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ANNIHILATORS OF POWER VALUES OF GENERALIZED DERIVATIONS ON MULTILINEAR POLYNOMIALS

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Abstract

Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, I a nonzero right ideal of R. Let $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, $m \geq 1$ a fixed integer, a a fixed element of R, g a generalized derivation of R. If $ag(f(r_1, \ldots, r_n))^m = 0$ for all $r_1, \ldots, r_n \in I$, then one of the following holds:

(1) $aI = ag(I) = (0)$;
(2) $g(x) = qx$, for some $q \in U$ and $aqI = 0$;
(3) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I;
(4) $g(x) = cx + [q, x]$ for all $x \in R$, where $c, q \in U$ such that $cI = 0$ and $[q, I]I = 0$.

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1. Introduction

Let R be a prime ring of characteristic different from 2. Throughout this paper $Z(R)$ always denotes the center of R, U the Utumi quotient ring of R and $C = Z(U)$, the center of U (C is usually called the extended centroid of R). We introduce on R an additive mapping d which satisfies the following rule: $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Such a mapping is called a derivation of R. Starting from this definition we also define a generalized derivation g of R as follows: g is an additive map on R, and there is a derivation d of R such that $g(xy) = g(x)y + xd(y)$ for all x, y in R. The simplest example of a generalized derivation is a map of the form $g(x) = ax + xb$, for some $a, b \in R$: such generalized derivations are called inner. Generalized inner derivations have primarily been studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see, for example, [11, 14, 16]). Here we shall consider some related problems concerning annihilators of power values of generalized derivations in prime rings.

Bresar [2] proves that if R is a semiprime ring, d a nonzero derivation of R and $a \in R$ such that $ad(x)^m = 0$, for all $x \in R$, where m is a fixed integer, then $ad(R) = 0$

when R is $(m-1)!$-torsion free. Lee and Lin [15] prove Bresar's result without the $(m-1)!$-torsion free assumption on R. They studied the Lie ideal case and, for the prime case, they showed that if R is a prime ring with a derivation $d \neq 0$, L a Lie ideal of R, $a \in R$ such that $ad(u)^m = 0$, for all $u \in L$, where m is fixed, then $ad(L) = 0$, except in the case where $\mathrm{char}(R) = 2$ and $\dim_C RC = 4$. In addition, if $[L, L] \neq 0$, then $ad(R) = 0$.

Chang and Lee [4] establish a unified version of the previous results for prime rings. Specifically, they prove the following theorem: let R be a prime ring, ϱ a nonzero right ideal of R, d a nonzero derivation of R, $a \in R$ such that $ad([x, y])^m \in Z(R)$ $(d([x, y])^m a \in Z(R))$. If $[\varrho, \varrho]\varrho \neq 0$ and $\dim_C RC > 4$, then either $ad(\varrho) = 0$ $(a = 0)$ or d is the inner derivation induced by some $q \in U$ such that $q\varrho = 0$.

In the first part of [3], Chang generalizes the above results by proving that if R is a prime ring with extended centroid C, I a nonzero right ideal of R, d a nonzero derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C, $a \in R$ and $m \geq 1$ a fixed integer such that $ad(f(r_1, \ldots, r_n))^m = 0$ for all $r_1, \ldots, r_n \in I$, then either $aI = d(I)I = (0)$ or $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I.

Here we shall continue the investigation of the properties of a subset S of R related to its left annihilator $\mathrm{Ann}_R(S) = \{x \in R \mid xS = (0)\}$. Specifically, we shall study the case where $S = \{g(f(x_1, \ldots, x_n))^m \mid x_1, \ldots, x_n \in R\}$, in which g is a generalized derivation on R, $f(x_1, \ldots, x_n)$ is a multilinear polynomial in n noncommuting variables over C and m is a fixed integer. We shall prove the following results.

THEOREM. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, I a nonzero right ideal of R. Let $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, $m \geq 1$ a fixed integer, a a fixed element of R, g a generalized derivation of R. If $ag(f(r_1, \ldots, r_n))^m = 0$ for all $r_1, \ldots, r_n \in I$, then one of the following holds:*

(1) $aI = ag(I) = (0)$;
(2) $g(x) = qx$, *for some* $q \in U$ *and* $aqI = 0$;
(3) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ *is an identity for* I;
(4) $g(x) = cx + [q, x]$ *for all* $x \in R$, *where* $c, q \in U$ *such that* $cI = 0$ *and* $[q, I]I = 0$.

Observe that if R is a domain, by supposing $a \neq 0$, we get $(g(f(r_1, \ldots, r_n)))^m = 0$, for any $r_1, \ldots, r_n \in I$. In this situation, thanks to a result contained in [1], one of the following holds:

(1) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I;
(2) $g(x) = qx$ for all $x \in R$, where $q \in U$ such that $qI = 0$;
(3) $g(x) = cx + [q, x]$ for all $x \in R$, where $c, q \in U$ such that $cI = 0$ and $[q, I]I = 0$.

In any case we are done. In light of this we shall always assume that R is not a domain.

We also recall that Lee [14] proved that every generalized derivation can be uniquely extended to a generalized derivation of U, and thus we implicitly assume that all generalized derivations of R are defined on the whole of U. Lee obtained the following result.

FACT 1. Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = cx + d(x)$, for some $c \in U$ and a derivation d on U.

Moreover, in all that follows we shall use the following notation:

$$f(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n + \sum_{\sigma \in S_n, \sigma \neq \mathrm{id}} \alpha_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$ and S_n the symmetric group of degree n. For any derivation d of R, we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma)$. Thus $d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$, for all $r_1, r_2, \ldots, r_n$ in R.

REMARK 2. Notice that one may write

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} h_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) x_i$$

where any h_i is a multilinear polynomial in $n-1$ variables over C, in which x_i never occurs. In this case, the hypothesis that $f(x_1, \ldots, x_n)$ is not an identity for I implies that there exists at least one $i \in \{1, \ldots, n\}$ such that $h_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is not an identity for I.

2. The case of inner generalized derivations

In this section we shall consider the generalized derivation $g(x) = cx + xb$, induced by suitable fixed elements $b, c \in R$. To prove our main result a number of lemmas are needed.

