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STABILITY OF PRODUCTION ECONOMIES

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Abstract

In this paper, the concept of essential equilibria for production economies is first given. We then prove that in 'most' production economies (in the sense of Baire category) all equilibria are essential.

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1. Introduction

In Section 8 of [4], Dierker introduced the concept of essential equilibria for pure exchange economies and proved that in 'most' pure exchange economies (in the sense of Baire category) all equilibria are essential.

The concept of essentiality for equilibria is a stability property. In [4], the stability of equilibria with perturbations on demand function and initial endowment of each consumer was studied.

In this paper, the concept of essential equilibria for production economies is first given. We then study the stability of equilibria with perturbations on utility-maximizing consumptions and initial endowment of each consumer and on profit-maximizing productions of each producer. We also prove that in 'most' production economies (in the sense of Baire category) all equilibria are essential.

2. Preliminaries

Let (X, d) be a metric space and $K(X)$ be the space of all non-empty compact subsets of X equipped with the Hausdorff metric h which is induced by the metric d. For each $\epsilon > 0$ and $A \in K(X)$, let $U(\epsilon, A) = \{x \in X : d(u, x) < \epsilon$ for some $u \in A\}$. Let Y be a Hausdorff topological space and $F : Y \to K(X)$ be a multivalued mapping. Then F is said to be *upper semicontinuous* (respectively, *lower semicontinuous*) *at* $y \in Y$ if for each $\epsilon > 0$, there is an open neighborhood $O(y)$ of y in Y such that $F(y') \subset U(\epsilon, F(y))$ (respectively, $F(y) \subset U(\epsilon, F(y')))$ for all $y' \in O(y)$; F is said to be *upper semicontinuous* (respectively, *lower semicontinuous*) *on* Y if it is upper semicontinuous (respectively, lower semicontinuous) at each point $y \in Y$ and F is said to be *continuous at* y if F is both upper semicontinuous and lower semicontinuous at $y \in Y$. Since F is compact-valued, if F is upper semicontinuous on Y, F is also called a *usco mapping*. Recall that a subset $Q \subset Y$ is called a *residual set* in Y if it is a countable intersection of open dense subsets of Y.

The following lemma is due to Fort [6, Theorem 2]:

LEMMA 1. *Let X be a metric space, Y be a Hausdorff topological space and $F : Y \to K(X)$ be a usco mapping. Then the set of all points where F is lower semicontinuous is a residual set in Y.*

Recall that a topological space X is a *Baire space* [5, p. 249] if the intersection of each countable family of open dense sets in X is dense in X.

LEMMA 2. *Let X be a metric space, Y be a Baire space and $F : Y \to K(X)$ be a usco mapping. Then the set of points where F is lower semicontinuous is a dense residual set in Y.*

PROOF. Since Y is a Baire space, a residual set in Y is dense, the result now follows from Lemma 1.

In [9], Oxtoby introduced the notion of a pseudo-complete space as follows: A Hausdorff topological space Y is called *quasi-regular* if every non-empty open set in Y contains the closure of some non-empty open set in Y. A family $\mathbb{B}$ of non-empty open sets in Y is called a *pseudo-base* if every non-empty open set in Y contains at least one member of $\mathbb{B}$. Y is called *pseudo-complete* if Y is quasi-regular and if there exists a sequence $\{\mathbb{B}(k)\}_{k=1}^{\infty}$ of pseudo-bases in Y with the property that $\bigcap_{k=1}^{\infty} U_k \neq \emptyset$ whenever $U_k \in \mathbb{B}(k)$ and $U_k \supset \overline{U_{k+1}}$ for $k = 1, 2, \ldots$, where $\overline{U_{k+1}}$ denotes the closure of U_{k+1} in Y.

