T. Hosoh
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AMPLE VECTOR BUNDLES ON A RATIONAL SURFACE

TOSHIO HOSOH

Introduction.

On a complete non-singular curve defined over the complex number field $\boldsymbol{C}$, a stable vector bundle is ample if and only if its degree is positive [3]. On a surface, the notion of the H-stability was introduced by F. Takemoto [8] (see § 1). We have a simple numerical sufficient condition for an H-stable vector bundle on a surface S defined over $\boldsymbol{C}$ to be ample; let E be an H-stable vector bundle of rank 2 on S with $\Delta(E) = c_1(E)^2 - 4c_2(E) \geqq 0$, then E is ample if and only if $c_1(E) > 0$ and $c_2(E) > 0$, provided S is an abelian surface, a ruled surface or a hyper-elliptic surface [9]. But the assumption above concerning $\Delta(E)$ evidently seems too strong. In this paper, we restrict ourselves to the projective plane $\boldsymbol{P}^2$ and a rational ruled surface Σ_n defined over an algebraically closed field k of arbitrary characteristic. We shall prove a finer assertion than that of [9] for an H-stable vector bundle of rank 2 to be ample (Theorem 1 and Theorem 3). Examples show that our result is best possible though it is not a necessary condition (see Remark (1) § 2).

In § 1, we shall recall the definition of H-stable vector bundles and their elementary properties proved by F. Takemoto [8].

In § 2, we shall prove the following;

THEOREM 1. *If E is an H-stable vector bundle of rank 2 on $\boldsymbol{P}^2$ with $c_1(E) \geqq (-1/2)\Delta(E)$, then E is ample.*

In § 3, we shall prove a similar sufficient condition for an H-stable vector bundle of rank 2 on Σ_n to be ample (Theorem 3).

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§ 1. Preliminaries

Let k be an algebraically closed field of arbitrary characteristic.

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Throughout this paper, the ground field k will be fixed. Let E be a vector bundle (i.e. a locally free sheaf) on a non-singular irreducible projective algebraic variety X defined over k. We shall use the following notation;

$h^i(X,E) := \dim_k H^i(X,E)$; the dimension of $H^i(X,E)$.
$E^* := \mathrm{Hom}_{O_X}(E, O_X)$; the dual vector bundle of E.
$\chi(E) := \sum_i (-1)^i h^i(X,E)$; the Euler-Poincaré characteristic of E.
$c_i(E)$; the i-th Chern class of E.

Let H be an ample line bundle (i.e. invertible sheaf) on X and $s = \dim X$. We recall the deflnition of H-stable vector bundles [8].

DEFINITION. A vector bundle E on X is H-stable if for every non-trivial, non-torsion, quotient sheaf F of $E, d(E,H)/r)E) < d(F,H)/r(F)$, where $d(F,H) = (c_1(F), H^{s-1})$ with the intersection pairing (,) and where $r(F)$ is the rank of F.

The following lemma is an immediate consequence of the definition.

LEMMA (1.1). (1) *A vector bundle is H-stable if and only if it is $H^{\otimes n}$-stable for any natural number n.*

(2) *If L is a line bundle, then E is H-stable if and only if $E \otimes L$ is H-stable.*

(3) *If E is H-stable and $d(E,H) \leqq 0$, then $H^0(X,E) = (0)$.*

We say that a vector bundle E is simple if any global endomorphism of E is constant, i.e. $H^0(X, \mathrm{End}(E)) = k$. We know that an H-stable vector bundle is simple ([8] Corollary (1.8)). In the case of rank 2 vector bundles on $\boldsymbol{P}^2$, also the converse is true ([8] Proposition (4.1)), i.e.;

LEMMA (1.2). *Let E be a vector bundle of rank 2 on $\boldsymbol{P}^2$, then the following conditions are equivalent*

(1). *E is simple.* (2). *E is $O^{\boldsymbol{P}^2}(1)$-stable.*

There is a very usefull criterion for a rank 2 vector bundle to be not simple ([7] Theorem 1.);

LEMMA (1.3). *Let E be a vector bundle of rank 2 on X, then the following conditions are equivalent.*

(1). *E is not simple.*

(2). *There exists a line bundle L on X such that for $E' = E \otimes L$,*

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$h^0(X, E') \neq 0$ *and* $h^0(X, E'^*) \neq 0$.

Let E be a vector bundle on $X, \boldsymbol{P}(E)$ the projective bundle on X associated to E and $O_{\boldsymbol{P}(E)}(1)$ the tautological line bundle on $\boldsymbol{P}(E)$ i.e. $\pi_*(O_{\boldsymbol{P}(E)}(1)) \cong E, \pi$ being the natural projection of $\boldsymbol{P}(E)$ onto X. If L is a line bundle on X, then the line bundle $O_{\boldsymbol{P}(E)}(1) \otimes \pi^*(L)$ is also the tautological line bundle on $\boldsymbol{P}(E \otimes L) \cong \boldsymbol{P}(E)$. If M is a line bundle on $\boldsymbol{P}(E), M$ is isomorphic to a line bundle $O_{\boldsymbol{P}(E)}(1)^{\otimes n} \otimes \pi*(N)$ for some integer n and some line bundle N on X (see EGA II. 4.1). A rational ruled surface is isomorphic to $\Sigma_n = \boldsymbol{P}(O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1})$ for some non-negative integer n. We denote the projection from Σ_n to $\boldsymbol{P}^1$ by π_n.

The following lemma plays an important role in the sequel.

