ON THE NONLINEAR DIRICHLET PROBLEM WITH $p(x)$-LAPLACIAN

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Using a dual variational method which we develop, we show the existence and stability of solutions for a family of Dirichlet problems

$$-\operatorname{div}\Big(\mathcal{V}\big(x, \nabla u(x)\big)\Big) = F_u^k\big(x, u(x)\big),$$

$$u(x)|_{\partial\Omega} = 0, \ u \in W_0^{1,p(x)}(\Omega),$$

$k = 0, 1, \ldots$ in a bounded domain in $\mathbb{R}^N$ and with the nonlinearity satisfying some general growth conditions. The assumptions put on $\mathcal{V}$ are satisfied by $p(x)$-Laplacian operators.

1. INTRODUCTION

In the paper we consider a problem

(1.1) $$-\operatorname{div}\Big(\mathcal{V}\big(x, \nabla u(x)\big)\Big) = F_u^k\big(x, u(x)\big),$$

$$u(x)|_{\partial\Omega} = 0, \ u \in W_0^{1,p(x)}(\Omega)$$

for $k = 0, 1, \ldots$ and with suitable assumptions on $\mathcal{V}$ which are valid for the $p(x)$-Laplacian operator. Following some ideas from [11] we construct a dual variational method which applies to more general type of nonlinearities than those that are subject to a Palais–Smale type condition. We relate critical values and critical points to the action functional for which (1.1) is the Euler–Lagrange equation and the dual action functional (introduced in the paper and different from the Clarke dual functional) on specially constructed subsets of their domains.

A variational approach concerning existence results for a class of problems involving critical and subcritical growth is shown in [2]. Critical point theory in certain Sobolev spaces is used to obtain existence and multiplicity results in sublinear and superlinear cases as well [5], where the problem is written as

$$-\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) + |u|^{p(x)-2}u = f(x, u),$$

$u \in W^{1,p(x)}(\mathbb{R}^N)$. Standard variational arguments in Sobolev spaces with variable exponents are used to show the existence of radial solutions $u \in W^{1,p(x)}(\mathbb{R}^N)$ with $\sup_{x \in \mathbb{R}^N} p(x) < N$ in [17] where we assert $\inf_{x \in \Omega} p(x) > N$ in a bounded domain. The singularity of positive radial solutions to

$$- \operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = f(x, u)$$

with p, f radial in x, f continuous and growth speed rate less than $N\big(p(x) - 1\big)/(N - p(x))$ as $u \to \infty$ one can find in [16].

Such problems as ours are studied in [3, 7] and these can be applied in elastic mechanics and electrorheological fluid dynamics. Although in both [3] and [7] the problems are not studied in the form which we investigate. See [13, 18] and references therein. In [7] the authors consider the problem

$$- \operatorname{div}\Big(\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big) = F_x\big(x, u(x)\big),$$
$$u(x)|_{\partial\Omega} = 0, \ u \in W_0^{1,p(x)}(\Omega)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded region and $W_0^{1,p(x)}(\Omega)$ denotes the generalised Orlicz–Sobolev space, (see [6, 8]), while in [3] the equation studied is

$$- \operatorname{div}\Big(a(x)\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x) + b(x)\big|u(x)\big|^{p(x)-2}\Big) = F_x\big(x, u(x)\big),$$
$$u(x)|_{\partial\Omega} = 0, \ u \in W_0^{1,p(x)}(\Omega)$$

with $a, b \in L^\infty(\Omega)$ such that $a(x) \geqslant a_0 > 0$ and $b(x) \geqslant b_0 \geqslant 0$ almost everywhere on Ω.

Both papers deal with sublinear and superlinear nonlinearities. In the sublinear case the direct method of the calculus of variations is used while in the superlinear case they apply the mountain pass geometry by showing that a type of Palais–Smale condition is satisfied.

Our method is based on investigating the primal action functional; that is, a functional for which the original problem is the Euler–lagrange equation and the suitably constructed dual one. The duality relations between the two functionals, derived with the aid of Fenchel–Young conjugacy [4] allows for relating the critical values to both functionals on suitably constructed subsets of their domains. Later we relate the relevant critical points at which the critical values are obtained. When this is done we are in a position to chose a minimising sequence and prove that it is convergent to a solution to the original problem.

The novelty of our approach allows also for considering the stability of solutions. Following some general framework for studying stability of solutions for variational problems in sublinear case [12, 14], [15] we provide suitable results for the Dirichlet problem involving $p(x)$-Laplacian problem also. The approach we use has been sketched in the first author's previous work [10].

2. THE ASSUMPTIONS AND AUXILIARY RESULTS

In what follows by C_S we denote the best Sobolev constant

$$\|u\|_{p(x)} \leqslant C_S \|\nabla u\|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Since $W_0^{1,p(x)}(\Omega)$ is continuously embedded into $W_0^{1,p^-}(\Omega)$, [6] and by the Sobolev Imbedding Theorem [1], we denote by C_1 and C_2 the following constants

(2.1) $$\|\nabla u\|_{p^-} \leqslant C_1 \|\nabla u\|_{p(x)},$$

(2.2) $$\max_{x \in \Omega} |u(x)| \leqslant C_2 \|\nabla u\|_{p^-} \text{ for all } u \in W_0^{1,p^-}(\Omega).$$

Here $p, q \in C(\overline{\Omega})$, $1/p(x) + 1/q(x) = 1$ for $x \in \Omega$ and $p^- = \inf_{x \in \Omega} p(x)$, $p^+ = \sup_{x \in \Omega} p(x)$, $p^- > N$, $N > 2$.