3. The model

The mathematical model of a production economy is defined as follows (see [3]):

Suppose that there are l commodities. Let $P = \{x = (x_1, \ldots, x_l) \in \mathbb{R}^l : x_h > 0,\ h = 1, \ldots, l\}$ and $L = (0, \infty)$. The set $\Delta = \{x \in P : \sum_{h=1}^{l} x_h = 1\}$ is called the

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price simplex. Consider a production economy with m consumers and n producers. Suppose $e_i \in P$ is the initial endowment of the ith consumer, and for a given price vector $p \in \Delta$, the ith consumer chooses his utility-maximizing consumptions $\xi_i(p, p \cdot e_i) \subset \overline{P}, i = 1, 2, \ldots, m$ ($p \cdot e_i$ is the inner product of p and e_i) and the jth producer chooses his profit-maximizing productions $\eta_j(p) \subset \mathbb{R}^l$, $j = 1, \ldots, n$. The excess demand correspondence is defined by

$$\zeta(p, e) = \sum_{i=1}^{m} \xi_i(p, p \cdot e_i) - \sum_{j=1}^{n} \eta_j(p) - \sum_{i=1}^{m} e_i$$

where $e = (e_1, \ldots, e_m) \in P^m$ and $p \in \Delta$.

LEMMA 3. *Suppose that the following conditions hold:*

(i) *for each $p \in \Delta$ and each $w \in L$, $\xi_i(p, w)$ is non-empty compact convex and ξ_i is upper semicontinuous on $\Delta \times L$ for $i = 1, 2, \ldots, m$;*
(ii) *for each $p \in \Delta$, $\eta_j(p)$ is non-empty compact convex and η_j is upper semi-continuous and bounded from above on Δ for $j = 1, \ldots, n$;*
(iii) *for each $p \in \Delta$ and each $z \in \zeta(p, e)$, $p \cdot z = 0$ (Walrasian Law);*
(iv) *for each sequence $\{(p^k, w^k)\}_{k=1}^{\infty}$ in $\Delta \times L$ with $(p^k, w^k) \to (p, w) \in (\overline{\Delta} \setminus \Delta) \times L$, there is some i such that $d(0, \xi_i(p^k, w^k)) = \inf_{u \in \xi_i(p^k, w^k)} \|u\| \to \infty$.*

Then there exists $p^* \in \Delta$ such that $0 \in \zeta(p^*, e)$.

PROOF. Fix $e \in P^m$; by (i) and (ii), for each $p \in \Delta$,

$$T(p) = \zeta(p, e) = \sum_{i=1}^{m} \xi_i(p, p \cdot e) - \sum_{j=1}^{n} \eta_j(p) - \sum_{i=1}^{m} e_i$$

is non-empty, compact convex and by Theorem 7.3.15 (ii) of [8], $T(p)$ is upper semicontinuous on Δ. Let $\{p^k\}_{k=1}^{\infty}$ be any sequence in Δ with $p^k \to p \in \overline{\Delta} \setminus \Delta$. If $z^k \in T(p^k)$ for $k = 1, 2, \ldots$, then since $w_i^k = p^k \cdot e_i \in L$ for $k = 1, 2, \ldots$, we have $w_i^k \to p \cdot e_i = w \in L$. By (i), (ii) and (iv), $d(0, T(p^k)) \to \infty$. Set $\hat{p} = (1/l, \ldots, 1/l) \in \Delta$; then there is k_1 such that $\hat{p} \cdot z^k > 0$ for all $k \geq k_1$.

Thus all conditions of Lemma 1 of [7] (or Theorem 18.13 of [1]) are satisfied so that there exists $p^* \in \Delta$ such that $0 \in T(p^*) = \zeta(p^*, e)$.

REMARK. The condition (iv) in Lemma 3 is a variant of Assumption (A) of Debreu in [2, p. 388] (see also Tarafdar and Thompson [10]). The condition (iv) in Lemma 3 expresses the idea that every commodity is demanded by some consumer.