LEMMA (1.4) *Let s be a section of the projection $\pi_n\colon \Sigma_n \to \boldsymbol{P}^1$, then;*

(1) *If the self-intersection number (s, s) is non-positive, then $(s, s) = -n$ and the direct image $\pi_{n*}(O_{\Sigma_n}(s))$ is isomorphic to the the vector bundle $O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1}$.*

(2) *If the self-intersection number (s, s) is non-negative, then $(s, s) \geqq n$ and the direct image $\pi_{n*}(O_{\Sigma_n}(s))$ is generated by its global sections.*

Proof. We have an exact sequence on Σ_n;

$$0 \longrightarrow O_{\Sigma_n} \longrightarrow O_{\Sigma_n}(s) \longrightarrow O_{\Sigma_n}(s)|_s \longrightarrow 0$$

Since $R^1\pi_{n*}(O_{\Sigma_n}) = (0), \pi_{n*}(O_{\Sigma_n}) \cong O_{\boldsymbol{P}^1}, \pi_{n*}(O_{\Sigma_n}(s)|_s) \cong \boldsymbol{P}^1((s, s))$ and $\pi_{n*}(O_{\Sigma_n}(s)) \cong (O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1}) \otimes O_{\boldsymbol{P}^1}(a)$ for some integer a, we have the following exact sequence;

$$0 \longrightarrow O_{\boldsymbol{P}^1} \longrightarrow (O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1}) \otimes O_{\boldsymbol{P}^1}(a) \longrightarrow O_{\boldsymbol{P}^1}((s, s)) \longrightarrow 0 \qquad (*)$$

(1) If $(s, s) \leqq 0$, then the exact sequence $(*)$ is split because $h^1(\boldsymbol{P}^1, O_{\boldsymbol{P}^1}(t)) = 0$ for $t \geqq 0$. Hence we have;

$$(O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1}) \otimes O_{\boldsymbol{P}^1}(a) \cong O_{\boldsymbol{P}^1}((s, s)) \oplus O_{\boldsymbol{P}^1}\,.$$

This is possible if and only if $a = 0$ and $O_{\boldsymbol{P}^1}((s, s)) \cong O_{\boldsymbol{P}^1}(-n)$, hence $(s, s) = -n$ and $\pi_{n*}(O_{\Sigma_n}(s)) \cong O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1}$.

(2) If $(s, s) \geqq 0$, then $O_{\boldsymbol{P}^1}((s, s))$ is generated by its global sections. Hence we have that $\pi_{n*}(O_{\Sigma_n}(s))$ is generated by its global sections by virtue of the exact sequence $(*)$. This is possible if and only if $a - n \geqq 0$. On the other hand, $O_{\boldsymbol{P}^1}((s, s))$ is isomorphic to $O_{\boldsymbol{P}^1}(2a - n)$ by $(*)$, which

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implies $(s, s) = 2a - n = 2(a - n) + n \geqq n$.

The section on Σ_n corresponding to the exact sequence;

$$0 \longrightarrow O_{P^1} \longrightarrow O_{P^1}(-n) \oplus O_{P^1} \longrightarrow O_{P^1}(-n) \longrightarrow 0$$

is called a minimal section of Σ_n and denoted by M. Let N be a fibre of π_n, then every divisor D on Σ_n is linearly equivalent to $aM + bN$ where $a = (D, N)$ and $b = (D, M) + an$. A canonical divisor on Σ_n is linearly equivalent to $-2M - (n + 2)N$.

§ 2. Simple vector bundles on P^2

Let E be a vector bundle of rank r on P^2 and ℓ be a line on P^2, then the restriction $E|_\ell$ of E to ℓ is isomorphic to a direct sum of line bundles L_i's $(1 \leqq i \leqq r)$ [2]; we set;

$$\alpha_E(\ell) = \min \{\deg (L_i) ; 1 \leqq i \leqq r\}$$

Evidently the number $\alpha_E(\ell)$ is bounded above and below when ℓ runs through lines on P^2. Hence we set;

$$M(E) := \max \{\alpha_E(\ell) ; \ell \text{ is a line on } P^2\}$$
$$m(E) := \min \{\alpha_E(\ell) ; \ell \text{ is a line on } P^2\}$$

If E is a vector bundle on P^2, we put $E(n) = E \otimes O_{P^2}(1)^{\otimes n}$.

LEMMA (2.1) *Let E be a vector bundle on P^2, then;*

(1) *If $M(E) \geqq -1$, then $h^1(P^2, E(1)) \leqq h^1(P^2, E)$.*

(2) *If $M(E) \geqq -1 > m(E)$, then $h^1(P^2, E(1)) < h^1(P^2, E)$.*

(3) *If $M(E) \geqq -1$ and $h^1(P^2, E(1)) = h^1(P^2, E)$, then $E(1)$ is generated by its global sections.*

Proof. (1) Let ℓ be a line with $\alpha_E(\ell) = M(E)$, then there is the following short exact sequence;

$$O \longrightarrow O_{P^2}(-1) \longrightarrow O_{P^2} \longrightarrow O_\ell \longrightarrow 0 \qquad (*)$$

Tensoring $E(1)$ with $(*)$, we get the short exact sequence;

$$0 \longrightarrow E \longrightarrow E(1) \longrightarrow E(1)|_\ell —— 0$$

and the long exact sequence of cohomologies;

$$\cdots \longrightarrow H^1(P^2, E) \longrightarrow H^1(P^2, E(1)) \longrightarrow H^1(\ell, E(1)|_\ell) \longrightarrow \cdots$$

Since $\alpha_{E(1)}(\ell) = \alpha_E(\ell) + 1 \geqq 0$, we have $h^1(\ell, E(1)|_\ell) = 0$, whence $h^1(P^2, E(1))$

$\leqq h^1(\boldsymbol{P}^2, E)$.