LEMMA 3. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, I a nonzero right ideal of R. Let $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, $m \geq 1$ a fixed integer, a, b, c fixed elements of R, such that $a(cf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b)^m = 0$ for all $r_1, \ldots, r_n \in I$. If $aI = 0$ then one of the following holds:*

(1) $acI = (0)$;

(2) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ *is an identity for I.*

PROOF. We assume, of course, that $f(x_1, \ldots, x_n)$ is not an identity for I. Since $aI = 0$, then $f(x_1 a, x_2, \ldots, x_n) = h_1(x_2, \ldots, x_n) x_1 a$, for $x_1, \ldots, x_n \in I$. In light of Remark 2, without loss of generality we suppose that $h_1(x_2, \ldots, x_n) x_1$ is not an identity for I. By hypothesis, I satisfies

$$a(cf(x_1 a, \ldots, x_n) + f(x_1 a, \ldots, x_n)b)^m$$

which is

$$a(ch_1(x_2, \ldots, x_n)x_1a)^m.$$

Thus $(ach_1(x_2, \ldots, x_n)I)^{m+1}$ is a generalized identity for I. By [9] we have that $ach_1(x_2, \ldots, x_n)I = 0$ and by the result in [7] it follows that either $acI = 0$ or $h_1(x_2, \ldots, x_n)x_1$ is an identity for I; in this last case we get a contradiction. □

REMARK 4. In order to prove our main result in the case of inner generalized derivations, in all that follows we may always suppose that $aI \neq 0$, if not we are done by the previous lemma.

LEMMA 5. *Let $k \geq 3$ and $R = M_k(F)$ be the ring of all $k \times k$ matrices over a field F of characteristic different from 2, I a nonzero right ideal of R, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over F and $m \geq 1$ a fixed integer. If a, b, c are fixed elements of R such that $a(cf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b)^m = 0$ for all $r_1, \ldots, r_n \in I$, then one of the following holds:*

(1) $aI = acI = (0)$;

(2) $[b, I]I = 0$ *and* $(c + b)I = 0$;

(3) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ *is an identity for* I.

PROOF. If $aI = 0$ we are done by Lemma 3. Therefore we may suppose that $aI \neq 0$. Denote by e_{uv} the usual unit matrix with 1 as (u, v)th entry and zero elsewhere, $a = \sum_{uv} a_{uv}e_{uv}$, $b = \sum_{uv} b_{uv}e_{uv}$, for $a_{uv}, b_{uv} \in F$.

Since there exists a set of matrix units that contains the idempotent generator of a given minimal right ideal, we observe that any minimal right ideal is part of a direct sum of minimal right ideals adding to R. In light of this and applying [10, Proposition 5, p. 52], we may assume that any minimal right ideal of R is a direct sum of minimal right ideals, each of the form $e_{ii}R$.

We know that I has a number of uniquely determined simple components: they are minimal right ideals of R and I is their direct sum. So we may write $I = eR$ for some $e = \sum_{i=1}^{p} e_{ii}$ and $p \in \{1, 2, \ldots, k\}$. Moreover $p \geq 2$, unless $[I, I]I = 0$, and *a fortiori* $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is an identity for I.

Suppose that b is not a diagonal matrix, for instance there exist $i \neq j$ such that $b_{ji} \neq 0$.

By [3, Lemma 3], if $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is not an identity for I, then for all $\alpha \in F$, $s \leq p$ and $t \neq s$ there exist $r_1, \ldots, r_n \in I$ such that $f(r_1, \ldots, r_n) = \alpha e_{st}$. By our hypothesis and considering this last evaluation of the polynomial on I, we have that

$$\alpha^m a(ce_{st} + e_{st}b)^m = 0$$

and right-multiplying by any e_{hh}, for $h \neq t$, it follows that $\alpha^m a(e_{st}b)^m e_{hh} = 0$, which means that

$$a_{qs}b_{th} = 0 \quad \forall q, \forall s \leq p, \forall t \neq s, \forall h \neq t. \tag{1}$$

In particular $b_{ji} \neq 0 \Longrightarrow a_{rs} = 0, \forall r, \forall s \leq p, \forall s \neq j$.

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If $j \geq p+1$, then by (1) $a = \sum_{r=1,t=p+1}^{k} a_{rt}e_{rt}$, that is $aI = 0$, a contradiction.

Thus we have $j \leq p$ and again by (1), $a = \sum_{r=1}^{k} a_{rj}e_{rj} + \sum_{r=1,t=p+1}^{k} a_{rt}e_{rt}$. Notice that if there exists $l \neq j$ such that $b_{ls} \neq 0$ for some $s \neq l$, then by (1) it follows that $a_{rj} = 0$ for all $r = 1, \ldots, k$, and as a consequence we should again have the contradiction $aI = 0$. Hence $b_{ls} = 0$ for all $l \neq j$ and $s \neq l$, that is, $b = \sum_{r=1,r\neq j}^{k} b_{jr}e_{jr} + \sum_{r=1}^{k} b_{rr}e_{rr}$.

Consider the following automorphisms of R:

$$\lambda(x) = (1+e_{ji})x(1-e_{ji}) = x + e_{ji}x - xe_{ji} - e_{ji}xe_{ji},$$
$$\mu(x) = (1-e_{ji})x(1+e_{ji}) = x - e_{ji}x + xe_{ji} - e_{ji}xe_{ji},$$

and note that $\lambda(I), \mu(I) \subseteq I$ are both right ideals of R satisfying respectively the following generalized identities:

$$\lambda(a)(\lambda(c)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)\lambda(b))^m,$$
$$\mu(a)(\mu(c)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)\mu(b))^m.$$

Denote $\lambda(a) = \sum \alpha_{uv}e_{uv}$, $\mu(a) = \sum \alpha'_{uv}e_{uv}$, $\lambda(b) = \sum \beta_{uv}e_{uv}$ and $\mu(b) = \sum \beta'_{uv}e_{uv}$.

By calculation, $\beta_{ji} = b_{ji} + b_{ii} - b_{jj}$ and $\beta'_{ji} = b_{ji} - b_{ii} + b_{jj}$. If $\beta_{ji} = \beta'_{ji} = 0$, since $\mathrm{char}(F) \neq 2$, we get the contradiction $b_{ji} = 0$. Thus either $\beta_{ji} \neq 0$ or $\beta'_{ji} \neq 0$. By (1) we have that either any sth column of $\lambda(a)$ is zero, for any $s \leq p$ and $s \neq j$, or any sth column of $\mu(a)$ is zero, for any $s \leq p$ and $s \neq j$.