Let C be the set of all $G = (\xi_1, \ldots, \xi_m; \eta_1, \ldots, \eta_n)$ which satisfy all the conditions of Lemma 3. Let h be the Hausdorff metric on $K(\mathbb{R}^l)$ induced by the usual metric d

on $\mathbb{R}^l$. Let $B = \{O(G, \epsilon) : G \in C \text{ and } 0 < \epsilon < 1\}$, where $O(G, \epsilon)$ is the set

$$\Big\{\hat{G} = (\hat{\xi}_1, \dots, \hat{\xi}_m; \hat{\eta}_1, \dots, \hat{\eta}_n) \in C : \max_{1 \le i \le m} \sup_{(p,w)\in\Delta\times L} h(\hat{\xi}_i(p, w), \xi_i(p, w)) + \max_{1\le j \le n} \sup_{p\in\Delta} h(\hat{\eta}_j(p), \eta_j(p)) < \epsilon\Big\}.$$

Then B is a base for the topology of uniform convergence on C. If $\epsilon > 0$, denote by $\overline{O}(G, \epsilon)$ the set

$$\Big\{\hat{G} = (\hat{\xi}_1, \dots, \hat{\xi}_m; \hat{\eta}_1, \dots, \hat{\eta}_n) \in C : \max_{1 \le i \le m} \sup_{(p,w)\in\Delta\times L} h(\hat{\xi}_i(p, w), \xi_i(p, w)) + \max_{1\le j \le n} \sup_{p\in\Delta} h(\hat{\eta}_j(p), \eta_j(p)) \le \epsilon\Big\};$$

it can be shown that the closure of $O(G, \epsilon)$ is $\overline{O}(G, \epsilon)$.

LEMMA 4. *C is a Baire space.*

PROOF. By the formulation (5.1) of [9], it is sufficient to prove that C is pseudo-complete.

It is obvious that C is quasi-regular. Let $B(k) = \{O(G, a) : G \in C \text{ and } 0 < a < 1/k\}$; then $B(k)$ is a pseudo-base for each $k = 1, 2, \dots$. For each $k = 1, 2, \dots$, let $U_k \in B(k)$ be such that $U_k \supset \overline{U_{k+1}}$; we need to prove that $\bigcap_{k=1}^{\infty} U_k \neq \emptyset$. Denote $U_k = O(G^k, a_k)$; then $0 < a_k < 1/k$, $k = 1, 2, \dots$ and

$$O(G^1, a_1) \supset \overline{O}(G^2, a_2) \supset O(G^2, a_2) \supset \overline{O}(G^3, a_3) \supset \cdots$$

where $G^k = (\xi_1^k, \dots, \xi_m^k; \eta_1^k, \dots, \eta_n^k)$ for $k = 1, 2, \dots$. Now for any $k = 1, 2, \dots$ and any $q = 1, 2, \dots$, since $G^{k+q} \in O(G^k, a_k)$, we have

$$\max_{1 \le i \le m} \sup_{(p,w)\in\Delta\times L} h(\xi_i^k(p, w), \xi_i^{k+q}(p, w)) < a_k < 1/k.$$

For any $i = 1, 2, \dots, m$, $\{\xi_i^k(p, w)\}_{k=1}^{\infty}$ is a Cauchy sequence uniformly in $(p, w) \in \Delta \times L$, by Theorem 4.3.9 and Theorem 4.3.13 in [8]; for each $(p, w) \in \Delta \times L$, there is a non-empty compact subset $\xi_i(p, w)$ of $\overline{P}$ such that $h(\xi_i^k(p, w), \xi_i(p, w)) \to 0$ as $k \to \infty$, uniformly in $(p, w) \in \Delta \times L$.

Similarly, for each $j = 1, 2, \dots, n$ and each $p \in \Delta$, there is a non-empty compact convex subset $\eta_j(p)$ of $\mathbb{R}^l$ such that $h(\eta_j^k(p), \eta_j(p)) \to 0$ as $k \to \infty$, uniformly in $p \in \Delta$, and also for each $k = 1, 2, \dots$, we have

$$(*) \qquad \max_{1 \le i \le m} \sup_{(p,w)\in\Delta\times L} h(\xi_i^k(p, w), \xi_i(p, w))) + \max_{1\le j \le n} \sup_{p\in\Delta} h(\eta_j^k(p), \eta_j(p)) \le a_k.$$