(2) By (1), we have $h^1(\boldsymbol{P}^2, E(1)) \leqq h^1(\boldsymbol{P}^2, E)$. Let ℓ be a line on $\boldsymbol{P}^2$ with $\alpha_E(\ell) = M(E)$, then as above we obtain the following long exact sequence of cohomologies;

$$\cdots \longrightarrow H^0(\boldsymbol{P}^2, E(1)) \longrightarrow H^0(\ell, E(1)|_\ell) \longrightarrow H^1(\boldsymbol{P}^2, E) \longrightarrow H^1(\boldsymbol{P}^2, E(1)) \longrightarrow H^1(\ell, E(1)|_\ell) = (0) .$$

If $h^1(\boldsymbol{P}^2, E(1)) = h^1(\boldsymbol{P}^2, E)$, then $H^1(\boldsymbol{P}^2, E) \cong H^1(\boldsymbol{P}^2, E(1))$. Thus $\varphi: H^0(\boldsymbol{P}^2, E(1)) \to H^0(\ell, E(1)|_\ell)$ is surjective. By the way, let ℓ' be a line on $\boldsymbol{P}^2$ with $\alpha_E(\ell') = m(E)$ and x be the closed point of the intersection of ℓ and ℓ', then $\psi: H^0(\ell, E(1)|_\ell) \to E(1) \otimes k(x)$ is surjective since $\alpha_{E(1)}(\ell) = \alpha_E(\ell) + 1 \geqq 0$. On the other hand $\psi': H^0(\ell', E(1)|_{\ell'}) \to E(1) \otimes k(x)$ is not surjective because $\alpha_{E(1)}(\ell') = \alpha_E(\ell') + 1 \leqq -1$. Furthermore we have the following commutative diagram;

$$\begin{array}{ccc} H^0(\boldsymbol{P}^2, E(1)) & \xrightarrow{\varphi} & H^0(\ell, E(1)|_\ell) \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \psi} \\ H^0(\ell', E(1)|_{\ell'}) & \xrightarrow{\psi'} & E(1) \otimes k(x) \end{array}$$

On the one hand, $\psi \circ \varphi$ is surjective because so are φ and ψ. On the other hand, $\psi' \circ \varphi'$ is not surjective because not so is ψ'. This is a contradiction.

(3) Let x be any closed point of $\boldsymbol{P}^2$ and ℓ be a line passing through x. The assumptions $\alpha_E(\ell) \geqq m(E) \geqq -1$ and $h^1(\boldsymbol{P}^2, E(1)) = h^1(\boldsymbol{P}^2, E)$ imply that $H^0(\boldsymbol{P}^2, E(1)) \to H^0(\ell, E(1)|_\ell)$ is surjective and $H^1(\ell, E(1)|_\ell) \to E(1) \otimes k(x)$ is surjective for any closed point x. By this and Nakayama's lemma $E(1)$ is generated by its global sections.

Let X be a scheme defined over k and E_1, E_2 vector bundles on X. If E_1 is ample and E_2 is generated by its global sections, then $E_1 \otimes E_2$ is ample ([4] Corollary 1.9.). We get therefore the following proposition as a corollary to the above lemma.

PROPOSITION (2.2) *Let E be a vector bundle on $\boldsymbol{P}^2$ with $M(E) \geqq -1$, then $E(a)$ is ample for any integer $a \geqq h^1(\boldsymbol{P}^2, E) + 2$.*

Proof. Put $b = h^1(\boldsymbol{P}^2, E)$, then by Lemma (2.1) we have;

$$b = h^1(\boldsymbol{P}^2, E) \geqq h^1(\boldsymbol{P}^2, E(1)) \geqq \cdots \geqq h^1(\boldsymbol{P}^2, E(b)) \geqq 0 .$$

Hence there must be an integer c $(0 \leqq c \leqq b)$ such that $h^1(\boldsymbol{P}^1, E(c)) = h^1(\boldsymbol{P}^2, E(c+1))$. By Lemma (2.1), $E(c+1)$ is generated by its global sections. Hence $E(a)$ is ample for any integer $a \geqq b+2$ because $O_{\boldsymbol{P}^2}(n)$ is ample for any integer $n \geqq 1$.

For a vector bundle E of rank 2 on a scheme we know that $E^* \cong E \otimes (\det E)^*$ ([6] Lemma 3.7). We shall use this fact in the next lemma.

If E is a vector bundle on $\boldsymbol{P}^2$, we identify the Chern class $c_i(E)$ of E with an integer by its degree.

LEMMA (2.3) *Let E be a simple vector bundle of rank 2 on $\boldsymbol{P}^2$, then;*
(1) *If $c_1(E) \leqq 0$, then $H^0(\boldsymbol{P}^2, E) = (0)$.*
(2) *If $c_1(E) \geqq -6$, then $H^2(\boldsymbol{P}^2, E) = (0)$.*

Proof. We have $E^* \cong E \otimes (\det E)^* \cong E(c)$, where $c = -c_1(E)$. If $c_1(E) \leqq 0$, then E can be regarded as a subsheaf of E^*. Hence $H^0(\boldsymbol{P}^2, E) \subset H^0(\boldsymbol{P}^2, E^*)$. If $H^0(\boldsymbol{P}^2, E) \neq (0)$, then $H^0(\boldsymbol{P}^2, E^*) \neq (0)$. This contradicts to Lemma (1.3) and proves (1). The second assertion follows from (1) by the Serre duality.