We also assume

F1 $\text{vol}(\Omega) \leqslant (1/p^- + 1/q^-)^{-1}$; there exist positive numbers $d_0, d_1, d_2, \ldots$ such that $C_1 C_2 \leqslant d_k \leqslant d_0$ for $k = 1, 2, \ldots$. For all $k = 0, 1, 2, \ldots$: $F_u^k(\cdot, d_k)$, $F_u^k(\cdot, -d_k) \in L^\infty(\Omega)$ and

(2.3) $$C_1 C_2 C_S \operatorname*{ess\,sup}_{x \in \Omega} |F_u^k(x, d_k)| \leqslant a_0 d_k,$$
$$C_1 C_2 C_S \operatorname*{ess\,sup}_{x \in \Omega} |F_u^k(x, -d_k)| \leqslant a_0 d_k.$$

F2   There exists positive number $d > d_0$ such that for all $k = 0, 1, 2, \ldots$ and $I = [-d, d]$: $F_u^k(\cdot, d), F_u^k(\cdot, -d) \in L^\infty(\Omega)$, $F^k: \Omega \times I \to \mathbb{R}$ are Caratheodory functions and convex in u for almost everywhere $x \in \Omega$, $F_u^k: \Omega \times I \to \mathbb{R}$ are Caratheodory functions, $F^k(x, u) := +\infty$ for $(x, u) \in \Omega \times (\mathbb{R} - I)$.

F3 $F_u^k(x, 0) \neq 0$, for almost everywhere $x \in \Omega$, $x \mapsto F^k(x, 0)$ and $x \mapsto (F^k)^*(x, 0)$ are integrable on Ω, $k = 0, 1, \ldots$.

F4 $V: \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function, convex in the second variable, $\mathcal{V} = (V_{u_1}, V_{u_2}, \ldots, V_{u_N}): \Omega \times \mathbb{R}^N \to \mathbb{R}$, $V_{u_i}, i = 1, \ldots, N$ are Caratheodory functions. $\mathcal{V}(x, 0) = 0$, $V(x, 0) = 0$ almost everywhere on Ω and $x \mapsto V(x, 0)$ is integrable on Ω for almost everywhere $x \in \Omega$.

F5   There exists a constant a_0 such that

$$a_0 \int_\Omega |\nabla u(x)|^{p(x)} \, dx \leqslant \int_\Omega |\mathcal{V}(x, \nabla u(x)) \nabla u(x)| \, dx$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Here $(F^k)^*$ denotes the Fenchel–Young conjugate of F^k, see [4]. Now for each $k = 0, 1, \ldots$ $F^k: \Omega \times \mathbb{R} \to \mathbb{R}$ is convex and lower semi-continuous.

Relation (1.1) is the Euler–Lagrange equation for a functional $J_k \colon W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ for $k = 0, 1, \ldots$

$$J_k(u) = \int_\Omega V\big(x, \nabla u(x)\big)\, dx - \int_\Omega F^k\big(x, u(x)\big)\, dx.$$

Let $W = \big\{ v \in L^{q(x)}(\Omega) \colon \operatorname{div} v \in L^{q(x)}(\Omega) \big\}$. The dual functional $J_{D_k} \colon W \to \mathbb{R}$ reads

$$J_{D_k}(v) = \int_\Omega (F^k)^*\big(x, -\operatorname{div} v(x)\big)\, dx - \int_\Omega V^*\big(x, v(x)\big)\, dx,$$

for $k = 0, 1, \ldots$. We assume that the operator $L : W_0^{1,p(x)}(\Omega) \to \big(W_0^{1,p(x)}(\Omega)\big)^*$ given by

$$\langle L(g), h \rangle = \int_\Omega \mathcal{V}\big(x, \nabla g(x)\big) \nabla h(x)\, dx$$

for all $g, h \in W_0^{1,p(x)}(\Omega)$ has the following properties

> F6 L is a continuous, bounded and strictly monotone operator having (S_+) property and being a homeomorphism.

Property (S_+) means that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ provided $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $\limsup\limits_{n\to\infty} \langle L(u_n) - L(u), u_n - u \rangle \leqslant 0$. Therefore

LEMMA 2.1.   *For any fixed $f \in L^\infty(\Omega)$ the Dirichlet problem*

$$-\operatorname{div}\Big(\mathcal{V}\big(x, \nabla u(x)\big) \Big) = f(x)$$
$$u(x)|_{\partial\Omega} = 0,\ u \in W_0^{1,p(x)}(\Omega)$$

has a unique weak solution.

Now we shall construct certain nonlinear subsets of spaces $W_0^{1,p(x)}(\Omega)$ and W on which we shall look for critical points and critical values of the action and dual action functional. Having established the relationship between the relevant critical points we get the solution to (1.1). For $k = 0, 1, \ldots$ we define

$$\widetilde{X_k} = \Big\{ u \in W_0^{1,p(x)}(\Omega) \colon \|\nabla u\|_{p(x)} \leqslant \frac{d_k}{C_1 C_2},\ \big|u(x)\big| \leqslant d_k \text{ for all } x \in \Omega \Big\}.$$

We consider a set X_k such that for all $u \in X_k$ the relation

$$(2.4) \qquad\qquad -\operatorname{div}\Big(\mathcal{V}\big(x, \nabla \widetilde{u}(x)\big) \Big) = F_u^k\big(x, u(x)\big),$$
$$\widetilde{u}(x)|_{\partial\Omega} = 0,\ u \in W_0^{1,p(x)}(\Omega)$$

implies $\widetilde{u} \in X_k$.

PROPOSITION 2.2. $X_k = \widetilde{X_k}$.