Set $G = (\xi_1(p, w), \ldots, \xi_m(p, w); \eta_1(p), \ldots, \eta_n(p))$; we shall prove that $G \in C$ and $G \in \bigcap_{k=1}^{\infty} U_k$. For each $\epsilon > 0$, there is k_1 such that

$$\sup_{(p,w)\in\Delta\times L} h(\xi_i^k(p, w), \xi_i(p, w)) < \epsilon/3$$

for all $k \geq k_1$ and $i = 1, 2, \ldots, m$. For each $(p, w) \in \Delta \times L$, since $\xi_i^{k_1}$ is upper semicontinuous at $(p, w) \in \Delta\times L$, there is $\delta > 0$ such that $\xi_i^{k_1}(p', w') \subset U(\epsilon/3, \xi_i^{k_1}(p, w))$ whenever $p' \in \Delta$ with $\|p - p'\| < \delta$, and $w' \in L$ with $|w - w'| < \delta$, for all $i = 1, 2, \ldots, m$. Thus,

$$\xi_i(p', w') \subset U(\epsilon/3, \xi_i^{k_1}(p', w')) \subset U(2\epsilon/3, \xi_i^{k_1}(p, w)) \subset U(\epsilon, \xi_i(p, w))$$

and hence ξ_i is upper semicontinuous at $(p, w) \in \Delta \times L$ for all $i = 1, 2, \ldots, m$. By the same method, η_j is upper semicontinuous at $p \in \Delta$ for all $j = 1, \ldots, n$. Let $\{(p^q, w^q)\}_{q=1}^{\infty}$ be a sequence in $\Delta \times L$ with $(p^q, w^q) \to (p, w) \in (\overline{\Delta} \setminus \Delta) \times L$. Since $\{G^k\}_{k=1}^{\infty} \subset C$, without loss of generality we may assume that $d(0, \xi_1^k(p^q, w^q)) \to \infty$ for all $k = 1, 2, \ldots$. Since $h(\xi_1^k(p^q, w^q), \xi_1(p^q, w^q)) < \epsilon/3$ for all $k \geq k_1$, we must also have $d(0, \xi_1(p^q, w^q)) \to \infty$ as $q \to \infty$. Now let $p \in \Delta$ and $z \in \zeta(p, e) = \sum_{i=1}^m \xi_i(p, p\cdot e_i) - \sum_{j=1}^n \eta_j(p) - \sum_{i=1}^m e_i$. We need to prove that $p \cdot z = 0$. Suppose to the contrary that $p \cdot z \neq 0$. Since $\xi_i^k \to \xi_i$, $\eta_j^k \to \eta_j$ uniformly on $\Delta \times L$, there are k_2 and $x_i^{k_2} \in \xi_i^{k_2}(p, p\cdot e_i)$, $\eta_j^{k_2} \in \eta_j^{k_2}(p)$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ such that

$$p \cdot \left(\sum_{i=1}^m x_i^{k_2} - \sum_{j=1}^m y_j^{k_2} - \sum_{i=1}^m e_i \right) \neq 0.$$

This contradicts the Walrasian Law that for each $p \in \Delta$ and each $z^{k_2} \in \zeta^{k_2}(p, e) = \sum_{i=1}^m \xi_i^{k_2}(p, p\cdot e_i) - \sum_{j=1}^n \eta_j^{k_2}(p) - \sum_{i=1}^m e_i$, $p \cdot z^{k_2} = 0$. Therefore $G \in C$.

By $(*)$, $G \in \overline{O}(G^k, a_k)$ for all $k = 1, 2, \ldots$. Hence $G \in \bigcap_{k=2}^{\infty} \overline{O}(G^k, a_k) \subset \bigcap_{k=1}^{\infty} O(G^k, a_k) = \bigcap_{k=1}^{\infty} U_k$ so that $\cap_{k=1}^{\infty} U_k \neq \emptyset$. Thus C is pseudo-complete.