Let E be a vector bundle of rank 2 on a non-singular projective surface S. Define an integer $\Delta(E)$ to be $c_1(E)^2 - 4c_2(E)$. It is easy to see that $-\Delta(E)$ is the second Chern class of $\operatorname{End}(E)$. Hence, if L is a line bundle on S, then $\Delta(E \otimes L) = \Delta(E)$. For given two integers c_1 and c_2, let $F(c_1, c_2)$ be the set of all simple vector bundles of rank 2 on $\boldsymbol{P}^2$ with i-th Chern class c_i $(i = 1, 2)$. Then $F(c_1, c_2)$ is not empty if and only if $c = c_1^2 - 4c_2$ is negative and is not equal to -4([6] Theorem 4.6). For a line bundle L on $\boldsymbol{P}^2$, we put $F(c_1, c_2)(L) = \{E \otimes L; E \in F(c_1, c_2)\}$. If c_1 is odd (resp. even), then for $L = O_{\boldsymbol{P}^2}(-(c_1+1)/2)$(resp. $O_{\boldsymbol{P}^2}(-c_1/2)$), $F(c_1, c_2)(L) = F(-1, n)$ (resp. $F(0, m)$) where $1 - 4n = c_1^2 - 4c_2$ (resp. $-4m = c_1^2 - 4c_2$). $F(-1, n)$(resp. $F(0, m)$) is not empty if and only if $n \geqq 1$ (resp. $m \geqq 2$).

Now we can compute a lower bound of $m(\ )$ for simple vector bundles of rank 2 on $\boldsymbol{P}^2$ with fixed Chern classes.

PROPOSITION (2.4) *If E is in $F(-1, n)$ (resp. $F(0, m)$), then;*

$$-n \leqq m(E) \leqq M(E) \leqq -1 \qquad (\textit{resp.}\ -m+1 \leqq m(E) \leqq M(E) \leqq 0)\,.$$

Proof. $M(E) \leqq -1$ (resp. $M(E) \leqq 0$) is obvious, because $c_1(E) = -1$ (resp. $c_1(E) = 0$). The Riemann-Roch theorem asserts that for a vector bundle E' of rank 2 on $\boldsymbol{P}^2$,

$$\chi(E') = 2 + \frac{3c_1(E')}{2} + \frac{c_1(E')^2 - 2c_2(E')}{2}.$$

Applying this to E we have $\chi(E) = 1 - n$ (resp. $2 - m$). On the other hand, by Lemma (2.3) $H^0(\boldsymbol{P}^2, E) = H^2(\boldsymbol{P}^2, E) = (0)$. Thus we obtain $h^1(\boldsymbol{P}^2, E) = n - 1$ (resp. $m - 2$). Let ℓ be any line on $\boldsymbol{P}^2$, then we have the following short exact sequence;

$$0 \longrightarrow E(-1) \longrightarrow E \longrightarrow E|_\ell \longrightarrow 0$$

and the long exact sequence of cohomologies;

$$\cdots \longrightarrow H^1(\boldsymbol{P}^2, E) \longrightarrow H^1(\ell, E|_\ell) \longrightarrow H^2(\boldsymbol{P}^2, E(-1)) \longrightarrow \cdots$$

Since $H^2(\boldsymbol{P}^2, E(-1)) = (0)$ by Lemma (2.3), we obtain $h^1(\ell, E|_\ell) \leqq n - 1$ (resp. $m - 2$). Hence $\alpha_E(\ell) \geqq -n$ (resp. $-m + 1$) for any line ℓ.

LEMMA (2.5) *Let E be in $F(-1, n)$ (resp. $F(0, m)$). We put $b =$* $\min\{x; H^0(\boldsymbol{P}^2, E(x)) \neq (0)\}$ *(b is positive because $c_1(E(b))$ must be positive by Lemma* (2.3)*). Then $E(a)$ is ample for any integer $a \geqq n - b^2 + b + 1$ (resp. $m - b^2 + 1$).*

Proof. First we shall prove that $M(E(b)) \geqq 0$. Let L be the tautological line bundle on $\boldsymbol{P}(E(b))$, then $H^0(\boldsymbol{P}(E(b)), L) \cong H^0(\boldsymbol{P}^2, E(b)) \neq (0)$. Take a member D of the linear system $|L|$, then Supp (D) contains only a finite number of fibres of the projection $\pi: \boldsymbol{P}(E(b)) \to \boldsymbol{P}^2$. For if otherwise, there is an effective divisor C on $\boldsymbol{P}^2$ such that $D - \pi^{-1}(C) > 0$, i.e. $H^0(\boldsymbol{P}(E(b)), L \otimes \pi^*(O_{\boldsymbol{P}^2}(-C))) \neq (0)$. Meanwhile this is isomorphic to $H^0(\boldsymbol{P}^2, E(b) \otimes O_{\boldsymbol{P}^2}(-C))$. Thus by the definition of b, C must be linearly equivalent to zero, which is not the case. Hence for a generic line ℓ on $\boldsymbol{P}^2$, $D|_{\pi^{-1}(\ell)}$ is a section of the rational ruled surface $\pi^{-1}(\ell) \cong \boldsymbol{P}(E(b)|_\ell)$. On the otherhand, the self-intersection number $(D|_{\pi^{-1}(\ell)}, D|_{\pi^{-1}(\ell)})_{\pi^{-1}(\ell)} = c_1(E(b)) > 0$. Hence by Lemma (1.4), $(\pi|_\ell)_*(O_{\pi^{-1}(\ell)}(D|_{\pi^{-1}(\ell)})) \cong E(b)|_\ell$ is generated by its global sections. This shows that $M(E(b)) \geqq 0$.