PROOF: Fix $k = 0, 1, \ldots$. We take any $h \in \widetilde{X_k}$. The solution $u \in W_0^{1,p(x)}(\Omega)$ to

(2.5)
$$-\operatorname{div}\left(\mathcal{V}\big(x, \nabla u(x)\big)\right) = F_u^k\big(x, h(x)\big)$$
$$u(x)|_{\partial\Omega} = 0$$

exists by Lemma 2.1. Multiplying (2.5) by $u(x)$, calculating integrals and using assumptions (F5) and (F1) we have

$$a_0 \int_\Omega |\nabla u(x)|^{p(x)}\, dx \leqslant \int_\Omega \left|\mathcal{V}(x, \nabla u(x))\nabla u(x)\right|\, dx$$
$$\leqslant \max\left\{ \int_\Omega \left|F_u^k(x, d_k)u(x)\right|\, dx, \int_\Omega \left|F_u^k(x, -d_k)u(x)\right| dx \right\}$$
$$\leqslant \left(\frac{1}{p^-} + \frac{1}{q^-}\right) \operatorname{vol}(\Omega) \cdot \frac{a_0 d_k}{C_1 C_2 C_S} \|u\|_{p(x)}.$$

So
$$\int_\Omega |\nabla u(x)|^{p(x)}\, dx \leqslant \frac{d_k}{C_1 C_2} \|\nabla u\|_{p(x)}.$$

If $\int_\Omega |\nabla u(x)|^{p(x)}\, dx \leqslant 1$ then $\|\nabla u\|_{p(x)} \leqslant 1 \leqslant (d_k)/(C_1 C_2)$.

If $\int_\Omega |\nabla u(x)|^{p(x)}\, dx \geqslant 1$ we get $\int_\Omega |\nabla u(x)|^{p(x)}\, dx \geqslant \|\nabla u\|_{p(x)}^{p^-}$ so we have

$$\|\nabla u\|_{p(x)} \leqslant \|\nabla u\|_{p(x)}^{p^- - 1} \leqslant \frac{d_k}{C_1 C_2}.$$

Using inequalities (2.2) and (2.1) we obtain

$$\frac{d_k}{C_1 C_2} \geqslant \|\nabla u\|_{p(x)} \geqslant \frac{1}{C_1} \|\nabla u\|_{p^-} \geqslant \frac{1}{C_1 C_2} \max_{x \in \Omega} |u(x)| \geqslant \frac{1}{C_1 C_2} |u(x)|$$

for all $x \in \Omega$. Therefore $|u(x)| \leqslant d_k$. Thus $u \in \widetilde{X_k}$ and we may put $X_k = \widetilde{X_k}$. □

The dual functional J_{D_k} will be considered on a set

$$X_k^d = \Big\{ v \in W : \exists u, \widetilde{u} \in X_k \text{ related by (2.4) such that}$$
$$-\operatorname{div} v(x) = F_u^k\big(x, u(x)\big) \text{ and } v(x) = \mathcal{V}\big(x, \widetilde{u}(x)\big) \Big\}.$$

J_k and J_{D_k} are well defined on X_k and X_k^d due to the following

LEMMA 2.3. *For any* $k = 0, 1, \ldots$ *there exist constants* $\gamma_k, \eta_k > 0$ *such that* $\int_\Omega \left|F^k\big(x, u(x)\big)\right| dx \leqslant \gamma_k$ *for all* $u \in X_k$ *and* $\left|\int_\Omega (F^k)^*\big(x, -\operatorname{div} v(x)\big) dx\right| \leqslant \eta_k$ *for all* $v \in X_k^d$.

PROOF: We have from (F1) and (F3) for almost everywhere $x \in \Omega$, since $u \in X_k$, that

$$\int_\Omega \Big| F^k(x, u(x)) \Big| dx \leqslant \int_\Omega \big| F^k(x, 0) \big| \, dx + \int_\Omega \Big| F_u^k(x, u(x)) \Big| \big| u(x) \big| \, dx$$

$$\leqslant \int_\Omega \big| F^k(x, 0) \big| dx + d_k \int_\Omega \big| F_u^k(x, d_k) \big| \, dx = \gamma_k.$$

Now only the boundedness of the integral $\int_\Omega \Big| (F^k)^*(x, -\operatorname{div} v(x)) \Big| \, dx$ remains to be shown. Since $(F^k)^*$ is convex, we have for almost everywhere $x \in \Omega$ that

$$-F^k(x, 0) \leqslant (F^k)^*\big(x, -\operatorname{div} v(x)\big) \leqslant (F^k)^*(x, 0) + \xi(x)\big(-\operatorname{div} v(x)\big),$$

where $\xi(x) \in \partial (F^k)^*\big(x, -\operatorname{div} v(x)\big)$. From the definition of X_k^d it follows that there exists $u \in X_k$ such that

$$-\operatorname{div} v(x) = F_u^k\big(x, u(x)\big).$$

By (F2) and (F3)

$$-\int_\Omega F^k(x, 0) \, dx \leqslant \int_\Omega \Big| (F^k)^*\big(x, -\operatorname{div} v(x)\big) \Big| \, dx \leqslant \int_\Omega \big| (F^k)^*(x, 0) \big| \, dx$$

$$+ \int_\Omega \big| u(x) \big| \cdot \big| -\operatorname{div} v(x) \big| \, dx \leqslant \int_\Omega \big| (F^k)^*(x, 0) \big| \, dx + d_k \int_\Omega \big| F_u^k(x, d_k) \big| dx.$$

