Bull. Aust. Math. Soc. **109** (2024), 1–5
doi:10.1017/S0004972723000084

A NOTE ON REGULAR SETS IN CAYLEY GRAPHS

JUNYANG ZHANG and YANHONG ZHU

(Received 4 December 2022; accepted 3 January 2023; first published online 9 February 2023)

Abstract

A subset R of the vertex set of a graph Γ is said to be (κ, τ)-regular if R induces a κ-regular subgraph and every vertex outside R is adjacent to exactly τ vertices in R. In particular, if R is a (κ, τ)-regular set of some Cayley graph on a finite group G, then R is called a (κ, τ)-regular set of G. Let H be a nontrivial normal subgroup of G, and κ and τ a pair of integers satisfying $0 \le \kappa \le |H| - 1$, $1 \le \tau \le |H|$ and $\gcd(2, |H| - 1) \mid \kappa$. It is proved that (i) if τ is even, then H is a (κ, τ)-regular set of G; (ii) if τ is odd, then H is a (κ, τ)-regular set of G if and only if it is a $(0, 1)$-regular set of G.

2020 *Mathematics subject classification*: primary 05C25; secondary 05E18, 94B25.

Keywords and phrases: regular set, perfect code, Cayley graph, finite group.

1. Introduction

In the paper, all groups considered are finite groups with identity element denoted as 1, and all graphs considered are finite, undirected and simple. Let R be a subset of the vertex set of a graph Γ, and κ and τ a pair of nonnegative integers. We call R a (κ, τ)-*regular set* (or *regular set* for short if there is no need to emphasise the parameters κ and τ in the context) of Γ if every vertex in R is adjacent to exactly κ vertices in R and every vertex outside R is adjacent to exactly τ vertices in R. In particular, we call R a perfect code of Γ if $(\kappa, \tau) = (0, 1)$ and a total perfect code of Γ if $(\kappa, \tau) = (1, 1)$. The concept of (κ, τ)-*regular set* was introduced in [3] and further studied in [1, 2, 4, 5]. Very recently, regular sets in Cayley graphs were studied in [8, 9].

Let G be a group and X an inverse closed subset of $G \setminus \{1\}$. The Cayley graph $\mathrm{Cay}(G, X)$ on G with connection set X is the graph with vertex set G and edge set $\{\{g, gx\} \mid g \in G, x \in X\}$. A subset R of G is called a (κ, τ)-*regular set* of G if there is a Cayley graph Γ on G such that R is a (κ, τ)-regular set of Γ. Regular sets of Cayley graphs are closely related to codes of groups. Let C and Y be two subsets of G and λ a positive integer. If for every $g \in G$ there exist precisely λ pairs $(c, y) \in C \times Y$ such that

The first author was supported by the Natural Science Foundation of Chongqing (CSTB2022NSCQ-MSX1054) and the Foundation of Chongqing Normal University (21XLB006).

$g = cy$, then C is called a *code* of G with respect to Y [6]. In particular, if $\lambda = 1$ and Y is an inverse closed subset of G containing 1, then C is called a *perfect code* of G [7]. Let H be a subgroup of G. It is straightforward to check that H is a $(0, \tau)$-regular set of G if and only if H is a code of G with respect to some inverse closed subset of G. In fact, if H is a $(0, \tau)$-regular set of the Cayley graph $\mathrm{Cay}(G, X)$, then H is a code of G with respect to $Y := X \cup Z$ for any inverse closed subset Z of H with cardinality τ. However, if H is a code of G with respect to Y, then H is a $(0, \tau)$-regular set of the Cayley graph $\mathrm{Cay}(G, X)$, where $X = Y \setminus H$ and $\tau = |H||Y|/|G|$.

It is natural to ask when a normal subgroup of a group is a regular set. This question was studied by Wang *et al.* in [9]. They proved that, for any finite group G, if a nontrivial normal subgroup H of G is a perfect code of G, then for any pair of integers κ and τ with $0 \leq \kappa \leq |H| - 1$, $1 \leq \tau \leq |H|$ and $\gcd(2, |H| - 1) \mid \kappa$, H is also a (κ, τ)-regular set of G. It was also shown in [9] that there exist normal subgroups of some groups which are (κ, τ)-regular sets for some pair of integers κ and τ but not perfect codes of the group. In this paper, we extend the main results in [9] by proving the following theorem.