The Chern classes of $E(b - 1)$ are;

$$c_1(E(b-1)) = 2b - 3 \qquad (\text{resp. } 2b - 2)$$
$$c_2(E(b-1)) = b^2 - 3b + 2 + n \qquad (\text{resp. } b^2 - 2b + 1 + m)$$

By the Riemann-Roch theorem, we obtain;

$$\chi(E(b-1)) = b^2 - n \qquad (\text{resp. } b^2 + b - m)$$

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On the other hand $H^0(\boldsymbol{P}^2, E(b-1)) = H^2(\boldsymbol{P}^2, E(b-1)) = (0)$. Hence we have $h^1(\boldsymbol{P}^2, E(b-1)) = n - b^2$ (resp. $m - b^2 - b$).

Combining these results, by Proposition (2.2) $E(b-1)(a')$ is ample for any integer $a^1 \geqq n - b^2 + 2$ (resp. $m - b^2 - b + 2$), i.e. $E(a)$ is ample for any integer $a \geqq n - b^2 + b + 1$ (resp. $m - b^2 + 1$).

COROLLARY (2.6) *If* $m(E) = -n$ (*resp.* $-m+1$), *then*;

(1) $M(E) \geqq -1$.

(2) $h^1(\boldsymbol{P}^2, E(a)) = n - 1 - a$ (*resp.* $m - 2 - a$) *for* $0 \leqq a \leqq n-1$ (*resp.* $0 \leqq a \leqq m-2$).

(3) *For an integer a the following conditions are equivalent to each other*;

i) $E(a)$ *is ample.*

ii) $a \geqq n+1$ (*resp.* m).

iii) $c_1(E(a)) \geqq -(1/2)\Delta(E(a))$.

Proof. (3) ii) $\Leftrightarrow$ iii). $c_1(E(a)) = 2a - 1$ (resp. $2a$) and $\Delta(E(a)) = 1 - 4n$ (resp. $-4m$). Hence $c_1(E(a)) \geqq -(1/2)\Delta(E(a))$ if and only if $a \geqq n+1$ (resp. m).

ii) $\Rightarrow$ i). $n + 1 \geqq n - b^2 + b + 1$ (resp. $m \geqq m - b^2 + 1$) for any $b \geqq 1$. Hence $E(a)$ is ample by Lemma (2.5).

i) $\Rightarrow$ ii). If $E(a)$ is ample, then $m(E(a)) = m(E) + a \geqq 1$. Hence $a \geqq -m(E) + 1 \geqq n + 1$ (resp. m).

(1) In the proof of (3), b must be equal to 1. Hence $M(E(1)) \geqq 0$ as we have shown in the proof of Lemma (2.5), i.e. $M(E) \geqq -1$.

(2) By the assumption $m(E) = -n$ (resp. $-m+1$) and (1), we have $M(E(a)) \geqq -1 > m(E(a))$ for $0 \leqq a \leqq n-2$ (resp. $0 \leqq a \leqq m-3$). Hence by Lemma (2.1), we obtain;

$$h^1(\boldsymbol{P}^2, E) > h^1(\boldsymbol{P}^2, E(1)) > \cdots > h^1(\boldsymbol{P}^2, E(n-1))$$
$$(\text{resp. } h^1(\boldsymbol{P}^2, E) > h^1(\boldsymbol{P}^2, E(1)) > \cdots > h^1(\boldsymbol{P}^2, E(m-2)))\,.$$

Since $h^1(\boldsymbol{P}^2, E) = n - 1$ (resp. $m - 2$), this shows the assertion.

In the proof of Corollary (2.6) (3), we did not use the assumption $m(E) = -n$ (resp. $m(E) = -m + 1$) to show iii) $\Rightarrow$ i). Thus, we have proved the following;

THEOREM 1. *If E is a simple vector bundle of rank 2 on $\boldsymbol{P}^2$ with $c_1(E) \geqq -(1/2)\Delta(E)$, then E is ample.*

Remark (1) Theorem 1. is best possible in the following senses;

i) For any integer $n \geqq 1$, there exists a simple vector bundle E in $F(-1, n)$ such that $m(E) = -n$, i.e. $E(a)$ is ample if and only if $c_1(E(a)) \geqq -(1/2)\Delta(E(a))$ (see Corollary (2.6) (3)).

ii) For any integers c_1 and c_2, let $F'(c_1, c_2)$ be the set of all vector bundles of rank 2 on $\boldsymbol{P}^2$ with its i-th Chern class being c_i, then $\inf\{m(E); E \text{ in } F'(c_1, c_2)\} = -\infty$ i.e. for any integer a, there exists a vector bundle E in $F'(c_1, c_2)$ such that $m(E) < a$. Hence we can not drop the hypothesis "simple".

For the construction of examples satisfying i) or ii), see [6] Theorem 4.6, Theorem 3.13.

Remark (2) If E is a simple vector bundle of rank 2 on $\boldsymbol{P}^2$ with $c_1(E) \geqq -(1/2)\Delta(E)$, then E can be written in the form $E' \otimes L$ where E' is generated by its global sections and L is a very ample line bundle, hence if k is the complex number field $\boldsymbol{C}$, E is positive in the sense of Griffiths [1].

§ 3. $H_{\alpha,\beta}$-stable vector bundles on a rational ruled surface.