Hence $\eta_k = \max\Big\{ -\int_\Omega F^k(x, 0) \, dx, \int_\Omega \big| (F^k)^*(x, 0) \big| \, dx + d_k \int_\Omega \big| F_u^k(x, d_k) \big| \, dx \Big\}.$ □

THEOREM 2.4. (Duality Principle)

$$\inf_{u \in X_k} J_k(u) = \inf_{v \in X_k^d} J_{D_k}(v).$$

PROOF: We fix $k = 0, 1, \ldots$ and define a functional $J_k^\# \colon X_k \times X_k^d \to \mathbb{R}$,

$$J_k^\#(u, v) = -\int_\Omega u(x)\big(-\operatorname{div} v(x)\big) \, dx + \int_\Omega V\big(x, \nabla u(x)\big) \, dx$$

$$+ \int_\Omega (F^k)^*\big(x, -\operatorname{div} v(x)\big) \, dx.$$

We observe that for any $u \in X_k$

(2.6) $$\inf_{v \in X_k^d} J_k^\#(u, v) = J_k(u).$$

Fix $u \in X_k$.

$$\inf_{v \in X_k^d} J_k^{\#}(u, v) = -\sup_{v \in X_k^d} \left\{ \int_\Omega u(x)\big(-\operatorname{div} v(x)\big) \, dx \right.$$

$$\left. - \int_\Omega (F^k)^*\big(x, -\operatorname{div} v(x)\big) \, dx \right\} + \int_\Omega V\big(x, \nabla u(x)\big) \, dx$$

$$\geqslant -\sup_{f \in L^{q(x)}(\Omega)} \left\{ \int_\Omega u(x) f(x) \, dx - \int_\Omega (F^k)^*\big(x, f(x)\big) \, dx \right\}$$

$$+ \int_\Omega V\big(x, \nabla u(x)\big) \, dx$$

$$= -\int_\Omega F^k\big(x, u(x)\big) \, dx + \int_\Omega V\big(x, \nabla u(x)\big) \, dx = J_k(u).$$

On the other hand, for any $u \in X_k$ there exists $\widetilde{v} \in X_k^d$ such that

$$-\operatorname{div} \widetilde{v}(x) = F_u^k\big(x, u(x)\big),$$

$$\int_\Omega F^k\big(x, u(x)\big) \, dx + \int_\Omega (F^k)^*\big(x, -\operatorname{div} \widetilde{v}(x)\big) \, dx = \int_\Omega u(x) \operatorname{div} \widetilde{v}(x) \, dx,$$

so

$$J_k(u) = \int_\Omega V\big(x, \nabla u(x)\big) \, dx + \int_\Omega (F^k)^*\big(x, -\operatorname{div} \widetilde{v}(x)\big) \, dx - \int_\Omega u(x) \operatorname{div} \widetilde{v}(x) \, dx$$

$$= J_k^{\#}(u, \widetilde{v}) \geqslant \inf_{u \in X_k} J_k^{\#}(u, v)$$

and (2.6) follows.

Now we show that for any $v \in X_k^d$

$$(2.7) \qquad\qquad\qquad \inf_{u \in X_k} J_k^{\#}(u, v) = J_{D_k}(v).$$

Fix $v \in X_k^d$.

$$\inf_{u \in X_k} J_k^{\#}(u, v) = -\sup_{u \in X_k} \left\{ \int_\Omega \nabla u(x) v(x) \, dx - \int_\Omega V\big(x, \nabla u(x)\big) \, dx \right\}$$

$$+ \int_\Omega (F^k)^*\big(x, -\operatorname{div} v(x)\big) \, dx$$

$$\geqslant -\sup_{f \in L^{p(x)}(\Omega)} \left\{ -\int_\Omega f(x) v(x) \, dx - \int V\big(x, f(x)\big) \, dx \right\}$$

$$+ \int_\Omega (F^k)^*\big(x, -\operatorname{div} v(x)\big) \, dx$$

$$= -\int_\Omega V^*\big(x, v(x)\big) \, dx + \int_\Omega (F^k)^*\big(x, -\operatorname{div} v(x)\big) \, dx = J_{D_k}(v).$$

On the other hand, there exist $u_p, \widetilde{u}_p \in X_k$, related by (2.4), such that

$$-\operatorname{div} v(x) = F_u^k\big(x, u(x)\big),$$

$$v(x) = \mathcal{V}\big(x, \widetilde{u}_p(x)\big).$$

Therefore

$$
\begin{aligned}
J_{D_k}(v) &= \int_\Omega (F^k)^*\big(x, -\operatorname{div} v(x)\big)\, dx - \int_\Omega V^*\big(x, v(x)\big)\, dx \\
&= \int_\Omega (F^k)^*\big(x, -\operatorname{div} v(x)\big)\, dx - \int_\Omega v(x) \nabla \widetilde{u}_p(x) + \int_\Omega V\big(x, \nabla \widetilde{u}(x)\big)\, dx \\
&= J_k^\#(\widetilde{u}, v) \leqslant \inf_{u \in X_k} J_k^\#(u, v)
\end{aligned}
$$

Hence (2.7) holds. This and (2.6) provide

$$
\inf_{u \in X_k} J_k(x) = \inf_{v \in X_k^d} J_{D_k}(v).
$$

$\square$

We shall use the duality results to derive necessary conditions for the existence of solutions to (1.1).

