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LETTERS TO THE EDITOR

A NOTE ON THE OCCURRENCE TIMES OF A PÓLYA–LUNDBERG PROCESS

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Recently Albrecht (1983) has pointed out that for the occurrence-time sequence $\{T_n\}$ of an appropriate version of a mixed Poisson process $\bar{N}(t)=N(t\Delta)$ (where $N(t)$ is a Poisson process with unit intensity and Δ is the mixing variable) the relation

$$\frac{T_n}{n}\to\frac{1}{\Delta}\quad \text{a.s.}\quad (n\to\infty) \tag{1}$$

holds. We shall show here by a martingale argument that such a relation is always valid for any version $N^*(t)$ of a Pólya–Lundberg process which is a mixed Poisson process with a gamma mixing variable with mean $\lambda>0$ and variance $\alpha\lambda^2$, $\alpha>0$.

Theorem. The sequence $\{S_n\}$ defined by $S_n=n/(1+\alpha\lambda T_n)$, $n\geqq 1$ forms a submartingale with $E(S_n)\leqq 2/\alpha$, $n\geqq 2$.

Proof. By the Markov property of $\{T_n\}$ (Pfeifer (1982)) we need only show

$$E\left(\frac{1}{1+\alpha\lambda T_{n+1}}\,\middle|\, T_n=t\right)\geqq\frac{n}{n+1}\frac{1}{1+\alpha\lambda t}\quad \text{a.s.} \tag{2}$$

which follows easily from the transition probabilities

$$P(T_{n+1}>s\mid T_n=t)=\left(\frac{1+\alpha\lambda t}{1+\alpha\lambda s}\right)^{n+1/\alpha}\quad \text{a.s.,}\quad s\geqq t\geqq 0. \tag{3}$$

Also, $E((n-1)/T_n)=\lambda$ for $n\geqq 2$, hence $E(S_n)\leqq 2/\alpha$.

By the martingale convergence theorem now $S_n\to S$ a.s. $(n\to\infty)$ for some random variable S, from which we also have

$$\frac{T_n}{n}\to\frac{1}{\alpha\lambda S}\quad \text{a.s.}\quad (n\to\infty), \tag{4}$$

i.e. $\alpha\lambda S$ is a canonical representation of the mixing variable.

Note that using estimations given in Albrecht (1983) (4) also implies that for the process $N^*(t)$ itself

$$\frac{N^*(t)}{t}\to\alpha\lambda S\quad \text{a.s.}\quad (t\to\infty). \tag{5}$$

References

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