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A DIVISOR PROBLEM FOR VALUES OF POLYNOMIALS

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ABSTRACT. In this article we investigate the average order of the arithmetical function

$$d^*(n) = \#\{ (u,v) \in \mathbb{N}^2 \ : \ p_1(u)p_2(v) = n \},$$

where $p_1(t), p_2(t)$ are polynomials in $\mathbb{Z}[t]$, of equal degree, positive and increasing for $t \geq 1$. Using the modern method for the estimation of exponential sums ("Discrete Hardy-Littlewood Method"), we establish an asymptotic result which is as sharp as the best one known for the classical divisor problem.

RÉSUMÉ. Dans cet article, on étudie l'ordre moyen de la fonction arithmétique

$$d^*(n) = \#\{ (u,v) \in \mathbb{N}^2 \ : \ p_1(u)p_2(v) = n \},$$

où $p_1(t), p_2(t)$ sont des polynômes dans $\mathbb{Z}[t]$, de degrés égaux, qui sont positifs et croissants pour $t \geq 1$. En utilisant la méthode moderne pour l'estimation de sommes exponentielles ("méthode discrète de Hardy-Littlewood"), on obtient un comportement asymptotique, aussi précis que le meilleur résultat connu, concernant le problème classique des diviseurs.

1. Introduction. The classical Dirichlet divisor problem concerns the number of ways to write a positive integer n as a product of two natural numbers u_1, u_2. (For its history and the present "state of art", see the recent textbooks of Fricker [1] and Krätzel [4].)

It seems natural to consider variants of this problem where one or both of u_1, u_2 are subject to certain restrictions. For instance, the case that u_1, u_2 (or at least one of them) lie in a given arithmetic progression has been discussed at length in the recent literature: See [7], [8], [9], [10], and [11].

In this paper, we consider the situation that u_1, u_2 are values of given polynomials of equal degree (with integer coefficients), corresponding to positive integer arguments. To be precise, let p_1, p_2 be two polynomials, both of degree $k \geqq 1$, with integer coefficients,

$$p_s(t) = a_k^{(s)} t^k + \ldots + a_1^{(s)} t + a_0^{(s)} \qquad (a_k^{(s)} \neq 0)$$

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($s = 1$ or 2, throughout the paper), and suppose that the corresponding polynomial functions are positive and strictly monotonic increasing on $[1,\infty)$. Denote by $d^*(n)$ the number of ways to write the natural number n in the form $p_1(u)p_2(v)$ with positive integers u, v, i.e.

$$d^*(n) = d^*_{p_1,p_2}(n) = \#\{\,(u,v) \in \mathbb{N}^2 \;:\; p_1(u)p_2(v) = n\,\}\,.$$

In this article we study the "average order" of this arithmetic function $d^*(n)$, i.e. we consider the summatory function

$$D^*(x) = \sum_{n \le x} d^*(n)\,,$$

where x is a large real variable. It is clear that $D^*(x)$ is the number of lattice points (of the standard lattice $\mathbb{Z}^2$) in the planar domain

$$\{\,(u,v) \in \mathbb{R}^2 \;:\; u > 0,\, v > 0, \quad p_1(u)p_2(v) \le x\}\,.$$

(Enlightening surveys of the theory of lattice points in large domains may be found in the textbooks [1] and [4]. In these monographs, the definitions of the order symbols O, o, $\ll$, and $\asymp$, which are used in the sequel, can be looked up as well.) We note that, from this purely geometric viewpoint, it is possible to drop the requirement that the coefficients $a_l^{(s)}$ be integers. Doing so, it obviously involves no loss of generality to assume that

$$a_k^{(1)} = a_k^{(2)} = 1\,,$$

as we shall do in what follows.

The objective of the present paper is a proof of the following asymptotic result.

THEOREM. *For $x \longrightarrow \infty$, fixed $k \geqq 1$, and fixed coefficients $a_l^{(s)}$ ($s = 1,2,\ l = 0,\ldots,k-1$), we have*

$$D^*(x) = x^{1/k}\log(x^{1/k}) - Cx^{1/k} + O\Big(x^{\frac{7}{22k}}(\log x)^{\frac{67}{22}}\Big)\,,$$

where

$$C = 1 + \sum_{s=1,2}\Big(-\tfrac{1}{2}(p_s(1))^{-1/k} + \frac{1}{k}\int_1^\infty \psi(u)\frac{p_s'(u)}{(p_s(u))^{1+1/k}}du + \int_1^\infty \frac{d}{du}\Big(\frac{u}{(p_s(u))^{1/k}}\Big)\log u\,du\Big)$$

($\psi(u) = u - [u] - \frac{1}{2}$ throughout), and the O-constant depends on p_1 and p_2.