THEOREM 2.5. (Variational Principle) *Assume* (F1)–(F5) *and that for any* $k = 0, 1, \dots$ *there exists* $\overline{u}_k \in X_k$ *such that* $-\infty < J_k(\overline{u}_k) = \inf\limits_{u \in X_k} J_k(u) < \infty$. *Then there exist* $\overline{v}_k \in X_k^d$, $k = 0, 1, \dots$ *such that*

(2.8) $$-\operatorname{div} \overline{v}_k(x) = F_u^k\big(x, \overline{u}_k(x)\big),$$

(2.9) $$\mathcal{V}\big(x, \nabla u_k(x)\big) = \overline{v}_k(x).$$

Moreover

(2.10) $$\inf_{v \in X_k^d} J_{D_k}(v) = J_{D_k}(\overline{v}_k) = J_k(\overline{u}_k) = \inf_{u \in X_k} J_k(u).$$

PROOF: Since $\overline{u}_k \in X_k$, we may take $\overline{v}_k \in X_k^d$ such that

$$-\operatorname{div} \overline{v}_k(x) = F_u^k\big(x, \overline{u}_k(x)\big)$$

Thus (2.8) holds. By (2.8) and by the Fenchel–Young inequality we have

$$
\begin{aligned}
J_k(\overline{u}_k) &= \int_\Omega V\big(x, \nabla \overline{u}_k(x)\big)\, dx - \int_\Omega F^k\big(x, \overline{u}_k(x)\big)\, dx \\
&= -\int_\Omega \big(-\operatorname{div} \overline{v}_k(x)\big) \nabla \overline{u}_k(x)\, dx + \int_\Omega (F^k)^*\big(x, -\operatorname{div} \overline{v}_k(x)\big)\, dx \\
&\qquad + \int_\Omega V\big(x, \nabla \overline{u}_k(x)\big)\, dx \\
&\geqslant -\int_\Omega V^*\big(x, \overline{v}_k(x)\big)\, dx + \int_\Omega (F^k)^*\big(x, -\operatorname{div} \overline{v}_k(x)\big)\, dx = J_{D_k}(\overline{v}_k).
\end{aligned}
$$

Hence $J_k(\overline{u}_k) \geqslant J_{D_k}(\overline{v}_k)$. By Theorem 2.4 it follows that

$$
J_k(\overline{u}_k) = \inf_{u \in X_k} J_k(u) = \inf_{v \in X_k^d} J_{D_k}(v) \leqslant J_{D_k}(\overline{v}_k).
$$

Hence $J_k(\overline{u}_k) = J_{D_k}(\overline{v}_k)$ and

$$\int_\Omega V\big(x, \nabla \overline{u}_k(x)\big)\, dx - \int_\Omega F^k\big(x, \overline{u}_k(x)\big)\, dx$$
$$= -\int_\Omega V^*\big(x, \overline{v}_k(x)\big)\, dx + \int_\Omega (F^k)^*\big(x, -\operatorname{div} \overline{v}_k(x)\big)\, dx.$$

By the above and (2.8) it follows that

$$V\big(x, \nabla \overline{u}_k(x)\big) + V^*\big(x, \overline{v}_k(x)\big) = \overline{v}_k(x) \nabla \overline{u}_k(x).$$

Hence (2.9) holds. Assertion (2.10) follows by Duality Principle and since $J_k(\overline{u}_k) = J_{D_k}(\overline{v}_k)$. $\square$

3. THE EXISTENCE OF SOLUTIONS

THEOREM 3.1. *Assume (F1)–(F5). For all $k = 0, 1, 2 \ldots$ there exists $(u_k, v_k) \in X_k \times X_k^d$ such that*

(3.1) $-\operatorname{div} v_k(x) = F_u^k\big(x, u_k(x)\big),$

(3.2) $\mathcal{V}\big(x, \nabla u_k(x)\big) = v_k(x),$

Moreover

(3.3) $\displaystyle \inf_{v \in X_k^d} J_{D_k}(v) = J_{D_k}(v_k) = J_k(u_k) = \inf_{u \in X_k} J_k(u).$

PROOF: First we show that J_k is bounded from below on X_k. Let us fix $k = 0, 1, \ldots$. By convexity of V, F_u^k, (F1), and integration by parts we get for any $u_k \in X_k$

$$\int_\Omega V\big(x, \nabla u_k(x)\big)\, dx \geqslant \int_\Omega V(x, 0)\, dx + \int_\Omega \mathcal{V}(x, 0) \nabla u_k(x)\, dx$$

By (F4) it follows that $\displaystyle \int_\Omega V\big(x, \nabla u_k(x)\big)\, dx$ is bounded from below on X_k.

Moreover, by Lemma 2.3 we obtain the boundedness of $\displaystyle \int_\Omega F^k\big(x, u_k(x)\big)\, dx$. As a consequence J_k is bounded from below on X_k.

By the definition of X_k it follows that it is weakly compact in $W^{1,p(x)}(\Omega)$. Therefore we can find a minimising sequence $\{u_k^{(n)}\}$ for J_k in X_k which is, possibly up to subsequence, convergent weakly in $W^{1,p(x)}(\Omega)$, strongly in $L^{p(x)}(\Omega)$ and thus strongly in $L^{p^-}(\Omega)$. Therefore it is convergent almost everywhere. We denote its limit by $u_k^{(0)}$. Therefore

$$\lim_{n \to \infty} \int_\Omega F^k\big(x, u_k^{(n)}(x)\big)\, dx = \int_\Omega F^k\big(x, u_k^{(0)}(x)\big)\, dx.$$

Moreover, since $\{\nabla u_k^{(n)}\}$ is convergent weakly in $L^{p(x)}(\Omega)$ and V is convex and lower semicontinuous we get

$$\liminf_{n\to\infty} \int_\Omega V\big(x, \nabla u_k^{(n)}(x)\big)dx \geqslant \int_\Omega V\big(x, \nabla u_k^{(0)}(x)\big)\,dx.$$