For a non-negative integer n, let Σ_n be the rational ruled surface $\boldsymbol{P}(O_{\boldsymbol{P}^1}(-n) \oplus O_{\boldsymbol{P}^1})$, M a minimal section on Σ_n and N a fibre of the projection $\pi_n : \Sigma_n \to \boldsymbol{P}^1$. Then every line bundle on Σ_n is isomorphic to $O_{\Sigma_n}(aM + bN)$ for some integers a and b. We denote the line bundle $O_{\Sigma_n}(aM + bN)$ by $L_{a,b}$.

LEMMA (3.1) (1) *$L_{a,b}$ is ample if and only if a is positive and $b - na$ is positive.*

(2) *$L_{a,b}$ is generated by its glebal sections if and only if a is non-negative and $b - na$ is non-negative.*

Proof. If $L_{a,b}$ is ample, then the intersection numbers $(L_{a,b}, N) = a$ and $(L_{a,b}, M) = b - na$ are positive by the Nakai criterion. Conversely if a is positive and $b - na$ is positive, then the self-intersection number $(L_{a,b}, L_{a,b}) = -a^2n + 2ab > -a^2n + 2a^2n = a^2n \geqq 0$. Any curve C on Σ_n is linearly equivalent to $a'M + b'N$ for some non-negative integers a' and b' such that $(a', b') \neq (0, 0)$. Hence the intersection number $(L_{a,b}, C) = a'(L_{a,b}, M) + b'(L_{a,b}, M) = a'(-na + b) + b'a$ is positive. Therefore $L_{a,b}$ is ample by the Nakai criterion.

(2) If $L_{a,b}$ is generated by its global sections then the tensor product $L_{a,b} \otimes L_{1,n+1} = L_{a+1,b+n+1}$ is ample since $L_{1,n+1}$ is ample by (1). Hence

$a+1$ is positive and $-n(a+1)+b+n+1$ is positive i.e. a and $b-na$ are non-negative. Conversely if a and $b-na$ are non-negative, then $L_{a,b}$ is generated by its global sections. In fact, $L_{1,n}$ is generated by its global sections and $L_{0,1}$ is so. Hence $L_{a,b} = L_{1,n}^{\otimes a} \otimes L_{0,1}^{\otimes(b-na)}$ is generated by its global sections.

We denote the divisor $\alpha(M+nN)+\beta N$ by $H_{\alpha,\beta}$. Then the intersection numbers $(H_{\alpha,\beta}, N)$ and $(H_{\alpha,\beta}, M)$ are α and β respectively and Lemma (3.1) (1) is restated as follows; $H_{\alpha,\beta}$ is ample if and only if $\alpha > 0$ and $\beta > 0$. We also denote $H_{1,1} = M + (n+1)N$ by H, then H is very ample and any irreducible member of the linear system $|H|$ is isomorphic to the projective line $\boldsymbol{P}^1$. Let E be a vector bundle of rank r on Σ_n and ℓ be an irreducible member of the linear system $|H|$, then the restriction $E|_\ell$ of E to ℓ is isomorphic to direct sum $L_1 \oplus \cdots \oplus L_r$ of line bundles L_i's on ℓ. We set;

$$\alpha_E(\ell) := \min\{\deg L_i \,;\, 1 \leqq i \leqq r\}$$

and

$$M(E) = \max\{\alpha_E(\ell)\,;\, \ell \text{ is an irreducible member of } |H|\}$$
$$m(E) = \min\{\alpha_E(\ell)\,;\, \ell \text{ is an irreducible member of } |H|\}$$

If E is a vector bundle on Σ_n and D is a divisor on Σ_n, we put $E(D) = E \otimes O_{\Sigma_n}(D)$.

LEMMA (3.2) *Let E be a vector bundle on Σ_n then;*

(1) *If $M(E) \geqq -n-2$, then $h^1(\Sigma_n, E) \geqq h^1(\Sigma_n, E(H))$.*

(2) *If $M(E) \geqq -n-2 > m(E)$, then $h^1(\Sigma_n, E) > h^1(\Sigma_n, E(H))$.*

(3) *If $m(E) \geqq -n-2$ and $h^1(\Sigma_n, E) = h^1(\Sigma_n, E(H))$, then $E(H)$ is generated by its global sections.*

Proof. The self-intersection number (H, H) is $n+2$, so the proof is similar to that of Lemma (2.1). Hence we omit it.

The following proposition can be proved as a corollary to Lemma (3.2) and the proof is similar to that of Proposition (2.2).

PROPOSITION (3.3) *If E is a vector bundle on Σ_n with $M(E) \geqq -n-2$, then $E(aH)$ is ample for any integer $a \geqq h^1(\Sigma_n, E) + 2$.*

For any integers a, b and c, we set;

$$F_n(a, b; c) := \{E; E \text{ is a simple vector bundle of rank 2 on } \Sigma_n \text{ with } c_1(E) = aM + bN \text{ and } c_2(E) = c\}$$

If L is a line bundle on Σ_n, we also set;

$$F_n(a, b; c)(L) := \{E \otimes L; E \text{ is in } F_n(a, b; c)\}$$

Then for any integers a, b and c there exists a line bundle L on Σ_n such that;

(1) If a is even and b is even
$F_n(a, b; c)(L) = F_n(0, 0; r)$ where $-4r = -a^2n + 2ab - 4c$.
(2) If a is even and b is odd
$F_n(a, b; c)(L) = F_n(0, -1; r)$ where $-4r = -a^2n + 2ab - 4c$.
(3) If a is odd and b is even
$F_n(a, b; c)(L) = F_n(-1, 0; r)$ where $-n - 4r = -a^2n + 2ab - 4c$.
(4) If a is odd and b is odd
$F_n(a, b; c)(L) = F_n(-1, -1; r)$ where $-n + 2 - 4r = -a^2n + 2ab - 4c$.

