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HYPERBOLIC GROUPS ARE HYPERHOPFIAN

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Abstract

The main result indicates that every finitely generated, residually finite, torsion-free, cohopfian group having no free Abelian subgroup of rank two is hyperhopfian. The argument relies on earlier work and ideas of Hirshon. As a corollary, fundamental groups of all closed hyperbolic manifolds are hyperhopfian.

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1. Introduction

A group Γ is said to be *hopfian* if every epimorphism $\Gamma \to \Gamma$ is an automorphism; dually, it is said to be *cohopfian* if every monomorphism $\Gamma \to \Gamma$ is an automorphism. In a related vein, Γ is said to be *residually finite* if for every γ $(\neq 1) \in \Gamma$ there exists a homomorphism $\alpha_\gamma : \Gamma \to G$ to a finite group G with $\alpha_\gamma(\gamma) \neq 1_G$. It is well-known that finitely generated, residually finite groups are hopfian.

The note focuses on a related hopfian property. Say that Γ is *hyperhopfian* if every homomorphism $\varphi : \Gamma \to \Gamma$ with $\varphi(\Gamma)$ normal in Γ and $\Gamma/\varphi(\Gamma)$ cyclic is an isomorphism (onto). Although finitely generated Abelian groups are definitely not hyperhopfian, abundant evidence suggests that the class of hyperhopfian groups is large. Silver [11] has shown that most classical knot groups are hyperhopfian. In his initial study of the property, the author [4] proved that nontrivial free products of finitely generated, residually finite groups are hyperhopfian, provided that the order of at least one factor is greater than 2; moreover, every hopfian group endowed with a finite presentation having at least 2 more generators than relators is hyperhopfian. Elsewhere [5] he established that fundamental groups of closed 3-manifolds having either $SL_2(\mathbb{R})^{\sim}$ or Sol geometric structure are hyperhopfian, as are some arising from manifolds having Nil structure (Chinen [2] has corrected the analysis in [5] concerning Nil groups).

The main result here affirms that all finitely generated, residually finite, torsion-free, cohopfian groups containing no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ are hyperhopfian. Consequently, fundamental groups of all closed hyperbolic n-manifolds are hyperhopfian. This resolves the solitary unsettled issue apparent in [5, Table 2] for the $n = 3$ case.

The impetus behind this involves ideas developed by Hirshon [7] which led to his result, for example, that each endomorphism of a finitely generated, residually finite, torsion-free group is monic. The author is indebted to Yongkuk Kim for bringing [7] to his attention.

Hyperhopfian groups play a key role at a certain juncture in geometric topology. Work of [1, 3, 7] indicates that all reasonable closed n-manifolds N having hyperhopfian fundamental groups are codimension-2 fibrators, which means that all proper mappings $p : M \to B$ defined on an $(n+2)$-manifold M such that each $p^{-1}(b)$ has the homotopy type of N are approximate fibrations.

2. Results

LEMMA 1 ([7, Lemma 1]). *Let Γ be a finitely generated, residually finite group and $\Gamma = \Gamma_1, \Gamma_2, \ldots$ a sequence of subgroups of Γ with $\Gamma_{i+1} \subset \Gamma_i$ for $i = 1, 2, \ldots$. Let θ_i be an endomorphism of Γ_i such that $\theta_i(\Gamma_i) = \Gamma_{i+1}$, and let $\ker(\theta_i)$ denote its kernel. Then $\bigcap_i \ker(\theta_i) = \{1\}$.*

Given a group, Γ, we use Γ' to denote its commutator subgroup.

THEOREM 2. *Let $\psi : \Gamma \to \Gamma$ be an endomorphism of a finitely generated, residually finite group Γ with $\Gamma' \subset \psi(\Gamma)$. Then there exists an integer $p \geq 0$ for which ψ restricts to a monomorphism on $\psi^p(\Gamma)$.*

PROOF. This follows just as in the proof of [7, Theorem 1]; we include details for completeness. Let $\Gamma_i = \psi^{i-1}(\Gamma)$ and N_i the intersection of all conjugates of Γ_{i+1} in Γ_i. Let θ_i denote $\psi|\Gamma_i$. Now $\theta_i(\Gamma_i) = \Gamma_{i+1}$ and $\theta_i(N_i) = N_{i+1}$, so θ_i induces an epimorphism $\overline{\theta_i} : \Gamma_i/N_i \to \Gamma_{i+1}/N_{i+1}$. Note that $\Gamma' = \Gamma_1' \subset N_1$, so Γ_1/N_1, and therefore each Γ_i/N_i, is Abelian. As a result, ultimately $\overline{\theta_i}$ is an isomorphism—that is, there exists an integer p such that for $i \geq p$ we have

$$\ker(\psi) \cap \Gamma_i = \ker(\theta_i) \subset N_i \subset \Gamma_{i+1}.$$

Hence, for $i \geq p$

$$\ker(\theta_i) = \ker(\psi) \cap \Gamma_i = \ker(\psi) \cap \Gamma_{i+1} = \ker(\theta_{i+1}),$$

from which it follows that $\bigcap_i \ker(\theta_i) = \ker(\theta_p)$. Finally, $\ker(\theta_p) = \{1\}$, by Lemma 1. □

Wise [12] recently produced an example illustrating the need for some restriction on ψ such as $\Gamma' \subset \psi(\Gamma)$ in Theorem 2 above or $[\Gamma : \psi(\Gamma)] < \infty$ in [7, Theorem 1].