Suppose that $i \leq p$: in this case either the ith column of $\lambda(a)$ is zero, or the ith column of $\mu(a)$ is zero, that is, either $\alpha_{ri} = 0$ for all $r = 1, \ldots, k$, or $\alpha'_{ri} = 0$ for all $r = 1, \ldots, k$. For $r \neq j$, either $0 = \alpha_{ri} = a_{ri} - a_{rj} = a_{rj}$ or $0 = \alpha'_{ri} = a_{ri} + a_{rj} = -a_{rj}$; in any case $a_{rj} = 0$ (in particular $a_{ij} = 0$). For $r = j$, either $0 = \alpha_{ji} = a_{ji} + a_{ii} - a_{jj} - a_{ij} = -a_{jj}$ or $0 = \alpha'_{ji} = a_{ji} - a_{ii} + a_{jj} - a_{ij} = a_{jj}$; in any case $a_{jj} = 0$. Therefore $a_{rj} = 0$ for all $r = 1, \ldots, k$, that is, $a = \sum_{r=1,t=p+1}^{k} a_{rt}e_{rt}$ and $aI = 0$, a contradiction. Thus we may assume that $i \geq p+1$. Since $k \geq 3$ and $p \geq 2$, there exists at least one index $q \neq i, j$ such that $q \leq p$.

Introduce now the following automorphisms of R:

$$\sigma(x) = (1+e_{qj})x(1-e_{qj}) = x + e_{qj}x - xe_{qj} - e_{qj}xe_{qj},$$
$$\tau(x) = (1-e_{qj})x(1+e_{qj}) = x - e_{qj}x + xe_{qj} - e_{qj}xe_{qj},$$

and also for this case note that $\sigma(I), \tau(I) \subseteq I$ are both right ideals of R satisfying respectively the following generalized identities:

$$\sigma(a)(\sigma(c)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)\sigma(b))^m,$$
$$\tau(a)(\tau(c)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)\tau(b))^m.$$

Denote $\sigma(a) = \sum \alpha''_{uv}e_{uv}$, $\tau(a) = \sum \alpha'''_{uv}e_{uv}$, $\sigma(b) = \sum \beta''_{uv}e_{uv}$ and $\tau(b) = \sum \beta'''_{uv}e_{uv}$.

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Remark that $\beta''_{qi} = b_{ji} \neq 0$ and also $\beta'''_{qi} = -b_{ji} \neq 0$. Therefore, by (1), any rth column of $\sigma(a)$ and $\tau(a)$ consists zeros, for all $r \leq p$ and $r \neq q$. In particular, $\alpha''_{sj} = \alpha'''_{sj} = 0$ for all $s = 1, \ldots, k$. By calculation,

$$0 = \alpha''_{qj} = a_{qj} + a_{jj},$$
$$0 = \alpha'''_{qj} = a_{qj} - a_{jj},$$

which means that $a_{qj} = 0$ and $a_{jj} = 0$, for all $q \neq i, j$ and $q \leq p$.

The previous argument says that if $b_{ji} \neq 0$ for some $i \neq j$, then $j \leq p$, $i \geq p + 1$ and $a = a_{ij}e_{ij} + \sum_{r=1,t=p+1}^{k} a_{rt}e_{rt}$ and we assume $a_{ij} \neq 0$, if not $aI = 0$.

Finally, for all $s \neq j$ with $s \leq p$, denote

$$\varphi(x) = (1 + e_{js})x(1 - e_{js}) = x + e_{js}x - xe_{js} - e_{js}xe_{js}.$$

Of course $\varphi(I) \subseteq I$ is a right ideal of R satisfying the generalized identity

$$\varphi(a)(\varphi(c)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)\varphi(b))^m.$$

Since $aI \neq 0$, then $\varphi(a)\varphi(I) \neq 0$. Consider now the (j, i)th entry of the matrix $\varphi(b)$, which is $b_{ji} + b_{si}$. If $b_{ji} + b_{si} = 0$, then $0 \neq b_{ji} = -b_{si}$, and as we said above, $a = a_{is}e_{is} + \sum_{r=1,t=p+1}^{k} a_{rt}e_{rt}$, that is, $a_{ij} = 0$, which is a contradiction.

On the other hand, if $b_{ji} + b_{si} \neq 0$, then any element on the sth column of $\varphi(a)$ is zero. This means that the (i, s)th entry of $\varphi(a)$ is zero, that is, $0 = a_{is} - a_{ij} = a_{ij}$, a contradiction again.

Therefore the assumption that $b_{ji} \neq 0$ leads to a number of contradictions. Hence b must be a diagonal matrix, and so must $\varphi(b)$. In particular, we note that the (j, s)th entry of the matrix $\varphi(b)$ is zero, that is, $b_{ss} - b_{jj} = 0$ and $b_{jj} = b_{ss} = \beta$, for all $s \leq p$. Thus $b = \sum_{r=1}^{p} \beta e_{rr} + \sum_{r=p+1}^{k} b_{rr}e_{rr}$. This means that $bI = \beta I$ and $[b, I]I = 0$.

Let $i \neq t$ and $i, t \leq p$. As above, there exist $r_1, \ldots, r_n \in I$ such that $f(r_1, \ldots, r_n) = \beta e_{it}$. Notice that $be_{it} = e_{it}b = \alpha e_{it}$. Thus, by our assumption, $a(ce_{it} + \alpha e_{it})^m = 0$. Denoting $a' = a(c + \alpha)$,, we may write $a'e_{it}(ce_{it})^{m-1} = 0$. Here we denote $c = \sum_{uv} c_{uv}e_{uv}$, $a' = \sum_{uv} \gamma_{uv}e_{uv}$, for $c_{uv}, \gamma_{uv} \in F$. Hence,

$$\gamma_{ri}c_{ti} = 0, \quad \forall r = 1, \ldots, k \quad \forall t \neq i, \quad \text{and} \quad t, i \leq p. \tag{2}$$

In particular if $\exists j \neq i$, $j \leq p$ such that $c_{ji} \neq 0$ then $\gamma_{ri} = 0$, $\forall r = 1, \ldots, k$.