REMARK. This estimate contains, as a special case, the to date sharpest result for the classical divisor problem: See Iwaniec and Mozzochi [3], Huxley [2], and W. Müller and Nowak [7]. In these papers a completely new method for the estimation of exponential sums has been developed ("Discrete Hardy-Littlewood Method") which, for quite a number of problems, not only yields improved results but also considerably simplifies the treatment.

This very fact provides some motivation to deal with our present topic just nowadays: With the elementary error term $O(x^{1/2k})$, our Theorem can be established using only tools which were available to Dirichlet already. However, the methods employed in modern times (from 1950 on, say) to improve the estimate in the classic divisor problem are technically too involved to permit an ("easy") extension to our general situation.

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2. Preliminaries.

LEMMA 1. *Let q_s denote the inverse function of p_s, defined on $[p_s(1),\infty)$, then for w sufficiently large we have a series expansion*

$$q_s(w) = w^{1/k} + b_0^{(s)} + \sum_{j=1}^{\infty} b_j^{(s)} w^{-j/k},$$

which may be differentiated termwise up to an arbitrary order. In particular,

$$b_0^{(s)} = -\frac{1}{k} a_{k-1}^{(s)}. \tag{2.1}$$

PROOF. This is a simple exercise in classic analysis.

The deduction of our Theorem is based on a simple elementary device ("hyperbola method") which gives

$$D^*(x) = \sum_{s=1,2}\Big(\sum_{n\le q_s(\sqrt{x})}[q_{3-s}(\frac{x}{p_s(n)})]\Big) - [q_1(\sqrt{x})][q_2(\sqrt{x})] =$$

$$= \sum_{s=1,2}(T_s(x) - R_s(x)) - \tfrac{1}{2}(q_1(\sqrt{x}) + q_2(\sqrt{x})) - [q_1(\sqrt{x})][q_2(\sqrt{x})] + O(1) \tag{2.2}$$

where

$$T_s(x) \overset{\text{def}}{=} \sum_{n\le q_s(\sqrt{x})} q_{3-s}(\frac{x}{p_s(n)}), \qquad R_s(x) \overset{\text{def}}{=} \sum_{n\le q_s(\sqrt{x})} \psi\Big(q_{3-s}(\frac{x}{p_s(n)})\Big).$$

3. Estimation of $R_s(x)$: The "Discrete Hardy-Littlewood Method".

LEMMA 2. *Let $M_x \ge 1$, $V_x > 0$, $f_x(u)$ a real function with derivatives up to an arbitrary order on $M_x \le u \le 2M_x$. (The subscript x indicates dependance on a large real parameter x.) Suppose that, for some $\sigma > 0$, and every $m \in \mathbb{N}$,*

$$f_x^{(m)}(u) = \Big(\frac{d^m}{du^m}(V_x u^{-\sigma})\Big)(1 + o(1)) \tag{3.1}$$

where $o(1)$ refers to $x \to \infty$ and is meant uniformly in $M_x \le u \le 2M_x$ (but not necessarily in $m \in \mathbb{N}$). Suppose furthermore that

$$V_x \gg M_x^{1+\sigma}.$$

Then it follows that

$$\sum_{M_x < n \leqq 2M_x} \psi(f_x(n)) \ll (V_x M_x^{1-\sigma})^{7/22}(\log(2V_x M_x^{1-\sigma}))^{45/22}.$$

PROOF. This is a suitable special case of the main result in Huxley [2] (refined slightly in W. Müller and Nowak [7]). In our present statement we have only formulated explicitly a sufficient condition for the function $f_x(u)$ on $[M_x, 2M_x]$ to admit the use of (one-dimensional) exponent pairs. (Cf. Krätzel [4], p. 52.)

We are going to apply this deep estimate to the function

$$f(u) = q_{3-s}\Big(\frac{x}{p_s(u)}\Big) \tag{3.2}$$

occuring in $R_s(x)$. Estimating the first terms trivially, we may suppose that

$$x^{1/4k} \ll u \ll x^{1/2k} \tag{3.3}$$

(cf. Lemma 1), hence

$$x^{1/2} \ll \frac{x}{p_s(u)} \ll x^{3/4}. \tag{3.4}$$

We only have to show that, in this range, $f(u)$ satisfies the requirement (3.1) of Lemma 2.

