)[\psi^{(n)}(x)-\psi^{(n)}(t)]\,dx\right| \\ &\le \left|K(t)\int_{t-\delta}^{t+\delta} G_m(x-t)[\psi^{(n)}(x)-\psi^{(n)}(t)]\,dx\right| \\ &\quad + |K(t)\psi^{(n)}(t)|\left[\int_{|x-t|>\delta} G_m(x-t)\,dt\right] \\ &\quad + \left|K(t)\int_{-\infty}^{t-\delta} G_m(x-t)\psi^{(n)}(x)\,dx\right| \\ &\quad + \left|K(t)\int_{t+\delta}^{\infty} G_m(x-t)\psi^{(n)}(x)\,dx\right| \\ &\equiv I_1(n, m)+I_2(n, m)+I_3(n, m)+I_4(n, m). \end{aligned}$$

Since $\psi(x) \in \mathscr{D}$ it has a compact support say (A, B) therefore $I_1(n, m)$ has $(A-\delta, B+\delta)$ as a support and

$$|I_1(n, m)| \le \max_{A-1<t<B+1} K(t) \max_{|x-t|<\delta} |\psi^{(n)}(x)-\psi^{(n)}(t)|$$

https://doi.org/10.4153/CMB-1970-038-9 Published online by Cambridge University Press

and of course we can choose $\delta < 1$ so small that

$$\max_{|x-t|<\delta} |\psi^{(n)}(x)-\psi^{(n)}(t)| < \frac{\varepsilon}{4}\Big(\max_{A-1<t<B+1} K(t)\Big)^{-1}.$$

Since $|K(t)\psi^{(n)}(t)|$ is bounded and since by [3, Lemma 7.1] for $c=0$ we have for every fixed δ $\lim_{m\to\infty} \int_{|t-x|>\delta} G_m(x-t)\,dx=0$, $|I_2(n, m)|<\varepsilon/4$ for $m>m_0$. To show that for $m>m, (n, \varepsilon)$, $|I_3(n, m)|<\varepsilon/4$ we recall again that supp $\psi(t)=(A, B)$ and therefore for $t<A+\delta$ $I_3(n, m)=0$ and it is easy to show that for $t<B+\delta$ $I_3(n, m)$ $<\varepsilon/4$ for $m\geq m_2$. For $t>B+\delta$ we write (assuming $B+\delta>1$ otherwise the proof is similar)

$$\begin{aligned} I_3(n, m) &= K(t)\int_A^B G_m(x-t)\psi^{(n)}(x)\,dx \\ &\leq \int_A^B G_m(x-t)\, e^{-c(x-t)}(e^{+cx}\psi^{(n)}(x))\,dx \leq \max_{A\leq x\leq B} e^{+cx}\psi^{(n)}(x) \\ &\quad\times \int_{-\infty}^{t-\delta} G_m(x-t)e^{-c(x-t)}dx \\ &\leq M\cdot \int_{|t|>\delta} G_m(t)\, e^{-ct}\,dt \end{aligned}$$

Using again [3, Lemma 7.1] we complete the proof of the theorem since the estimation of $I_4(n, m)$ is similar. Q.E.D.

References

1. Z. Ditzian and A. Jakimovski, *A remark on a class of convolution transforms*, Tôhoku Math. J. **20** (1968), 170–174.

2. ———, *Properties of kernels for a class of convolution transforms*, Tôhoku Math. J. **20** (1968), 175–198.

3. ———, *Convergence and inversion result for a class of convolution transforms*, Tôhoku Math. J. **21** (1969), 195–220.

4. I. I. Hirschman and D. V. Widder, *The convolution transform*, Princeton Univ. Press, 1955.

5. J. Pandey and A. Zemanian, *An extension of Tanno's form of the convolution transformation*, Tôhoku Math. J. **20** (1968), 425–430.

6. Y. Tanno, *On the convolution transform part I*, Kōdai Math. Sem. Rep. (1959), 40–50.

7. ———, *On the convolution transform* (*part II and III*) Science report of the Faculty of Literature and Science, Hirosaki Univ. Japan (1962), 5–20.

8. ———, *On a class of convolution transforms I*, Tôhoku Math. J. **18** (1966), 157–173.

9. A. Zemanian, *A generalized convolution transformation*, SIAM J. Appl. Math. **15** (1967), 324–346.

10. ———, *A generalized Weirstrass Transformation*, SIAM J. Appl. Math. **15**, 1088–1105.

University of Alberta,
Edmonton, Alberta