Suppose in all that follows that $c_{ji} \neq 0$ for some $j \neq i$ and $j \leq p$. Thus $\gamma_{ri} = 0$ for all $r = 1, \ldots, k$.

Let $s \neq i$ and $s \leq p$. We have of course two choices: either $s \neq j$ (only if $p \geq 3$) or $s = j$.

If $s \neq j$, consider the following automorphisms of R:

$$\omega(x) = (1 + e_{is})x(1 - e_{is}) = x + e_{is}x - xe_{is} - e_{is}xe_{is},$$
$$\chi(x) = (1 - e_{is})x(1 + e_{is}) = x - e_{is}x + xe_{is} - e_{is}xe_{is},$$

and remark that $\omega(e_{it}) = \chi(e_{it}) = e_{it}$.

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Hence $\omega(a(ce_{it}+\beta e_{it})^m)=0$ and $\omega(a')e_{it}(\omega(c)e_{it})^{m-1}=0$. Analogously $\chi(a(ce_{it}+\beta e_{it})^m)=0$ and $\chi(a')e_{it}(\chi(c)e_{it})^{m-1}=0$. Denote $\omega(c)=\sum\omega_{uv}e_{uv}$, $\chi(c)=\sum\chi_{uv}e_{uv}$, $\omega(a')=\sum\omega'_{uv}e_{uv}$ and $\chi(a')=\sum\chi'_{uv}e_{uv}$.

If $\omega_{js}\neq 0$ and $\chi_{js}\neq 0$ then, by (2), $\omega'_{rs}=0$ and $\chi'_{rs}=0$ for all index r. For $r\neq i$ we have $0=\omega'_{rs}=\gamma_{rs}-\gamma_{ri}=\gamma_{rs}$. If $r=i$, then

$$0=\omega'_{rs}=\omega'_{is}=\gamma_{rs}+\gamma_{ss}-\gamma_{rr}-\gamma_{sr}=\gamma_{rs}+\gamma_{ss}$$

and

$$0=\chi'_{rs}=\chi'_{is}=\gamma_{rs}-\gamma_{ss}+\gamma_{rr}-\gamma_{sr}=\gamma_{rs}-\gamma_{ss}.$$

Since $\mathrm{char}(F)\neq 2$ it follows that $\gamma_{rs}=0$.

On the other hand, if either $\omega_{js}=0$ or $\chi_{js}=0$, then either $c_{js}=c_{ji}$ or $c_{js}=-c_{ji}$, in any case $c_{js}\neq 0$ and, by (2), $\gamma_{rs}=0$ for all r and $s\leq p$.

Hence if $s\neq j$ and $s\leq p$, we always have that $\gamma_{rs}=0$ for all $r=1,\dots,k$.

Consider now the case where $s=j$. Rewrite the previous automorphisms as follows:

$$\omega(x)=(1+e_{ij})x(1-e_{ij})=x+e_{ij}x-xe_{ij}-e_{ij}xe_{ij},$$
$$\chi(x)=(1-e_{ij})x(1+e_{ij})=x-e_{ij}x+xe_{ij}-e_{ij}xe_{ij},$$

and again let $\omega(c)=\sum\omega_{uv}e_{uv}$, $\chi(c)=\sum\chi_{uv}e_{uv}$, $\omega(a')=\sum\omega'_{uv}e_{uv}$ and $\chi(a')=\sum\chi'_{uv}e_{uv}$.

If $\omega_{ij}=\chi_{ij}=0$ then, by calculation, it follows that

$$0=c_{ij}+c_{jj}-c_{ii}-c_{ji}=c_{ij}-c_{jj}+c_{ii}-c_{ji},$$

which means that $c_{ij}-c_{ji}=0$. This implies that $c_{ij}\neq 0$ and, by (2), $\gamma_{rj}=0$ for all index r.

If either $\omega_{ij}\neq 0$ or $\chi_{ij}\neq 0$ then, again by (2), either $\omega'_{rj}=0$ for all r, or $\chi'_{rj}=0$ for all r. From these it follows that if $r\neq i$, by calculation we have in any case that $\gamma_{rj}=0$. If $r=i$ the calculation says that $0=\gamma_{ij}\pm\gamma_{jj}=\gamma_{ij}$. Thus we have $\gamma_{tj}=0$ for all $t=1,\dots,k$.

In other words, we have seen that if there exist $i,j\leq p$ and $i\neq j$ such that $c_{ji}\neq 0$ then $a'I=0$, that is, $0=a(c+\beta)I=a(c+b)I$, because $bI=\beta I$, and we are done.

Now suppose that $c_{ji}=0$, for all $i\neq j$ and $i,j\leq p$. Let $s\geq p+1$ and denote

$$\varphi(x)=(1+e_{is})x(1-e_{is})=x+e_{is}x-xe_{is}-e_{is}xe_{is}$$

and $\varphi(c)=\sum\varphi_{uv}e_{uv}$,with $\varphi_{uv}\in F$. Of course $\varphi(I)\subseteq I$ is a right ideal of R and $\varphi(e_{it})=e_{it}$. By previous argument, the (i,t)th entry of the matrix $\varphi(c)$ is zero, that is $0=c_{it}+c_{st}=c_{st}$. This means that $c=\sum_{l=1}^{p}c_{ll}e_{ll}+\sum_{h=1,r=p+1}c_{hr}e_{hr}$.

On the other hand, for any index $q\neq i$ and $q\leq p$, if we consider

$$\varphi'(x)=(1+e_{iq})x(1-e_{iq})=x+e_{iq}x-xe_{iq}-e_{iq}xe_{iq}$$

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then the (i,q)th entry of the matrix $\varphi'(c)$ is zero, that is, $c_{qq} = c_{ii} = \gamma$. Since this holds for all $q \le p$, we write $c = \sum_{l=1}^{p} \gamma e_{ll} + \sum_{h=1,r=p+1} c_{hr} e_{hr}$. Thus $cI = \gamma I$. Since I satisfies the generalized identity $a(cf(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)b)^m$, then *a fortiori* I satisfies $a(cf(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)b)^m f(x_1,\ldots,x_n)$, that is, $a(\gamma+\beta) f(x_1,\ldots,x_n)^{m+1} = 0$.