M. Maruyama ([6] Theorem 4.15) proved that;

(1) $F_n(0, 0; r)$ is not empty if and only if $r \geqq 2$.
(2) $F_n(0, -1; r)$ is not empty if and only if $r \geqq 1$.
(3) $F_n(-1, 0; r)$ is not empty if and only if $r \geqq 1$.
(4) $F_n(-1, -1; r)$ is not empty if and only if $r \geqq 1$ when $n \neq 0$, $r \geqq 2$ when $n = 0$.

LEMMA (3.4) *Let E be a simple vector bundle of rank 2 on Σ_n with $c_1(E) = aM + bN$, then*
(1) *If $a \leqq 0$ and $b \leqq 0$, then $H^0(\Sigma_n, E) = (0)$.*
(2) *If $a \geqq -4$ and $b \geqq -2(n+2)$, then $H^2(\Sigma_n, E) = (0)$.*

Proof. The canonical line bundle on Σ_n is isomorphic to the line bundle $L_{-2,-n-2}$, so the proof is similar to that of Lemma (2.3).

We say that a set S of vector bundles on a k-scheme X is bounded if there exists an algebraic k-scheme T and a vector bundle V on $T \times X$ such that each E in S is isomorphic to $V_t = V|_{t \times X}$ for some closed point t in T.

THEOREM 2. *For any integers a, b and c, $F_n(a, b; c)$ is bounded.*

Proof. It is sufficient to prove the theorem for $-1 \leqq a$, $b \leqq 0$.

We shall prove the theorem for $F_n(0,0;r)$ only, since the other cases are similar. By a theorem of Kleiman ([5] Theorem 1.13), it is sufficient to show that there are integers m_1 and m_2 such that for any E in $F_n(0,0;r)$, i) $h^0(\Sigma_n, E) \leqq m_1$ and ii) $h^0(\ell, E|_\ell) \leqq m_2$ for a generic member ℓ of the linear system $|H|$. By Lemma (3.4), $h^0(\Sigma_n, E) = 0$ for any E in $F_n(0,0;r)$. We now show ii). The Riemann-Roch theorem asserts that for a vector bundle E' of rank 2 on Σ_n,

$$\chi(E') = 2 + \frac{(2M + (n+2)N, c_1(E'))}{2} + \frac{c_1(E')^2 - 2c_2(E')}{2} .$$

Applying this to E in $F_n(0,0;r)$, we have $\chi(E) = 2 - r$. On the other hand, by Lemma (3.4), $h^0(\Sigma_n, E) = h^2(\Sigma_n, E) = 0$. Thus we obtain $h^1(\Sigma_n, E) = r - 2$. Let ℓ be a generic member of the linear system $|H|$, then we have the following short exact sequence;

$$0 \longrightarrow E(-H) \longrightarrow E \longrightarrow E|_\ell \longrightarrow 0$$

and the long exact sequence of cohomologies;

$$\cdots \longrightarrow H^1(\Sigma_n, E) \longrightarrow H^1(\ell, E|_\ell) \longrightarrow H^2(\Sigma_n, E(-H)) \longrightarrow \cdots$$

Since $c_1(E(-H)) = -2M - 2(n+1)N$, $h^2(\Sigma_n, E(-H)) = 0$ by Lemma (3.4). Hence we obtain;

$$h^1(\ell, E|_\ell) \leqq r - 2 .$$

On the other hand, by the Riemann-Roch theorem for a vector bundle of rank 2 on the projective line, we have;

$$h^0(\ell, E|_\ell) - h^1(\ell, E|_\ell) = 2 + \deg(c_1(E|_\ell)) = 2 .$$

Hence we obtain $h^0(\ell, E|_\ell) \leqq r$.

LEMMA (3.5) *Let E be a simple vector bundle of rank 2 on Σ_n with $c_1(E) = aM + bN$ such that $-1 \leqq a, b \leqq 0$. Put $d = \min\{x; h^0(\Sigma_n, E(xH)) \neq 0\}$ (d is positive by Lemma (3.4)). If there exist integers α and β with $\alpha \geqq 1, \beta \geqq 1$ and $1/2 \leqq \beta/\alpha \leqq n+3$ if $n \neq 0$, $1/3 \leqq \beta/\alpha \leqq 3$ if $n = 0$ such that E is $H_{\alpha,\beta}$-stable, then $M(E(dH)) \geqq 0$.*

Proof. We shall prove the theorem for $a = 0$ and $b = 0$ only since the other cases are similar. Let X be the projective bundle $\boldsymbol{P}(E(dH))$ on $\Sigma_n, \pi: X \to \Sigma_n$ the projection and L the tautological line bundle on X. Let D' be a member of the linear system $|L|$ on X, then D' can be