Since P^m is a locally compact Hausdorff space, it is pseudo-complete [9, p. 164]. Let $Y = C \times P^m$, by Theorem 6 of [9], Y is pseudo-complete and by formulation (5.1) of [9], Y is a Baire space.

DEFINITION 1. If $E = (\xi_1, \ldots, \xi_m; \eta_1, \ldots, \eta_n; e_1, \ldots, e_m) \in Y$, then $p \in \Delta$ is called an *equilibrium point* of the economy E if $0 \in \sum_{i=1}^m \xi_i(p, p\cdot e_i) - \sum_{j=1}^n \eta_j(p) - \sum_{i=1}^m e_i$.

Denote by $W(E)$ the set of all equilibria of the economy $E \in Y$; then $W(E) \neq \emptyset$ by Lemma 3. In the next section, we shall study the stability of $W(E)$.

4. Main results

LEMMA 5. *For each $E \in Y$, $W(E)$ is a compact set.*

PROOF. Since $W(E) \subset \overline{\Delta}$ and $\overline{\Delta}$ is compact, it is sufficient to prove that $W(E)$ is closed in $\overline{\Delta}$.

Let $\{p^k\}_{k=1}^{\infty}$ be any sequence in $W(E)$ with $p^k \to p \in \overline{\Delta}$. Suppose $p \in \overline{\Delta} \setminus \Delta$. Since for each $i = 1, 2, \ldots, m$, $w_i^k = p^k \cdot e_i \in L$ so that $w_i^k \to p \cdot e_i = w_i \in L$, there is some $i \in \{1, \ldots, m\}$ such that

$$(**) \qquad d(0, \xi_i(p^k, w^k)) \to \infty$$

as $k \to \infty$. Since each η_j is bounded from above, it follows from $(**)$ that

$$d\left(0, \sum_{i=1}^{m} \xi_i(p^k, p^k \cdot e_i) - \sum_{j=1}^{n} \eta_j(p^k) - \sum_{i=1}^{m} e_i\right) \to \infty$$

as $k \to \infty$ which contradicts that

$$0 \in \sum_{i=1}^{m} \xi_i(p^k, p^k \cdot e_i) - \sum_{j=1}^{n} \eta_j(p^k) - \sum_{i=1}^{m} e_i$$

for $k = 1, 2, \ldots$. Hence $p \in \Delta$.

If $0 \notin \sum_{i=1}^{m} \xi_i(p, p \cdot e_i) - \sum_{j=1}^{n} \eta_j(p) - \sum_{i=1}^{m} e_i$, since ξ_i is upper semicontinuous at $(p, p \cdot e_i)$, η_j is upper semicontinuous at p and $p^k \to p$ and $p^k \cdot e_i \to p \cdot e_i$, it is easy to show that there is k_1 such that $0 \notin \sum_{i=1}^{m} \xi_i(p^k, p^k \cdot e_i) - \sum_{j=1}^{n} \eta_j(p^k) - \sum_{i=1}^{m} e_i$ for all $k \geq k_1$. This contradicts that $p^k \in W(E)$ for $k = 1, 2, \ldots$. Therefore we must have $0 \in \sum_{i=1}^{n} \xi_i(p, p \cdot e_i) - \sum_{j=1}^{n} \eta_j(p) - \sum_{i=1}^{n} e_i$. Hence $p \in W(E)$ so that $W(E)$ is closed in $\overline{\Delta}$.

By Lemma 5, the mapping $E \to W(E)$ indeed defines a multivalued mapping $W : Y \to K(\mathbb{R}^l)$.

LEMMA 6. *W is upper semicontinuous on Y.*

PROOF. Suppose that W were not upper semicontinuous at $E \in Y$; then there are $\epsilon_0 > 0$ and a sequence $\{E^k\}_{k=1}^{\infty}$ in Y with $E^k \to E$ such that for each $k = 1, 2, \ldots$, there exists $p^k \in W(E^k)$ with $p^k \notin U(\epsilon_0, W(E))$.