Thus J_k is lower semicontinuous on X_k. Consequently, for any $k = 0, 1, 2, \ldots$ one can find $u_k \in X_k$ such that $J_k(u_k) = \inf_{u \in X_k} J_k(u)$. By Theorem 2.5 it follows that there exists $v_k \in X_k^d$ satisfying (3.1). As in Theorem 2.5 we assert that (3.2) and (3.3) also hold. $\square$

COROLLARY 3.2. *For all $k = 0, 1, 2, \ldots$ there exists $u_k \in X_k$ such that*

$$-\operatorname{div}\Big(\mathcal{V}\big(x, \nabla u_k(x)\big)\Big) = F_u^k(x, u_k(x)),$$
$$u_k|_{\partial\Omega} = 0,$$
$$J_k(u_k) = \inf_{u \in X_k} J_k(u).$$

Moreover $-\operatorname{div}\Big(\mathcal{V}\big(\cdot, \nabla u_k(\cdot)\big)\Big) \in L^\infty(\Omega)$ *and*

$$\|\nabla u_k\|_{L^{p(x)}(\Omega)} \leqslant \frac{d_k}{C_1 C_2}, \ |u_k(x)| \leqslant d_k.$$

4. STABILITY OF SOLUTIONS

THEOREM 4.1. *Assume (F1)–(F5) and that for every $u \in W_0^{1,p(x)}(\Omega)$ there is a subsequence $\{k_i\}$ such that*

$$\lim_{i\to\infty} F_u^{k_i}\big(x, u(x)\big) = F_u^0\big(x, u(x)\big)$$

weakly in $L^\infty(\Omega)$. Then for each $k = 0, 1, 2, \ldots$ there exists a solution u_k to problem (1.1). There exists a subsequence $\{u_{k_n}\}_{n=1}^\infty$ of the sequence $\{u_k\}_{k=1}^\infty$ and $\overline{u} \in W_0^{1,p(x)}(\Omega)$ such that

$$u_{k_n} \to \overline{u} \in X_0, \text{ strongly in } W_0^{1,p(x)}(\Omega)$$

and

(4.1)
$$-\operatorname{div}\Big(\mathcal{V}\big(x, \nabla\overline{u}(x)\big)\Big) = F_u^0\big(x, \overline{u}(x)\big),$$
$$\overline{u}(x)|_{\partial\Omega} = 0.$$

Moreover

$$J_0(\overline{u}) = \inf_{u \in X_0} J_0(\overline{u}).$$

PROOF: By Corollary 3.2 it follows that for each $k = 0, 1, 2, \ldots$ there exists $u_k \in W_0^{1,p(x)}(\Omega)$ satisfying (1.1). Due to the fact that $X_k \subset X_0$ it follows that the sequence $\{\nabla u_k\}_{k=1}^\infty$ is bounded in $L^{p(x)}(\Omega)$ and we may choose a weakly convergent subsequence

in $W_0^{1,p(x)}(\Omega)$ which up to a subsequence may be assumed to be strongly convergent in $L^{p^-}(\Omega)$ and thus convergent almost everywhere. We denote its limit by $\overline{u}$. By the assumption we may take a subsequence $\{k_i\}_{i=1}^{\infty}$ such that $\lim\limits_{i\to\infty} F_u^{k_i}(x,\overline{u}(x)) = F_u^0(x,\overline{u}(x))$ weakly in $L^\infty(\Omega)$. We denote all the resulting subsequences by the subscript k for simplicity. Due to (2.3) we obtain

$$\operatorname*{ess\,sup}_{x\in\Omega}\left|F_u^k(x,\overline{u}(x)) - F_u^0(x,\overline{u}(x))\right| \leqslant \frac{2a_0 d_0}{C_1 C_2 C_S}$$

This and the definition of X_k imply that $\left\{-\operatorname{div}\left(\mathcal{V}(\cdot,\nabla u_k(\cdot))\right)\right\}_{k=1}^{\infty}$ is, up to a subsequence, weakly convergent in $L^{q(x)}(\Omega)$ to a certain function $d \in L^{q(x)}(\Omega)$. Thus

$$\int_\Omega \Big\langle -\operatorname{div}\big(\mathcal{V}(x,\nabla u_k(x))\big) - \operatorname{div}\big(\mathcal{V}(x,\nabla\overline{u}(x))\big), u_k(x) - \overline{u}(x)\Big\rangle dx \to 0.$$

Hence and by the fact that the operator $-\operatorname{div}\left(\mathcal{V}(\cdot,\nabla u_k(\cdot))\right)$ has (S_+) property it follows that $\{\nabla u_k\}_{k=1}^{\infty}$ is strongly convergent in $W_0^{1,p(x)}(\Omega)$.