LEMMA 3. *The function $f(u)$ defined in* (3.2) *satisfies (for every $m \in \mathbb{N}$)*

$$f^{(m)}(u) = (1 + o(1))\frac{d^m}{du^m}(x^{1/k}u^{-1}),$$

as $x \longrightarrow \infty$, uniformly in $x^{1/4k} \ll u \ll x^{1/2k}$.

PROOF. By (3.2) and Lemma 1, we have (dropping the dependance on $s = 1, 2$ for notational simplicity)

$$f(u) = \Big(\frac{x}{p(u)}\Big)^{1/k} + b_0 + \sum_{j=1}^{\infty} b_j\Big(\frac{p(u)}{x}\Big)^{j/k}. \tag{3.5}$$

Now it is clear that

$$\Big(\frac{x}{p(u)}\Big)^{1/k} = x^{1/k}u^{-1}\Big(1 + \frac{a_{k-1}}{u} + \ldots + \frac{a_0}{u^k}\Big)^{-1/k} = x^{1/k}\Big(\frac{1}{u} + \sum_{r=2}^{\infty}\alpha_r\frac{1}{u^r}\Big),$$

for x sufficiently large. Similarly, for every $j \in \mathbb{N}$,

$$\Big(\frac{p(u)}{x}\Big)^{j/k} = \frac{u^j}{x^{j/k}}\Big(1 + \frac{a_{k-1}}{u} + \ldots + \frac{a_0}{u^k}\Big)^{j/k} = \frac{u^j}{x^{j/k}}\Big(1 + \sum_{m=1}^{\infty}\beta_{m,j}\frac{1}{u^m}\Big).$$

Inserting this into (3.5), we get

$$f(u) = x^{1/k}\Big(\frac{1}{u} + \sum_{r=2}^{\infty}\alpha_r\frac{1}{u^r}\Big) + b_0 + \sum_{j=1}^{\infty} b_j\Big(\frac{u}{x^{1/k}}\Big)^j +$$

$$+\sum_{j=1}^{\infty}\sum_{m=1}^{\infty} b_j\beta_{m,j}\Big(\frac{u}{x^{1/k}}\Big)^j u^{-m}.$$

By absolute and uniform convergence, these power series permit interchange of the order of summation as well as iterated termwise differentiation. Recalling the bounds for u, we thus immediately infer the assertion of Lemma 3.

We are now ready to estimate $R_s(x)$. To this end we split up the range of summation by a geometric sequence

$$M_x^{(j)} = 2^{-j}q_s(\sqrt{x}) \qquad (j = 0, 1, \ldots, J)$$

with $J = J(x)$ such that $M_x^{(J)} \asymp x^{1/4k}$. We may apply Lemma 2 to each interval $I_j = (M_x^{(j)}, 2M_x^{(j)}]$, $j \geq 1$, since the validity of (3.1) is ensured by Lemma 3, with $V_x = x^{1/k}$, $\sigma = 1$. (Furthermore, $(M_x^{(j)})^{1+\sigma} \ll (q_s(\sqrt{x}))^2 \ll x^{1/k} = V_x$.) It follows that

$$\sum_{n \in I_j} \psi\Big(q_{3-s}(\frac{x}{p_s(n)})\Big) \ll x^{7/22k}(\log x)^{45/22}.$$

Finally we observe that $J(x) \ll \log x$, sum over $j = 1, \ldots, J$ and estimate the terms corresponding to $n \leq M_x^{(J)}$ trivially to arrive at

$$R_s(x) = O(x^{7/22k}(\log x)^{67/22}). \tag{3.6}$$

4. Asymptotic evaluation of $T_s(x)$.

By the Euler summation formula,

$$T_s(x) = \int_1^{q_s(\sqrt{x})} q_{3-s}(\frac{x}{p_s(u)})du - \psi(q_s(\sqrt{x}))q_{3-s}(\sqrt{x}) + \tfrac{1}{2}q_{3-s}(\frac{x}{p_s(1)}) +$$

$$+ \int_1^{q_s(\sqrt{x})} \psi(u)\frac{d}{du}\Big(q_{3-s}(\frac{x}{p_s(u)})\Big)du. \tag{4.1}$$