Since $\{p^k\}_{k=1}^{\infty} \subset \overline{\Delta}$ and $\overline{\Delta}$ is compact, we may assume that $p^k \to p \in \overline{\Delta}$. Note that we must have $p \notin U(\epsilon_0, W(E))$. If $p \in \overline{\Delta} \setminus \Delta$, then since $w_i^k = p^k \cdot e_i^k \in L$, $w_i^k \to p \cdot e_i \in L$, it follows that

$$d\left(0, \sum_{i=1}^{m} \xi_i(p^k, p^k \cdot e_i) - \sum_{j=1}^{n} \eta_j(p^k) - \sum_{i=1}^{m} e_i\right) \to \infty.$$

Since $\xi_i^k \to \xi_i$, $\eta_j^k \to \eta_j$ and $e_i^k \to e_i$ for $i = 1, 2, \ldots, m$ and $j = 1, \ldots, n$,

$$d\left(0, \sum_{i=1}^m \xi_i^k(p^k, p^k \cdot e_i^k) - \sum_{j=1}^n \eta_j^k(p^k) - \sum_{i=1}^m e_i^k\right) \to \infty.$$

This contradicts $0 \in \sum_{i=1}^m \xi_i^k(p^k, p^k \cdot e_i^k) - \sum_{j=1}^n \eta_j^k(p^k) - \sum_{i=1}^m e_i^k$ for $k = 1, 2 \ldots$. Hence $p \in \Delta$.

If $0 \notin \sum_{i=1}^m \xi_i(p, p \cdot e_i) - \sum_{j=1}^n \eta_j(p) - \sum_{i=1}^m e_i$, since $\sum_{i=1}^m \xi_i(p, p \cdot e_i) - \sum_{j=1}^n \eta_j(p) - \sum_{i=1}^m e_i$ is compact, there is $\delta > 0$ such that

$$0 \notin U\left(\delta, \sum_{i=1}^m \xi_i(p, p \cdot e_i) - \sum_{j=1}^n \eta_j(p) - \sum_{i=1}^m e_i\right).$$

Since $\xi_i^k \to \xi_i$, $\eta_j^k \to \eta_j$ and $e_i^k \to e_i$, for $\epsilon = \delta/(2m+n)$, there is k_1 such that

$$h(\xi_i^k(p^k, p^k \cdot e_i^k), \xi_i(p^k, p^k \cdot e_i^k)) < \epsilon/2, \quad h(\eta_j^k(p^k), \eta_j(p^k)) < \epsilon/2, \quad \|e_i^k - e_i\| < \epsilon$$

for all $k \geq k_1$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Since $p^k \to p$, $w_i^k = p^k \cdot e_i^k \in L$, $w_i^k \to p \cdot e_i \in L$, ξ_i is upper semicontinuous at $(p, p \cdot e)$ and η_j is upper semicontinuous at p, there is $k_2 \geq k_1$ such that $\xi_i(p^k, p^k \cdot e_i^k) \subset U(\epsilon/2, \xi_i(p, p \cdot e_i))$ and $\eta_j(p^k) \subset U(\epsilon/2, \eta_j(p))$ for all $k \geq k_2$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Thus

$$\xi_i^k(p^k, p^k \cdot e_i^k) \subset U(\epsilon/2, \xi_i(p^k, p^k \cdot e_i^k)) \subset U(\epsilon, \xi_i(p, p \cdot e_i))$$

and

$$\eta_j^k(p^k) \subset U(\epsilon/2, \eta_j(p^k)) \subset U(\epsilon, \eta_j(p))$$

for all $k \geq k_2$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. It follows that

$$\sum_{i=1}^m \xi_i^k(p^k, p^k \cdot e_i^k) - \sum_{j=1}^m \eta_j^k(p^k) - \sum_{i=1}^m e_i^k$$
$$\subset U\left(\delta, \sum_{i=1}^m \xi_i(p, p \cdot e_i) - \sum_{j=1}^m \eta_j(p) - \sum_{i=1}^m e_i\right)$$

for all $k \geq k_2$. This contradicts that

$$0 \in \sum_{i=1}^m \xi_i^k(p^k, p^k \cdot e_i^k) - \sum_{j=1}^n \eta_j^k(p^k) - \sum_{i=1}^m e_i^k$$

for all $k = 1, 2, \ldots$. Hence we must have

$$0 \in \sum_{i=1}^m \xi_i(p, p \cdot e_i) - \sum_{j=1}^m \eta_j(p) - \sum_{i=1}^m e_i$$

so that $p \in W(E)$ which again contradicts that $p \notin U(\epsilon_0, W(E))$. Therefore W must be upper semicontinuous at $E \in Y$.