We shall next prove that

$$-\operatorname{div}\left(\mathcal{V}(x,\nabla\overline{u}(x))\right) = F_u^0(x,\overline{u}(x)).$$

By the convexity of F^k we get for any $u \in W_0^{1,p(x)}(\Omega)$

$$\int_\Omega \Big\langle F_u^k(x,u_k(x)) - F_u^k(x,u(x)), u_k(x) - u(x)\Big\rangle dx \geqslant 0.$$

Corollary 3.2 provides

$$\int_\Omega \Big\langle -\operatorname{div}\left(\mathcal{V}(x,\nabla u_k(x))\right) - F_u^k(x,u(x)), u_k(x) - u(x)\Big\rangle dx \geqslant 0.$$

Since $u_k \to \overline{u}$ strongly in $L^{p(x)}(\Omega)$ and $F_u^k(\cdot,u(\cdot)) \rightharpoonup F_u^0(\cdot,u(\cdot))$ weakly in $L^{q(x)}(\Omega)$ we easily get that

$$\int_\Omega \Big\langle -F_u^k(x,u(x)), u_k(x) - u(x)\Big\rangle dx \to \int_\Omega \Big\langle -F_u^0(x,u(x)), \overline{u}(x) - u(x)\Big\rangle dx.$$

Moreover

$$\int_\Omega \Big\langle -\operatorname{div}\big(\mathcal{V}(x,\nabla u_k(x))\big), -u(x)\Big\rangle dx = \int_\Omega \Big\langle \mathcal{V}(x,\nabla u_k(x)), -\nabla u(x)\Big\rangle dx$$

$$\to \int_\Omega \Big\langle \mathcal{V}(x,\nabla\overline{u}(x)), -\nabla u(x)\Big\rangle dx = \int_\Omega \Big\langle -\operatorname{div}\big(\mathcal{V}(x,\nabla\overline{u}(x))\big), -u(x)\Big\rangle dx.$$

We further observe that

$$\int_\Omega \Big\langle -\operatorname{div}\big(\mathcal{V}(x,\nabla u_k(x))\big), u_k(x)\Big\rangle dx = \int_\Omega \mathcal{V}(x,\nabla u_k(x))\nabla u_k(x)\,dx$$

$$\to \int_\Omega -\operatorname{div}\big(\mathcal{V}(x,\nabla\overline{u}(x))\big)\nabla\overline{u}(x)\,dx.$$

Hence

(4.2) $$\int_\Omega \Big\langle -\operatorname{div}\Big(\mathcal{V}\big(x, \nabla \overline{u}(x)\big)\Big) - F_u^0(x, u), \overline{u}(x) - u(x)\Big\rangle dx \geqslant 0$$

for any $u \in W_0^{1,p(x)}(\Omega)$.

Now we apply the Minty "trick" that is, we consider the points $\overline{u} + tu$, where $u \in W_0^{1,p(x)}(\Omega)$, $|u(x)| \leqslant d_0$ on Ω and $t > 0$ are such that $\overline{u}(x) + tu(x) \in I$ almost everywhere. By inequality (4.2) we obtain

$$\int_\Omega \Big\langle -\operatorname{div}\Big(\mathcal{V}\big(x, \nabla \overline{u}(x)\big)\Big) - F_u^0\big(x, \overline{u}(x) + tu(x)\big), u(x)\Big\rangle dx \leqslant 0$$

Since the function $t \mapsto F^0\big(\cdot, \overline{u}(\cdot) + tu(\cdot)\big)$ is convex it follows that its derivative $t \mapsto \int_\Omega \Big\langle F_u^0\big(x, \overline{u}(x) + tu(x)\big), u(x)\Big\rangle dx$ is continuous at any sufficiently small t. Hence for any $u \in W_0^{1,p(x)}(\Omega)$ satisfying the above conditions

$$0 \geqslant \lim_{t \to 0} \int_\Omega \Big\langle -\operatorname{div}\Big(\mathcal{V}\big(x, \nabla \overline{u}(x)\big)\Big) - F_u^0\big(x, \overline{u}(x) + tu(x)\big), u(x)\Big\rangle dx$$
$$= \int_\Omega \Big\langle -\operatorname{div}\Big(\mathcal{V}\big(x, \nabla \overline{u}(x)\big)\Big) - F_u^0\big(x, \overline{u}(x)\big), u(x)\Big\rangle dx.$$

Since $-\operatorname{div}\Big(\mathcal{V}\big(\cdot, \nabla \overline{u}(\cdot)\big)\Big) - F_u^0\big(\cdot, \overline{u}(\cdot)\big) \in L^{p(x)}(\Omega)$ we obtain (4.1). □

5. Example

Consider the problem

(5.1) $$-\operatorname{div}\Big(\varphi\big(x, |\nabla u|^{p(x)-1}\big)|\nabla u|^{p(x)-2}\nabla u\Big) = F_u^k(x, u)$$
$$u|_{\partial\Omega} = 0, \ u \in W_0^{1,p(x)}(\Omega)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded region satisfying (F1), F is subject to F1, F2, F3 and φ satisfies

O1 $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function; there exist constants $a_0, a_1 > 0$ such that for almost everywhere $x \in \Omega$ and $a_0 \leqslant \varphi_k(x, a) \leqslant a_1$ for all $a \in \mathbb{R}_+$; there exists a constant $m > 0$ such that

$$\varphi(x, a)a - \varphi(x, b)b \geqslant m(a - b)$$

for all $a \geqslant b$, $a, b \in \mathbb{R}$ and almost everywhere $x \in \Omega$.

We observe that assumptions F4, F5, F6 are obviously satisfied. Indeed, exactly as in [9] we may prove that operator

$$-\operatorname{div}\Big(\varphi\big(\cdot, |\nabla u(\cdot)|^{p(\cdot)-1}\big)|\nabla u(\cdot)|^{p(\cdot)-2}\nabla u(\cdot)\Big)$$

is monotone and by its definition it follows that it is radially continuous and coercive. We need to write V explicitly

$$\int_\Omega V\big(x, h(x)\big)\, dx = \int_\Omega \int_0^{|h(x)|} \varphi_k\big(x, s^{p(x)-1}\big) s^{p(x)-1}\, ds\, dx.$$

5.1. EXISTENCE   We take up an following example

$$(5.2) \qquad -\operatorname{div}\Big(\varphi\big(x, |\nabla u|^{p(x)-1}\big)|\nabla u|^{p(x)-2}\nabla u\Big) = |x|^2 \cdot |u(x)|^{\alpha(x)-1} u(x) + |x|^2,$$

$$u|_{\partial\Omega} = 0,\ u \in W_0^{1,p(x)}(\Omega)$$

where $\Omega = B(0, 1/2)$ is a ball in $\mathbb{R}^3$, $p \in C(\overline{\Omega})$, $p^- > 3$. We also assume $\alpha(x) \geqslant \alpha^- > p^+$ and that α has an upper bound $\alpha^+ > 2$.