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written in the form $D' = D + \pi^{-1}(C)$ where D is an irreducible divisor on X and C is an effective divisor on Σ_n i.e. C is linearly equivalent to $xM + yN$ $(x \geqq 0,\ y \geqq 0)$. Put $E' = \pi_*(O_X(D)) \cong E(dH - xM - yN)$. Let ℓ be a generic member of the linear system $|H|$ on Σ_n, then $D|_{\pi^{-1}(\ell)}$ is a section of the rational ruled surface $\pi^{-1}(\ell)$ and the self-intersection number $(D|_{\pi^{-1}(\ell)}, D|_{\pi^{-1}(\ell)})_{\pi^{-1}(\ell)} = (c_1(E(dH - xM - yN)), H) = 2d(n+2) - 2(x+y)$. If $2d(n+2) - 2(x+y) \geqq 0$, then $\alpha_{E'}(\ell) \geqq 0$ by Lemma (1.4). Hence $\alpha_{E(dH)}(\ell) = \alpha_{E'}(\ell) + x + y \geqq 0$, therefore $M(E(dH)) \geqq 0$. If $2d(n+2) - 2(x+y) < 0$, then $\alpha_{E'}(\ell) = 2d(n+2) - 2(x+y)$ by Lemma (1.4). Hence $\alpha_{E(dH)}(\ell) = 2d(n+2) - (x+y)$. We shall show that $2d(n+2) \geqq x + y$. Now assume that $2d(n+2) < x + y$, then we shall show a contradiction. Since $h^0(\Sigma_n, E') \neq 0$ and E' is $H_{\alpha,\beta}$-stable, $(c_1(E'), H_{\alpha,\beta}) = 2\beta(d - x) + 2\alpha(d(n+1) - y) > 0$ by Lemma (1.1), hence $\beta d + \alpha d(n+1) > \beta x + \alpha y$. We shall consider two cases i) $\beta \leqq \alpha$ and ii) $\beta \geqq \alpha$ separately.

i) Assume that $\beta \leqq \alpha$. If $n \neq 0$, then $\beta d + \alpha d(n+1) \leqq \alpha d(n+2)$ and $\beta x + \alpha y \geqq \beta(x+y)$, hence $\alpha d(n+2) > \beta(x+y) > 2\beta d(n+2)$. This contradicts to $1/2 \leqq \beta/\alpha$. If $n = 0$, then $3\beta \geqq \alpha$. Hence $\beta d + \alpha d \leqq 4\beta d$ and $\beta x + \alpha y \geqq \beta(x+y) > 4\beta d$, therefore $4\beta d > 4\beta d$. This is a contradiction.

ii) Assume that $\beta \geqq \alpha$. Then $\beta d + \alpha d(n+1) \leqq \alpha d(n+3) + \alpha d(n+1) = 2\alpha d(n+2)$, and $\beta x + \alpha y \geqq \alpha(x+y) > 2\alpha d(n+2)$. Hence $2\alpha d(n+2) > 2\alpha d(n+2)$, this is a contradiction.

For any integers a, b and c, we set;

$$F_n^0(a, b; c) := \{E \text{ in } F_n(a, b; c);\ E \text{ is } H_{\alpha,\beta}\text{-stable for some } \alpha \text{ and } \beta \text{ with } 1/2 \leqq \beta/\alpha \leqq n+3 \text{ if } n \neq 0,\ 1/3 \leqq \beta/\alpha \leqq 3 \text{ if } n = 0\}.$$

COROLLARY (3.6) (1) *If E is in $F_n^0(0, 0; r)$ then $E(rH)$ is ample.*
(2) *If E is in $F_n^0(0, -1; r)$ then $E((r+1)H)$ is ample.*
(3) *If E is in $F_n^0(-1, 0; r)$ then $E((r+1)H)$ is ample.*
(4) *If E is in $F_n^0(-1, -1; r)$ then $E((r+1)H)$ is ample.*

Proof. The proof is similar to that of Corollary (2.6), so we omit it.

THEOREM 3. *Let E be a simple vector bundle of rank 2 on Σ_n with $c_1(E) = aM + bN$. Assume that E is $H_{\alpha,\beta}$-stable for some $\alpha \geqq 1$ and $\beta \geqq 1$ such that $1/2 \leqq \beta/\alpha \leqq n+3$ if $n \neq 0$, $1/3 \leqq \beta/\alpha \leqq 3$ if $n = 0$, then the intersection numbers $(c_1(E), N) = a, (c_2(E), M) = b - na$ and;*

(1) *If a is even, b is even and $a \geqq 2r, b - na \geqq 2r$ where $-4r =$*

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$\Delta(E)$, then E is ample.

(2) *If a is even, b is odd and $a \geqq 2(r+1), b - na \geqq 2(r+1) - 1$ where $-4r = \Delta(E)$, then E is ampe.*

(3) *If a is odd, b is even and $a \geqq 2(r+1) - 1, b - na \geqq 2(r+1) + n$ where $-n - 4r = \Delta(E)$, then E is ample.*

(4) *If a is odd, b is odd and $a \geqq 2(r+1) - 1, b - na \geqq 2(r+1) + n - 1$ where $-n + 2 - 4r = \Delta(E)$, then E is ample.*

Proof. We shall prove the case (1) only since the other cases are similar. Let E be an $H_{\alpha,\beta}$-stable vector bundle of rank 2 which satisfies the conditions of (1), then E is written in the form $E'(rH) \otimes L_{a',b'}$ where E' is in $F_n^0(0,0;r)$ and $a' = a/2 - r, b' = b/2 - r(n+1)$. $E'(rH)$ is ample by Corollary (3.6) and $L_{a',b'}$ is generated by its global sections by Lemma (3.1) because $a' = a/2 - r \geqq 0$ and $b' - na' = b/2 - r(n+1) - n(a/2 - r) = 1/2(b - na - 2r) \geqq 0$, therefore $E = E'(rH) \otimes L_{a',b'}$ is ample.

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Nagoya University