In view of Lemma 1,

$$\frac{d}{du}\Big(q_{3-s}(\frac{x}{p_s(u)})\Big) = -\frac{1}{k}x^{1/k}\frac{p_s'(u)}{(p_s(u))^{1+1/k}} + \sum_{j=1}^{\infty}\frac{j}{k}b_j^{(s)}x^{-j/k}(p_s(u))^{j/k-1}p_s'(u) =$$

$$= -\frac{1}{k}x^{1/k}\frac{p_s'(u)}{(p_s(u))^{1+1/k}} + O(x^{-1/k}), \tag{4.2}$$

uniformly for u in the range of integration given in (4.1). The error term here contributes only $O(x^{-1/2k})$ to the last integral in (4.1). Whereas to the main term in (4.2), observe that, for $y \geq 1$,

$$\int_y^{\infty} \psi(u)\frac{p_s'(u)}{(p_s(u))^{1+1/k}}du \ll \max_{u \geq y}\Big|\frac{p_s'(u)}{(p_s(u))^{1+1/k}}\Big| \ll \frac{1}{y^2}, \tag{4.3}$$

by the second mean-value theorem. Hence, if we define

$$C_0^{(s)} \stackrel{\text{def}}{=} \frac{1}{k}\int_1^{\infty} \psi(u)\frac{p_s'(u)}{(p_s(u))^{1+1/k}}du,$$

it follows from (4.1) - (4.3) that

$$\int_1^{q_s(\sqrt{x})} \psi(u)\frac{d}{du}\Big(q_{3-s}(\frac{x}{p_s(u)})\Big)du = -C_0^{(s)}x^{1/k} + O(1), \tag{4.4}$$

recalling that $q_s(\sqrt{x}) \asymp x^{1/2k}$. To deal with the first integral in (4.1), we use Lemma 1 in the weak form

$$q_{3-s}(\frac{x}{p_s(u)}) = x^{1/k}(p_s(u))^{-1/k} + b_0^{(3-s)} + O(x^{-1/2k}),$$

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uniformly for $1 \leqq u \ll x^{1/2k}$. This implies

$$\int_1^{q_s(\sqrt{x})} q_{3-s}\Big(\frac{x}{p_s(u)}\Big)du = x^{1/k}F_s(q_s(\sqrt{x})) + b_0^{(3-s)}x^{1/2k} + O(1) \tag{4.5}$$

with

$$F_s(y) \stackrel{\text{def}}{=} \int_1^y (p_s(u))^{-1/k}du \qquad (y \geq 1). \tag{4.6}$$

LEMMA 4. *The function $F_s(y)$ defined by* (4.6) *possesses the asymptotic expansion (as $y \longrightarrow \infty$)*

$$F_s(y) = \log y - C_1^{(s)} - \gamma_1^{(s)}\frac{1}{y} + O(\frac{1}{y^2}\log y),$$

with

$$C_1^{(s)} = \int_1^\infty \frac{d}{du}\Big(\frac{u}{(p_s(u))^{1/k}}\Big)\log u\, du, \qquad \gamma_1^{(s)} = -\frac{1}{k}a_{k-1}^{(s)}.$$

PROOF. Observe that

$$g(u) \stackrel{\text{def}}{=} \frac{u}{(p_s(u))^{1/k}} = \Big(1 + \frac{a_{k-1}^{(s)}}{u} + \ldots + \frac{a_0^{(s)}}{u^k}\Big)^{-1/k} = 1 + \sum_{j=1}^{\infty}\gamma_j^{(s)}u^{-j},$$

(the last equation being true for u sufficiently large). Hence, for $u \geq 1$,

$$g(u) = 1 + \gamma_1^{(s)}\frac{1}{u} + O(\frac{1}{u^2}), \qquad g'(u) = -\gamma_1^{(s)}\frac{1}{u^2} + O(\frac{1}{u^3}).$$

Applying integration by parts, we thus obtain

$$F_s(y) = \int_1^y \frac{1}{u}g(u)du = g(y)\log y - \int_1^y g'(u)\log u du =$$

$$= \log y + \gamma_1^{(s)}\frac{1}{y}\log y + O(\frac{1}{y^2}\log y) - C_1^{(s)} + \int_y^\infty \Big(-\gamma_1^{(s)}\frac{1}{u^2}\log u + O(\frac{\log u}{u^3})\Big)\,du =$$

$$= \log y - \gamma_1^{(s)}\frac{1}{y} - C_1^{(s)} + O(\frac{1}{y^2}\log y)$$

which proves Lemma 4.