If we assume that $(\gamma+\beta) \ne 0$, then by [7] we conclude that either $aI = 0$ or $[f(x_1,\ldots,x_n), x_{n+1}]x_{n+2}$ is an identity for I.

In case $\gamma = -\beta$, then $cI = -bI = \gamma I$ and $(c+b)I = 0$. □

LEMMA 6. *Let $R = M_2(F)$ be the ring of all 2×2 matrices over a field F of characteristic different from 2, I a nonzero right ideal of R, $f(x_1,\ldots,x_n)$ a noncentral multilinear polynomial over F and $m \ge 1$ a fixed integer. If a,b,c are fixed elements of R such that $a(cf(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)b)^m = 0$ for all $r_1,\ldots,r_n \in I$, then one of the following holds:*

(1) $a = 0$;
(2) $b \in Z(R)$ *and* $a(c+b) = 0$;
(3) $[I,I]I = (0)$.

PROOF. If $I = eR$ for an idempotent $e \in R$ of rank 1, then $[I,I]I = 0$. Then we assume that $I = e_{11}R + e_{22}R = R = M_2(F)$. Denote $a = \sum a_{uv}e_{uv}$ and $b = \sum b_{uv}e_{uv}$. Since $f(x_1,\ldots,x_n)$ is not central, by [17, Lemma 2 and proof of Lemma 3] for suitable $i \ne j$ and $\alpha \in F$ there exists $s_1,\ldots,s_n \in R$ such that $\alpha e_{ij} = f(s_1,\ldots,s_n)$. Moreover, since the set $f(R) = \{f(r_1,\ldots,r_n) \mid r_1,\ldots,r_n \in R\}$ is invariant under the action of all F-automorphisms, then for all $i \ne j$ and $\alpha \in F$ there exists $s_1,\ldots,s_n \in R$ such that $\alpha e_{ij} = f(s_1,\ldots,s_n)$. By our assumption $\alpha^m a(ce_{ij} + e_{ij}b)^m = 0$ and right-multiplying by e_{ii}, it follows that $ab_{ji}e_{ii} = 0$, that is, either $b_{ji} = 0$ or $a_{ii} = a_{ji} = 0$. Suppose that b is not a diagonal matrix; then without loss of generality we may assume that $b_{21} \ne 0$. Hence $a_{21} = a_{11} = 0$. For the same reason, if $a \ne 0$, we must suppose that $b_{12} = 0$.

Now let $f(r_1,\ldots,r_n) = \alpha e_{12} \ne 0$. Since the set $f(R)$ is invariant under the action of all inner automorphisms of R, $(1-e_{21})(\alpha e_{12})(1+e_{21}) \in f(R)$, then $\alpha(e_{11} + e_{12} - e_{21} - e_{22}) \in f(R)$. By the hypothesis

$$a(c(e_{11}+e_{12}-e_{21}-e_{22}) + (e_{11}+e_{12}-e_{21}-e_{22})b)^m = 0$$

and right-multiplying by $(e_{11} - e_{21})$, we get

$$\begin{aligned}
0 &= a(c(e_{11}+e_{12}-e_{21}-e_{22}) + (e_{11}+e_{12}-e_{21}-e_{22})b)^m (e_{11}-e_{21}) \\
&= a((e_{11}+e_{12}-e_{21}-e_{22})b)^m (e_{11}-e_{21}) \\
&= \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11}+b_{21} & b_{22} \\ -b_{11}-b_{21} & -b_{22} \end{bmatrix}^m \cdot \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} \cdot \begin{bmatrix} (b_{11}+b_{21}-b_{22})^m & 0 \\ -(b_{11}+b_{21}-b_{22})^m & 0 \end{bmatrix},
\end{aligned}$$

that is, either $a_{12} = a_{22} = 0$ or $b_{11} + b_{21} - b_{22=0}$.

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Following the same argument as above, we note that $(e_{11}-e_{12}+e_{21}-e_{22}) \in f(R)$ and

$$a(c(e_{11}-e_{12}+e_{21}-e_{22})+(e_{11}-e_{12}+e_{21}-e_{22})b)^m=0.$$

Right-multiplying by $(e_{11}+e_{21})$ gives

$$\begin{aligned}0&=a(c(e_{11}-e_{12}+e_{21}-e_{22})+(e_{11}-e_{12}+e_{21}-e_{22})b)^m(e_{11}+e_{21})\\&=\begin{bmatrix}0 & a_{12}\\ 0 & a_{22}\end{bmatrix}\cdot\begin{bmatrix}b_{11}-b_{21} & -b_{22}\\ b_{11}-b_{21} & -b_{22}\end{bmatrix}^m\cdot\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\\&=\begin{bmatrix}0 & a_{12}\\ 0 & a_{22}\end{bmatrix}\cdot\begin{bmatrix}(b_{11}-b_{21}-b_{22})^m & 0\\ (b_{11}-b_{21}-b_{22})^m & 0\end{bmatrix},\end{aligned}$$

that is, either $a_{12}=a_{22}=0$ or $b_{11}-b_{21}-b_{22}=0$.

If we suppose that $a\neq 0$, then both $b_{11}+b_{21}-b_{22}=0$ and $b_{11}-b_{21}-b_{22}=0$ hold. Therefore, we get the contradiction $b_{21}=0$.

In other words, if $a\neq 0$ then b is a diagonal matrix. Moreover, let

$$\varphi(x)=(1+e_{12})x(1-e_{12})=x+e_{12}x-xe_{12}-e_{12}xe_{12}$$

be an automorphism of R, with $\varphi(b)=\sum\varphi_{uv}e_{uv}$, for $\varphi_{uv}\in F$. Of course the following holds: for all $r_1,\ldots,r_n\in R$,

$$\varphi(a)(\varphi(c)f(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)\varphi(b))^m=0.$$

Since $a\neq 0$, we also have $\varphi(a)\neq 0$, therefore, by the previous argument, $\varphi_{12}=0$, that is, $b_{22}=b_{11}$ and b is central on R.

Thus $a((c+b)f(r_1,\ldots,r_n))^m=0$ for all $r_1,\ldots,r_n\in R$. Write $c+b=q=\sum q_{uv}e_{uv}$, and let $aq=\sum\gamma_{uv}e_{uv}$, for all $q_{uv},\gamma_{uv}\in F$.