Theorem 1.1. *Let G be a group and H a nontrivial normal subgroup of G. Let κ and τ be a pair of integers satisfying $0 \leq \kappa \leq |H| - 1$, $1 \leq \tau \leq |H|$ and $\gcd(2, |H| - 1) \mid \kappa$. The following two statements hold:*

(i) *if τ is even, then H is a (κ, τ)-regular set of G;*
(ii) *if τ is odd, then H is a (κ, τ)-regular set of G if and only if it is a perfect code of G.*

It was proved in [7, Theorem 2.2] that a normal subgroup H of G is a perfect code of G if and only if

\# for any $g \in G$ with $g^2 \in H$, there exists $h \in H$ such that $(gh)^2 = 1$.

Note that condition # always holds if H is of odd order or odd index [7, Corollary 2.3]. Therefore, Theorem 1.1 has the following direct corollary, which is also an immediate consequence of [7, Corollary 2.3] and [9, Theorem 1.2].

Corollary 1.2. *Let G be a group and H a nontrivial normal subgroup of G. If either $|H|$ or $|G/H|$ is odd, then H is a (κ, τ)-regular set of G for every pair of integers κ and τ satisfying $0 \leq \kappa \leq |H| - 1$, $1 \leq \tau \leq |H|$ and $\gcd(2, |H| - 1) \mid \kappa$.*

Remark 1.3. It is a challenging question whether Theorem 1.1 and Corollary 1.2 can be generalised to nonnormal subgroups H of G.

Remark 1.4. Let H be a nontrivial normal subgroup of G of even order not satisfying condition #. Let κ and τ be a pair of integers satisfying $0 \leq \kappa \leq |H| - 1$, $2 \leq \tau \leq |H|$ and $2 \mid \tau$. Then Theorem 1.1(i) and [7, Theorem 2.2] imply that H is a (κ, τ)-regular set but not a perfect code of G.

https://doi.org/10.1017/S0004972723000084 Published online by Cambridge University Press

2. Proof of Theorem 1.1

Throughout this section, we use $\dot{\bigcup}_{i=1}^{n} S_i$ to denote the union of the pair-wise disjoint sets $S_1, S_2, \ldots, S_n$. Let G be a group and H a nontrivial normal subgroup of G. Let κ and τ be a pair of integers satisfying $0 \le \kappa \le |H| - 1$, $1 \le \tau \le |H|$ and $\gcd(2, |H| - 1) \mid \kappa$. We first prove three lemmas and then complete the proof of Theorem 1.1.

LEMMA 2.1. *If τ is even, then H is a $(0, \tau)$-regular set of G.*

PROOF. Let $A := \{1, a_1, \ldots, a_s\}$ be a left transversal of H in G. Assume that the number of involutions contained in a_iH is n_i for $1 \le i \le s$. Let σ be a permutation on $\{1, \ldots, s\}$ such that $a_i^{-1}H = a_{\sigma(i)}H$. Since H is normal in G,

$$a_{\sigma^2(i)}H = a_{\sigma(i)}^{-1}H = Ha_{\sigma(i)}^{-1} = (a_{\sigma(i)}H)^{-1} = (a_i^{-1}H)^{-1} = Ha_i = a_iH.$$

It follows that σ is the identity permutation or an involution. Assume that σ fixes t integers in $\{1, \ldots, s\}$. Then $0 \le t \le s$ and $s - t$ is even. Set $\ell := (s - t)/2$. Without loss of generality, we assume that

$$\sigma(i) = \begin{cases} i & \text{if } i \le t, \\ i + \ell & \text{if } t < i \le t + \ell, \\ i - \ell & \text{if } t + \ell < i \le s. \end{cases}$$

Then a_iH is inverse closed if $i \le t$ and $(a_{t+j}H)^{-1} = a_{t+j+\ell}H$ for every positive integer j not greater than ℓ. In particular, $n_i = 0$ if $i > t$. For every $i \in \{1, \ldots, s\}$, take a subset X_i of a_iH of cardinality τ by the following rules:

- if $i \le t$ and $n_i \ge \tau$, then X_i consists of exactly τ involutions;
- if $i \le t$, $n_i < \tau$ and $\tau - n_i$ is even, then X_i consists of n_i involutions and $(\tau - n_i)/2$ pairs of mutually inverse elements of order greater than 2;
- if $i \le t$, $n_i < \tau$ and $\tau - n_i$ is odd, then X_i consists of $n_i - 1$ involutions and $(\tau + 1 - n_i)/2$ pairs of mutually inverse elements of order greater than 2;
- if $t < i \le t + \ell$, then X_i consists of exactly τ elements of order greater than 2;
- if $i > t + \ell$, then set $X_i = X_{i-\ell}^{-1}$.

Note that $X_1, \ldots, X_s$ are pair-wise disjoint. Set $X = \dot{\bigcup}_{i=1}^{s} X_i$. Then X is an inverse closed subset of G satisfying $X \cap H = \emptyset$ and $|X \cap gH| = \tau$ for every $g \in G \setminus H$. It follows that H is a $(0, \tau)$-regular set of the Cayley graph $\mathrm{Cay}(G, X)$ and therefore a $(0, \tau)$-regular set of G. □

LEMMA 2.2. *If τ is odd, then H is a $(0, \tau)$-regular set of G if and only if it is a perfect code of G.*

PROOF. The sufficiency follows from [9, Theorem 1.2]. Now we prove the necessity. Let H be a $(0, \tau)$-regular set of the Cayley graph $\mathrm{Cay}(G, X)$. Then $X = X^{-1}$, $X \cap H = \emptyset$ and $|X \cap gH| = \tau$ for every $g \in G \setminus H$. Let $A := \{1, a_1, \ldots, a_s\}$ be a left transversal of H in G and set $X_i = X \cap a_iH$ for every $i \in \{1, 2, \ldots, s\}$. Then $X = \dot{\bigcup}_{i=1}^{s} X_i$. If X_i

https://doi.org/10.1017/S0004972723000084 Published online by Cambridge University Press

contains an involution for each $i \in \{1, \dots, s\}$, then H is a perfect code of G with respect to $\{1, y_1, \dots, y_s\}$, where y_i is an involution in X_i, $i = 1, \dots, s$. Now we assume that there exists at least one integer $k \in \{1, \dots, s\}$ such that X_k contains no involution. Then $x^{-1} \neq x$ for every element $x \in X_k$. It follows that $|X_k \cup X_k^{-1}|$ is even. Since $|X_k| = \tau$ and τ is odd, we get $X_k \neq X_k^{-1}$. Since H is normal in G, we obtain $(a_k H)^{-1} = (H a_k)^{-1} = a_k^{-1} H$. Assume that $a_k^{-1} H = a_j H$ for some $j \in \{1, \dots, s\}$. Then $X_k^{-1} \subseteq a_j H$. Since $X = \dot{\bigcup}_{i=1}^{s} X_i$ and $X^{-1} = X$, we conclude that $X_k^{-1} = X_j$. Therefore, without loss of generality, we can assume that $X_i^{-1} = X_{i+\ell}$ if $1 \leq i \leq \ell$ and $X_i^{-1} = X_i$ if $2\ell < i \leq s$, where ℓ is a positive integer not greater than $s/2$. Note that X_i contains at least one involution if $X_i^{-1} = X_i$ (as it is of odd cardinality). For every $i \in \{1, \dots, s\}$, take an element $y_i \in X_i$ by the following rules:

- y_i is an arbitrary element in X_i if $i \leq \ell$;
- $y_i = y_{i-\ell}^{-1}$ if $\ell < i \leq 2\ell$;
- y_i is an involution if $i > 2\ell$.

Then H is a perfect code of G with respect to $\{1, y_1, \dots, y_s\}$. □

LEMMA 2.3. *H is a (κ, τ)-regular set of G if and only if H is a $(0, \tau)$-regular set of G.*

PROOF. ($\Rightarrow$) Let H be a (κ, τ)-regular set of the Cayley graph $\mathrm{Cay}(G, X)$. Then $|H \cap X| = \kappa$ and $|gH \cap X| = \tau$ for every $g \in G \setminus H$. Set $Y = X \setminus H$. Then $|H \cap Y| = 0$ and $|gH \cap Y| = \tau$ for every $g \in G \setminus H$. Since $X^{-1} = X$ and $H^{-1} = H$, we get $Y^{-1} = Y$. It follows that H is a $(0, \tau)$-regular set of the Cayley graph $\mathrm{Cay}(G, Y)$ and therefore a $(0, \tau)$-regular set of G.