DEFINITION 2. If $E \in Y$, then $p \in W(E)$ is said to be an *essential equilibrium point* of the economy E provided that for each $\epsilon > 0$, there is $\delta > 0$ such that for each $E' \in Y$ with $E' \in O(E, \delta)$, there exists $p' \in W(E')$ with $\|p - p'\| < \epsilon$. The economy E is said to be *essential* if every $p \in W(E)$ is essential.

THEOREM 1. *W is lower semicontinuous at $E \in Y$ if and only if E is essential.*

PROOF. Suppose that W is lower semicontinuous at $E \in Y$. Then for each $\epsilon > 0$, there is $\delta > 0$ such that for each $E' \in Y$ with $E' \in O(E, \delta)$, we have $W(E) \subset U(\epsilon, W(E'))$. It follows that for each $p \in W(E)$, there is $p' \in W(E')$ with $\|p - p'\| < \epsilon$. Thus each $p \in W(E)$ is an essential equilibrium point and hence E is essential.

Conversely, suppose that E is essential. If W were not lower semicontinuous at $E \in Y$, then there exist $\epsilon_0 > 0$ and a sequence $\{E^k\}_{k=1}^{\infty}$ in Y with $E^k \to E$ such that for each $k = 1, 2, \ldots$, there is $p^k \in W(E)$ with $p^k \notin U(\epsilon_0, W(E^k))$. Since $W(E)$ is compact, we may assume that $p^k \to p \in W(E)$. Since p is essential, $E^k \to E$ and $p^k \to p$, there is k_1 such that $\|p^k - p\| < \epsilon_0/2$ and $p \in U(\epsilon_0/2, W(E^k))$ for all $k \geq k_1$. Hence $p^k \in U(\epsilon_0, W(E^k))$ for all $k \geq k_1$ which contradicts that $p^k \notin U(\epsilon_0, W(E^k))$ for all $k = 1, 2, \ldots$. Hence W must be lower semicontinuous at E.

By Lemma 6 and Theorem 1, we know that $E \in Y$ is essential if and only if W is continuous at $E \in Y$: for each $\epsilon > 0$, there is $\delta > 0$ such that $h(W(E'), W(E)) < \epsilon$ for each $E' \in Y$ and $E' \in O(E, \delta)$, that is, the set $W(E)$ of equilibria is stable: $W(E')$ is close to $W(E)$ whenever E' is close to E.

THEOREM 2. *The set of essential economies in Y is a dense residual set in Y.*

PROOF. By Lemma 5 and Lemma 6, W is a usco mapping. Since Y is a Baire space, by Lemma 1, the set of points where W is lower semicontinuous is a dense residual set in Y. By Theorem 1, the set of essential economies in Y is a dense residual set in Y.

Thus, we proved that in 'most' production economies (in the sense of Baire category) all equilibria are essential.

Finally, we shall give a sufficient condition that $E \in Y$ is essential:

THEOREM 3. *If $E \in Y$ is such that $W(E)$ is singleton set, then E is essential.*

PROOF. Suppose $W(E) = \{p\}$. By Lemma 6, W is upper semicontinuous at E. Thus for each $\epsilon > 0$, there is $\delta > 0$ such that for each $E' \in Y$, $E' \in O(E, \delta)$ implies $W(E') \subset U(\epsilon, W(E))$. But $W(E) = \{p\}$, so $W(E) = \{p\} \subset U(\epsilon, W(E'))$. This show that W is lower semicontinuous at E. By Theorem 1, E is essential.

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