As above, for $i\neq j$, consider the evaluation $0\neq\alpha e_{ij}\in f(R)$ and, by hypothesis, $(aq)(\alpha e_{ij}q)^{m-1}\alpha e_{ij}=0$. By calculation, it follows that either $q_{ji}=0$ or $\gamma_{ii}=\gamma_{ji}=0$.

Suppose that the matrix q is not diagonal, for instance let $q_{21}\neq 0$. Thus $\gamma_{11}=\gamma_{21}=0$. If $aq=0$ we are done. If $aq\neq 0$, by previous argument $q_{12}=0$.

Also in this case we continue the proof by choosing some different evaluation of $f(x_1,\ldots,x_n)$. Let $s_1,\ldots,s_n\in R$ such that $f(s_1,\ldots,s_n)=(e_{11}+e_{12}-e_{21}-e_{22})$. By the hypothesis

$$a(q(e_{11}+e_{12}-e_{21}-e_{22}))^m=0$$

and right-multiplying by e_{11} we get

$$\begin{aligned}0&=aq((e_{11}+e_{12}-e_{21}-e_{22})q)^{m-1}(e_{11}+e_{12}-e_{21}-e_{22})e_{11}\\&=\begin{bmatrix}0 & \gamma_{12}\\ 0 & \gamma_{22}\end{bmatrix}\cdot\begin{bmatrix}q_{11}+q_{21} & q_{22}\\ -q_{11}-q_{21} & -q_{22}\end{bmatrix}^{m-1}\cdot\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}\\&=\begin{bmatrix}0 & \gamma_{12}\\ 0 & \gamma_{22}\end{bmatrix}\cdot\begin{bmatrix}(q_{11}+q_{21}-q_{22})^{m-1} & 0\\ -(q_{11}+q_{21}-q_{22})^{m-1} & 0\end{bmatrix},\end{aligned}$$

that is, $q_{11}+q_{21}-q_{22}=0$, since $aq\neq 0$.

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Again let $(e_{11} - e_{12} + e_{21} - e_{22}) \in f(R)$ and

$$a(q(e_{11} - e_{12} + e_{21} - e_{22}))^m = 0.$$

Right-multiplying by e_{11},

$$\begin{aligned} 0 &= aq((e_{11} - e_{12} + e_{21} - e_{22})q)^{m-1}(e_{11} - e_{12} + e_{21} - e_{22})e_{11} \\ &= \begin{bmatrix} 0 & \gamma_{12} \\ 0 & \gamma_{22} \end{bmatrix} \cdot \begin{bmatrix} q_{11} - q_{21} & -q_{22} \\ q_{11} - q_{21} & -q_{22} \end{bmatrix}^{m-1} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \gamma_{12} \\ 0 & \gamma_{22} \end{bmatrix} \cdot \begin{bmatrix} (q_{11} - q_{21} - q_{22})^{m-1} & 0 \\ (q_{11} - q_{21} - q_{22})^{m-1} & 0 \end{bmatrix}, \end{aligned}$$

that is, $q_{11} - q_{21} - q_{22} = 0$.

Then $q_{11} + q_{21} - q_{22} = q_{11} - q_{21} - q_{22} = 0$ implies the contradiction $q_{21} = 0$. Also in this case we conclude that, if $aq \neq 0$, then q must be a diagonal matrix. By using the same argument as above one can show that q is a central matrix for R (we omit this for the sake of brevity). All this means that $b + c \in Z(R)$. If $c + b = 0$ we are done. In the other case, $(c + b)^m a(f(r_1, \ldots, r_n))^m = 0$ for all $r_1, \ldots, r_n \in R$, and since $c + b$ is not a zero divisor it follows that $a(f(r_1, \ldots, r_n))^m = 0$ for all $r_1, \ldots, r_n \in R$. In this last case, by [7], either $a = 0$ or $f(x_1, \ldots, x_n)$ is an identity for R, a contradiction. □

LEMMA 7. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, I a nonzero right ideal of R. Let $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, $m \geq 1$ a fixed integer, a, b, c fixed elements of R, such that $a(cf(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)b)^m = 0$ for all $r_1, \ldots, r_n \in I$. If R does not satisfy any nontrivial generalized polynomial identity, then one of the following holds:*

(1) $aI = acI = (0)$;
(2) $b \in C$ *and* $a(c + b)I = 0$;
(3) $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ *is an identity for* I.

PROOF. Of course if $aI = 0$ we are done again by Lemma 3. Therefore we assume that $aI \neq 0$. Let $T = U *_C C\{X\}$ be the free product over C of the C-algebra U and the free C-algebra $C\{X\}$, with X the countable set consisting of noncommuting indeterminates $x_1, x_2, \ldots, x_n, \ldots$. Recall that if B is a basis of U over C, then any element of $T = U *_C C\{x_1, \ldots, x_n\}$ can be written in the form $G = \sum_i \alpha_i m_i$, where $\alpha_i \in C$ and m_i are B-monomials, that is, $m_i = q_0 y_1 \cdots y_n q_n$, with $q_i \in B$ and $y_i \in \{x_1, \ldots, x_n\}$. Chuang [6] shows that a generalized polynomial $G = \sum_i \alpha_i m_i$ is the zero element of T if and only if any α_i is zero. As a consequence, if $a_1, a_2 \in U$ are linearly independent over C and $a_1 G_1(x_1, \ldots, x_n) + a_2 G_2(x_1, \ldots, x_n) = 0 \in T$, for some $G_1, G_2 \in T$, then both $G_1(x_1, \ldots, x_n)$ and $G_2(x_1, \ldots, x_n)$ are the zero element of T.

Suppose that R does not satisfy any nontrivial generalized polynomial identity. Therefore, for all $u \in I$, $a(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^m$ is the zero element in the free product $T = U *_C C\{x_1, \ldots, x_n\}$, that is,

$$a(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^m = 0 \in T. \tag{3}$$

Suppose that there exists $u \in I$ such that $\{acu, au\}$ are linearly independent over C. In this case, from

$$\begin{aligned} &a(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b) \\ &\quad \cdot(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^{m-1} = 0 \in T \end{aligned}$$

it follows that

$$acf(ux_1, \ldots, ux_n)(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^{m-1} = 0 \in T,$$

that is,

$$\begin{aligned} &acf(ux_1, \ldots, ux_n)(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b) \\ &\quad \cdot(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^{m-2} = 0 \in T. \end{aligned}$$

Moreover, since $\{cu, u\}$ are linearly independent over C, we also get

$$a(cf(ux_1, \ldots, ux_n))^2(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^{m-2} = 0 \in T.$$

Eventually, we obtain

$$a(cf(ux_1, \ldots, ux_n))^m = 0 \in T,$$

which is a contradiction.