We shall show that (F1)–(F3) hold. Here

$$F(x, u) = \frac{1}{\alpha(x) + 1}|x|^2 \cdot \big|u(x)\big|^{\alpha(x)+1} + |x|^2 \cdot u(x)$$

and it is Caratheodory function, convex in u. Of course F_u is also Caratheodory, such that $F_u(x, 0) \neq 0$ almost everywhere on Ω and $F_u(\cdot, w) \in L^\infty(\Omega)$ for any fixed $w \in \mathbb{R}$. Moreover, the functions $x \mapsto F(x, 0)$ and $x \mapsto F^*(x, 0)$ are integrable on Ω. Thus (F2) and (F3) hold. Clearly $\operatorname{vol}(\Omega) \leqslant (1/p^- + 1/q^-)^{-1}$.

Assume that $C_1 C_2 > 1$ and

O2 $a_0 > (C_1 C_2 C_S)/2$ for all $x \in \Omega$.

To conclude that (F1) is satisfied we only need to show that there exists a constant $d \geqslant C_1 C_2$ such that inequalities in (2.3) hold. We have

$$C_1 C_2 C_S \operatorname*{ess\,sup}_{x \in \Omega}\big|F_u(x, d)\big| = C_1 C_2 C_S \frac{d^{\alpha^+} + 1}{4},$$

$$C_1 C_2 C_S \operatorname*{ess\,sup}_{x \in \Omega}\big|F_u(x, -d)\big| = C_1 C_2 C_S \frac{d^{\alpha^+} - 1}{4}$$

Therefore we need to find d satisfying $d^{\alpha^+} + 1 \leqslant (4a_0 d)/(C_1 C_2 C_S)$. Since $d \geqslant 1$ we have that $d^{\alpha^+} + 1 \leqslant 2d^{\alpha^+}$ and since

$$2d^{\alpha^+} = \frac{4a_0 d}{C_1 C_2 C_S}$$

when $d = \big((2a_0)/(C_1 C_2 C_S)\big)^{1/(\alpha^+-1)}$ we assert by (O2) that $d \geqslant 1$ and inequalities in (2.3) hold. Finally we put

$$d = \max\Big\{C_1 C_2, \Big(\frac{2a_0}{C_1 C_2 C_S}\Big)^{1/(\alpha^+-1)}\Big\}.$$

We may now construct the set X which reads

$$X = \Big\{u \in W_0^{1,p(x)}(\Omega)\colon \|\nabla u\|_{p(x)} \leqslant \frac{d}{C_1 C_2},\ \big|u(x)\big| \leqslant d \text{ for all } x \in \Omega\Big\},$$

where d is given above. Corollary 3.2 provides existence of solution to (5.2).

5.2. STABILITY We consider the problem

$$(5.3) \qquad -\operatorname{div}\Big(\varphi\big(x,|\nabla u|^{p(x)-1}\big)|\nabla u|^{p(x)-2}\nabla u\Big) = |x|^2 \cdot \big|u(x)\big|^{\alpha_k(x)-1} u(x) + |x|^2,$$

$$u|_{\partial\Omega} = 0, \ u \in W_0^{1,p(x)}(\Omega),$$

with $\Omega = B\big(0,(1/2)\big) \subset \mathbb{R}^3$, p as above, $\alpha_k(x) \geqslant \alpha_k^- > p^+$ and with an upper bound $\alpha_k^+ > 2$ for each $k = 0, 1, \dots$.

As previously the assumptions (F1)–(F6) are satisfied and for each $k = 0, 1, \dots$ there exists a solution to (5.3). We assume (O2) and

O3 $\alpha_1^+ > \alpha_2^+ > \cdots > \alpha_k^+ > \alpha_{k+1}^+ > \cdots > \alpha_0^+ > 2.$

We obtain as previously that $d_k = \max\Big\{ C_1 C_2, \big((2a_0)/(C_1 C_2 C_S)\big)^{1/(\alpha_k^+ - 1)}\Big\}$ for every $k = 0, 1, \dots$ and by (O3) it is an nondecreasing sequence. Consequently the sets

$$X_k = \Big\{ u \in W_0^{1,p(x)}(\Omega) \colon \|\nabla u\|_{p(x)} \leqslant \frac{d_k}{C_1 C_2}, \ \big|u(x)\big| \leqslant d_k \text{ for all } x \in \Omega \Big\}$$

form into an nondecreasing sequence $X_1 \subseteq X_2 \subseteq \dots \subseteq X_0$. By Theorem 4.1 it follows that the solutions u_k of (5.3) up to a subsequence converge to a certain $u_0 \in X_0$ being a solution to

$$-\operatorname{div}\Big(\varphi\big(x,|\nabla u|^{p(x)-1}\big)|\nabla u|^{p(x)-2}\nabla u\Big) = |x|^2 \cdot \big|u(x)\big|^{\alpha_0(x)-1} u(x) + |x|^2,$$

$$u|_{\partial\Omega} = 0, \ u \in W_0^{1,p(x)}(\Omega).$$

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