LEMMA 3. *Suppose*

$$1 \longrightarrow N \xrightarrow{j} \Gamma \xrightarrow{\rho} C \longrightarrow 1$$

is an exact sequence of groups such that C is cyclic, and suppose there exists a homomorphism φ defined on Γ with $\ker(\varphi) \neq \{1\} = \ker(\varphi) \cap j(N)$. *Then some finite index subgroup Γ^+ of Γ admits a direct product decomposition $\Gamma^+ \cong N \times C^+$, where C^+ is a nontrivial cyclic subgroup of C.*

This is obvious: here $C^+ = \rho(\ker(\varphi))$ and Γ^+ is the subgroup generated by $j(N) \cup \ker(\varphi)$.

Dualizing the hyperhopfian concept, we say that a group Γ is *hyper-cohopfian* if every monomorphism $\phi : \Gamma \to \Gamma$ with $\phi(\Gamma)$ normal in Γ and $\Gamma/\phi(\Gamma)$ cyclic is necessarily an epimorphism. Also, we call Γ *atoroidal* if it contains no free Abelian subgroup of rank 2.

THEOREM 4. *A group Γ is hyperhopfian if it is finitely generated, residually finite, torsion-free, hyper-cohopfian and atoroidal.*

PROOF. Let $\psi : \Gamma \to \Gamma$ be a homomorphism such that $\psi(\Gamma)$ is normal in Γ and $\Gamma/\psi(\Gamma)$ is cyclic. Our goal is to show that ψ is an automorphism; by residual finiteness, it suffices to demonstrate surjectivity of ψ.

Assume to the contrary that $\psi(\Gamma) \neq \Gamma$. If ψ were monic, the hypothesized cohopfian property would imply it is an automorphism. Hence, ψ must have nontrivial kernel. Since $\Gamma/\psi(\Gamma)$ is cyclic, $\Gamma' \subset \psi(\Gamma)$. In the notation of Theorem 2, we have $\Gamma_i = \psi^{i-1}(\Gamma) = N_i$, and $\theta_i = \psi|\Gamma_i$, as before. That result yields an integer p such that $\ker(\theta_{p+1}) = \{1\}$. Choose the least value of p for which this is true; that is,

$$\ker(\theta_p) \neq \{1\} = \ker(\theta_{p+1}) = \ker(\theta_p) \cap \Gamma_{p+1}.$$

Note that $p > 0$, and recall that the θ_i's induce epimorphisms $\Gamma_i/N_i = \Gamma_i/\Gamma_{i+1} \to \Gamma_{i+1}/\Gamma_{i+2}$, so each Γ_i/Γ_{i+1} is cyclic. Now application of Lemma 3 to the exact sequence

$$1 \longrightarrow \Gamma_{p+1} \longrightarrow \Gamma_p \longrightarrow \Gamma_p/\Gamma_{p+1} \longrightarrow 1$$

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provides a finite index subgroup Γ_p^+ of Γ_p admitting a direct product decomposition $\Gamma_p^+ \cong \Gamma_{p+1} \times C^+$, with $C^+ \neq \{1\}$ cyclic. Neither factor has torsion, as Γ is torsion-free. If Γ_{p+1} were nontrivial, Γ^+ would contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$; otherwise, p would be at least 2, Γ_p would be infinite cyclic, and inspection of

$$1 \longrightarrow \Gamma_p \longrightarrow \Gamma_{p-1} \longrightarrow \mathbb{Z} \longrightarrow 1$$

would give rise to an index ≤ 2 subgroup of Γ_{p-1} isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Either circumstance would contradict Γ being atoroidal. Consequently, ψ must be surjective. □

COROLLARY 5. *The fundamental group of every closed hyperbolic n-manifold, $n > 1$, is hyperhopfian.*

See Ratcliffe [9, Sections 5.5, 8.2, 11.6].

COROLLARY 6. *Every residually finite, torsion-free, hyper-cohopfian, (finitely presented) word-hyperbolic group is hyperhopfian.*

See Coornaert and Papadopoulos [3, page 9] about atoroidality. The need for a cohopfian hypothesis is illustrated, for example, by the infinite cyclic group.

COROLLARY 7. *Every non-elementary, residually finite, torsion-free, freely indecomposable, (finitely presented) word-hyperbolic group is hyperhopfian.*

Sela [10, Theorem 4.4] has shown such groups to be cohopfian.

PROPOSITION 8. *Suppose Γ is a finitely generated, residually finite, hyper-cohopfian group such that Γ/Γ' is finite, and suppose no finite index subgroup Γ^+ of Γ admits a direct product factorization $\Gamma^+ \cong N \times C$, where C is a nontrivial, finite cyclic group. Then Γ is hyperhopfian.*

PROOF. This follows essentially by the methods used to establish Theorem 4. The extra feature needed is the observation that

$$[\Gamma_i : \Gamma_{i+1}] \leq [\Gamma_1 : \Gamma_2] \leq |\Gamma/\Gamma'| < \infty$$

for $i = 1, 2, \ldots, p-1$. □

Certain finite groups Γ enjoy the feature of being hyperhopfian if and only if Γ itself admits no direct product factorization involving a cyclic factor. For instance, this holds when Γ has square free order, or when it acts freely on the 3-sphere [4, Section 4].

COROLLARY 9. *Let Γ be a finitely generated, residually finite, torsion-free group such that Γ/Γ' is finite. Then Γ is hyperhopfian if and only if it is hyper-cohopfian.*

It should be added that Corollary 9 also follows directly from Hirshon's work [7]. Finite cyclic groups indicate that the reverse implication fails in case Γ has torsion, and groups such as the one given by the presentation

$$\langle k, a, b \mid 1 = [k, a] = [k, b], [a, b] = k^3 \rangle$$

exemplify failure in case Γ/Γ' is infinite (see [5, pages 1464–1465]).

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