Consider now the case where there exists $\alpha \in C$ such that $acu = \alpha au$, for all $u \in I$, that is, $a(c - \alpha)I = 0$. Since we assume that $aI \neq 0$, there exists $u \in I$ such that $au \neq 0$.

By (3),

$$\begin{aligned} &(\alpha af(ux_1, \ldots, ux_n) + af(ux_1, \ldots, ux_n)b) \\ &\quad \cdot(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^{m-1} = 0 \in T, \end{aligned} \tag{4}$$

that is,

$$(af(ux_1, \ldots, ux_n)(\alpha + b))(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^{m-1} = 0 \in T.$$

If there exists $u \in I$ such that $\{cu, u\}$ are linearly independent over C, then from

$$\begin{aligned} &(af(ux_1, \ldots, ux_n)(\alpha + b))(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b) \\ &\quad \cdot(cf(ux_1, \ldots, ux_n) + f(ux_1, \ldots, ux_n)b)^{m-2} = 0 \in T \end{aligned}$$

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it follows that

$$(af(ux_1, \dots, ux_n)(\alpha + b))(f(ux_1, \dots, ux_n)b) \cdot (cf(ux_1, \dots, ux_n) + f(ux_1, \dots, ux_n)b)^{m-2} = 0 \in T,$$

and repeating this process,

$$(af(ux_1, \dots, ux_n)(\alpha + b))(f(ux_1, \dots, ux_n)b)^{m-1} = 0 \in T, \tag{5}$$

which is a contradiction unless (5) is trivial, that is, when either $au = 0$ or $(\alpha + b) = 0$. Since $au = 0$ forces a contradiction, we get $b = -\alpha \in C$ and $a(c + b)I = 0$.

If there exists $\gamma \in C$ such that $(c - \gamma)I = 0$, by (4) it follows that

$$(af(ux_1, \dots, ux_n)(\alpha + b))(f(ux_1, \dots, ux_n)(\gamma + b))^{m-1} = 0, \in T \tag{6}$$

which is a contradiction unless (6) is trivial, that is, when either $au = 0$ or $(\alpha + b) = 0$ or $(\gamma + b) = 0$. Also in this case we cannot consider the conclusion $au = 0$. Both the last two cases imply that $b \in C$ and $a(c + b)I = 0$. □

We conclude the section with the following final result about inner generalized derivations.

PROPOSITION. *Let R be a noncommutative prime ring of characteristic different from 2 with Martindale quotient ring Q and extended centroid C, I a nonzero right ideal of R. Let $f(x_1, \dots, x_n)$ be a noncentral multilinear polynomial over C, $m \geq 1$ a fixed integer, a, b, c fixed elements of R. If $a(cf(r_1, \dots, r_n) + f(r_1, \dots, r_n)b)^m = 0$ for all $r_1, \dots, r_n \in I$, then one of the following holds:*

(1) $aI = acI = (0)$;
(2) $[b, I]I = 0$ *and* $a(c + b)I = 0$;
(3) $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$ *is an identity for* I.

PROOF. Since if R does not satisfy any nontrivial generalized polynomial identity the result follows from Lemma 7, in all that follows we may assume that R satisfies some nontrivial generalized polynomial identity. By [18], RC is a primitive ring and so U has nonzero socle H with nonzero right ideal $J = IH$. Moreover, J and I satisfy the same differential identities with coefficients in U (see [13]). Since $Ja \neq 0$, we may replace a by $0 \neq ca \in J$ for some $c \in J$. Thus replace R by H, I by J and a by ca; then without loss of generality we may consider that R is a simple ring and equal to its own socle, $I = IR$ and $a \in I$.

Suppose that the conclusions of the proposition do not hold, hence there exist $u, v, z, t, w_1, \dots, w_{n+2} \in I$ such that:

(1) $au \neq 0$;
(2) either $a(c + b)v \neq 0$ or $[b, z]t \neq 0$;
(3) $[f(w_1, \dots, w_n), w_{n+1}]w_{n+2} \neq 0$.

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Moreover, choose F to be the algebraic closure of C or $F=C$, according to $|C|=\infty$ or $|C|<\infty$. Note that $IH\otimes_C F$ is a completely reducible right $H\otimes_C F$-module which satisfies the generalized polynomial identity $a(cf(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)b)^m$. Thus there exists an idempotent $h\in IH\otimes_C F$ such that $u, v, z, t, w_1, \ldots, w_{n+2}\in h(IH\otimes_C F)$. By Litoff's theorem (for a proof, see [8]) there exists $e^2=e\in H\otimes_C F$ such that

$$h, hc, ch, hb, bh, a, u, v, z, t, , w_1, \ldots, w_{n+2}\in e(H\otimes_C F)e$$

with $e(H\otimes_C F)e\cong M_k(F)$, for $k\geq 2$.

For all $r_1,\ldots,r_n\in he(H\otimes_C F)e\subseteq(IH\otimes_C F)\cap e(H\otimes_C F)e$,

$$\begin{aligned}0&=ea(cf(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)b)^m\\&=ea(chf(r_1,\ldots,r_n)+hf(r_1,\ldots,r_n)b)^m\\&=(eae)((ece)f(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)(ebe))^m.\end{aligned}$$

By Lemmas 5 and 6, we have that one of the following holds:

(1) $[ebe, he(H\otimes_C F)e]he(H\otimes_C F)e=0$ and $(eae)(ebe+ece)he(H\otimes_C F)e=0$, which implies the contradiction that either $0\neq[b,z]t=[ebe, heze]hete=0$ or $0\neq a(b+c)v=(eae)(b+c)(heve)=(eae)(ebe+ece)heve=0$;

(2) $(eae)he(H\otimes_C F)e=0$, which implies the contradiction $0\neq au=(eae)heue=0$;

(3) $he(H\otimes_C F)e$ satisfies $[f(x_1,\ldots,x_n),x_{n+1}]x_{n+2}$; on the other hand, we also have that

$$\begin{aligned}0&=[f(hew_1e,\ldots,hew_ne),hew_{n+1}e]hew_{n+2}e\\&=[f(w_1,\ldots,w_n),w_{n+1}]w_{n+2}\neq 0,\end{aligned}$$

a contradiction again. □

3. The proof of the theorem

Finally we extend the above result to any generalized derivation defined on R. In light of Fact 1, we consider $g(x)=cx+d(x)$, for some $c\in U$ and a derivation d on U. In order to prove our result we divide the proof into two cases.