($\Leftarrow$) Let H be a $(0, \tau)$-regular set of the Cayley graph $\mathrm{Cay}(G, Y)$. Then $|H \cap Y| = 0$ and $|gH \cap Y| = \tau$ for every $g \in G \setminus H$. Let m be the number of elements contained in H of order greater than 2. Then m is even and the number of involutions contained in H is $|H| - 1 - m$. Recall that $0 \leq \kappa \leq |H| - 1$ and $\gcd(2, |H| - 1) \mid \kappa$. If κ is odd, then $|H|$ is even and therefore contains at least one involution. Take an inverse closed subset Z of H of cardinality κ by the following rules:

- if $m \geq \kappa$ and κ is even, then Z consists of exactly $\kappa/2$ pairs of mutually inverse elements of order greater than 2;
- if $m \geq \kappa$ and κ is odd, then Z consists of $(\kappa - 1)/2$ pairs of mutually inverse elements of order greater than 2 and one involution;
- if $m < \kappa$, then Z consists of $m/2$ pairs of mutually inverse elements of order greater than 2 and $\kappa - m$ involutions.

Set $X = Y \cup Z$. Then $|H \cap X| = \kappa$ and $|gH \cap X| = \tau$ for every $g \in G \setminus H$. Therefore, H is a (κ, τ)-regular set of the Cayley graph $\mathrm{Cay}(G, X)$ and therefore a (κ, τ)-regular set of G. □

PROOF OF THEOREM 1.1. Lemmas 2.1 and 2.3 imply that H is a (κ, τ)-regular set of G if τ is even. Now assume τ is odd. By Lemmas 2.2 and 2.3, H is a (κ, τ)-regular set of G if and only if it is a perfect code of G. □

https://doi.org/10.1017/S0004972723000084 Published online by Cambridge University Press

References

[1] M. Anđelić, D. M. Cardoso and S. K. Simić, 'Relations between (κ, τ)-regular sets and star complements', *Czechoslovak Math. J.* **63**(138) (2013), 73–90.

[2] D. M. Cardoso, 'An overview of (κ, τ)-regular sets and their applications', *Discrete Appl. Math.* **269** (2019), 2–10.

[3] D. M. Cardoso and P. Rama, 'Equitable bipartitions of graphs and related results', *J. Math. Sci.* **120**(1) (2004), 869–880.

[4] D. M. Cardoso and P. Rama, 'Spectral results on graphs with regularity constraints', *Linear Algebra Appl.* **423** (2007), 90–98.

[5] D. M. Cardoso, I. Sciriha and C. Zerafa, 'Main eigenvalues and (κ, τ)-regular sets', *Linear Algebra Appl.* **423** (2010), 2399–2408.

[6] H. M. Green and M. W. Liebeck, 'Some codes in symmetric and linear groups', *Discrete Math.* **343**(8) (2020), Article no. 111719.

[7] H. Huang, B. Z. Xia and S. M. Zhou, 'Perfect codes in Cayley graphs', *SIAM J. Discrete Math.* **32** (2018), 548–559.

[8] Y. Wang, B. Z. Xia and S. M. Zhou, 'Subgroup regular sets in Cayley graphs', *Discrete Math.* **345**(11) (2022), Article no. 113023.

[9] Y. Wang, B. Z. Xia and S. M. Zhou, 'Regular sets in Cayley graphs', *J. Algebr. Comb.*, to appear. Published online (26 October 2022).

JUNYANG ZHANG, School of Mathematical Sciences,
Chongqing Normal University, Chongqing 401331, PR China
e-mail: jyzhang@cqnu.edu.cn

YANHONG ZHU, School of Mathematical Sciences,
Liaocheng University, Liaocheng 252000, PR China
e-mail: zhuyanhong911@163.com

https://doi.org/10.1017/S0004972723000084 Published online by Cambridge University Press