If the derivation d is inner, namely $d(x)=[q,x]$ for some $q\in U$, then $g(x)=(c+q)x-xq$ and the conclusion follows from our previous proposition.

Thus we consider only the case when d is not an inner derivation of R. In this case I satisfies the generalized identity

$$a\bigg(cf(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)+\sum_{i=1}^{n}f(x_1,\ldots,d(x_i),\ldots,x_n)\bigg)^m.$$

Assume that $aI\neq 0$ and $f(x_1,\ldots,x_n)x_{n+1}$ is not an identity for I. We prove that a number of contradictions follow.

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Of course there exists at least one element $u \in I$ such that $au \neq 0$; moreover, $u_1, \ldots, u_{n+1}$ such that $f(u_1, \ldots, u_n)u_{n+1} \neq 0$. Since R satisfies

$$a \cdot (cf(ux_1, \ldots, ux_n) + f^d(ux_1, \ldots, ux_n) \\ + \sum_{i=1}^{n} f(ux_1, \ldots, d(u)x_i + ud(x_i), \ldots, ux_n))^m$$

and d is not inner, by Kharchenko's [12] result, R satisfies

$$a \cdot (cf(ux_1, \ldots, ux_n) + f^d(ux_1, \ldots, ux_n) \\ + \sum_{i=1}^{n} f(ux_1, \ldots, d(u)x_i + uy_i, \ldots, ux_n))^m.$$

In particular, R satisfies the blended component

$$a(f(uy_1, \ldots, ux_n))^m,$$

which is a nontrivial generalized polynomial identity for R, since $au \neq 0$.

By [18] U is a primitive ring with socle $H = \mathrm{Soc}(U) \neq 0$ and $f(x_1, \ldots, x_n)x_{n+1}$ is not an identity for IH, since, by [6], I, IU and IH satisfy the same generalized identities. By the regularity of H, there exists an idempotent element $e \in IH$ such that $eR = uR + \sum_{i=1}^{n+1} u_i R$ and $u = eu$, $u_i = eu_i$, for all $i = 1, \ldots, n+1$. Hence

$$a(f(ex_1, \ldots, ex_n))^m$$

is a generalized identity for R, that is, $a(f(x_1, \ldots, x_n))^m$ is an identity for I. By using the main result in [7], either $ae = 0$ or eR satisfies $f(x_1, \ldots, x_n)x_{n+1}$. If $ae = 0$ we have the contradiction $0 = aeu = au \neq 0$. In the other case we get $0 = f(eu_1, \ldots, eu_n)eu_{n+1} = f(u_1, \ldots, u_n)u_{n+1} \neq 0$, again a contradiction.

Hence we may assume that $aI = 0$. In light of Remark 2, for all $x_1, \ldots, x_n \in I$ we may write $f(x_1a, x_2, \ldots, x_n) = h_1(x_2, \ldots, x_n)x_1a$. Notice that

$$a(cf(x_1a, x_2, \ldots, x_n) + d(f(x_1a, x_2, \ldots, x_n))) \\ = a(ch_1(x_2, \ldots, x_n) + d(h_1(x_2, \ldots, x_n)))x_1a,$$

and from this and main hypothesis we have that I satisfies the generalized identity

$$(a(ch_1(x_2, \ldots, x_n) + d(h_1(x_2, \ldots, x_n)))x_1a)^m,$$

that is, I satisfies

$$((ach_1(x_2, \ldots, x_n) + ad(h_1(x_2, \ldots, x_n)))x_1)^{m+1}.$$

By the main result in [9] it follows that I satisfies

$$ach_1(x_2, \ldots, x_n) + ad(h_1(x_2, \ldots, x_n))$$

and, using the fact that $aI = 0$, *a fortiori* I satisfies

$$ach_1(x_2, \ldots, x_n)x_1 + ad(h_1(x_2, \ldots, x_n)x_1).$$

The same argument shows that I satisfies the identities

$$\begin{aligned} &ach_2(x_1, \ldots, x_n)x_2 + ad(h_2(x_1, \ldots, x_n)x_2),\\ &ach_3(x_1, \ldots, x_n)x_3 + ad(h_3(x_1, \ldots, x_n)x_3),\\ &ach_4(x_1, \ldots, x_n)x_4 + ad(h_4(x_1, \ldots, x_n)x_4); \end{aligned}$$

and in general, for all $i = 1, \ldots, n$, I satisfies

$$ach_i(x_1, \ldots, x_n)x_i + ad(h_i(x_1, \ldots, x_n)x_i).$$

By the sum of all these addends, I satisfies

$$acf(x_1, \ldots, x_n) + ad(f(x_1, \ldots, x_n)). \tag{7}$$

Now let G be the additive subgroup generated by the set $f(I) = \{f(r_1, \ldots, r_n) \mid r_1, \ldots, r_n \in I\}$. If I does not satisfy $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$, by [5] there exists a right ideal $I_0 \subseteq I$ such that $[I_0, I] \subseteq G$ (in particular, if I satisfies some polynomial identity then I_0 coincides with I). Notice that (7) implies $acw + ad(w) = 0$ for all $w \in G$.

Therefore $ac[I_0, I] + ad([I_0, I]) = (0)$, that is, for all $0 \neq u_0 \in I_0$, $u \in I$ and $r, s \in R$,

$$0 = ac[u_0ra, us] + ad([u_0ra, us]) = -acusu_0ra - ad(u)su_0ra$$

and by the primeness of R we get $acu + ad(u) = 0$ which implies that $ag(I) = 0$, and we are done. $\square$

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