Appealing to Lemma 1, we see that

$$\log(q_s(\sqrt{x})) = \log(x^{1/2k} + b_0^{(s)} + O(x^{-1/2k})) =$$

$$= \log(x^{1/2k}) + \log(1 + b_0^{(s)}x^{-1/2k}) + \log(1 + O(x^{-1/k})) =$$

$$= \frac{1}{2k}\log x + b_0^{(s)}x^{-1/2k} + O(x^{-1/k}),$$

and

$$\frac{1}{q_s(\sqrt{x})} = (x^{1/2k} + O(1))^{-1} = x^{-1/2k} + O(x^{-1/k}).$$

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Thus it follows that

$$F_s(q_s(\sqrt{x})) = \frac{1}{2k}\log x - C_1^{(s)} + (b_0^{(s)} - \gamma_1^{(s)})x^{-1/2k} + O(x^{-1/k}\log x) =$$
$$= \frac{1}{2k}\log x - C_1^{(s)} + O(x^{-1/k}\log x)$$

in view of (2.1) and the definition of $\gamma_1^{(s)}$ in Lemma 4. Inserting this last result into (4.5), and going back to (4.4) and (4.1), we finally obtain

$$T_s(x) = \frac{1}{2k}x^{1/k}\log x - \left(C_0^{(s)} + C_1^{(s)} - \tfrac{1}{2}(p_s(1))^{-1/k}\right)x^{1/k} +$$
$$+b_0^{(3-s)}x^{1/2k} - \psi(q_s(\sqrt{x}))q_{3-s}(\sqrt{x}) + O(\log x)\,. \tag{4.7}$$

Whereas to the last terms in (2.2), we see that

$$\tfrac{1}{2}(q_1(\sqrt{x}) + q_2(\sqrt{x})) + [q_1(\sqrt{x})][q_2(\sqrt{x})] =$$
$$= \tfrac{1}{2}(q_1(\sqrt{x}) + q_2(\sqrt{x})) + (q_1(\sqrt{x}) - \psi(q_1(\sqrt{x})) - \tfrac{1}{2})(q_2(\sqrt{x}) - \psi(q_2(\sqrt{x})) - \tfrac{1}{2}) =$$
$$= q_1(\sqrt{x})q_2(\sqrt{x}) - \sum_{s=1,2}\psi(q_s(\sqrt{x}))q_{3-s}(\sqrt{x}) + O(1) =$$
$$= x^{1/k} + \sum_{s=1,2}\left(b_0^{(3-s)}x^{1/2k} - \psi(q_s(\sqrt{x}))q_{3-s}(\sqrt{x})\right) + O(1)\,,$$

by one more application of Lemma 1. Inserting this together with (3.6) and (4.7) into (2.2), we complete the proof of our Theorem.

Concluding remarks. 1. There is an alternative description of the constant C in our Theorem. Let

$$\zeta_{p_s}(z) = \sum_{n=1}^{\infty}(p_s(n))^{-z} \qquad (\mathrm{Re}\ z > \frac{1}{k})$$

be the zeta function associated with the monic polynomial p_s ($s = 1$ or 2). It is easy to see that ζ_{p_s} can be continued analytically at least to the half plane Re $z > 0$, with the exception of a simple pole with residue $\frac{1}{k}$ at $z = \frac{1}{k}$. Let us define

$$\gamma_s = \lim_{z\to\frac{1}{k}}\left(\zeta_{p_s}(z) - \frac{1}{k}(z - \frac{1}{k})^{-1}\right).$$

With these notations,

$$C = 1 - \gamma_1 - \gamma_2\,.$$

2. Our assumption that p_1, p_2 be of equal degree is not vital to obtain an asymptotic expansion of $T_s(x)$. It is, however, of importance for the estimation of the ψ-sum : If p_1, p_2 have different degrees k, l, the situation is similar to the "asymmetric divisor problem" involving the number of pairs $(u, v) \in \mathbb{N}^2$ with $u^k v^l \leq x$. (This is dealt with in detail in the book of Krätzel [4].) Here it is usually much more cumbersome to choose an optimal

"strategy" for the estimation of the fractional parts sum, depending on the relative size of k and l.

3. In a similar manner, one can investigate the average order of the arithmetical function

$$r_{p_1,p_2}(n) = \#\{(u,v) \in \mathbb{N}^2 : p_1(u) + p_2(v) = n\} .$$

This was done recently by G. Kuba and the second named author [6]. (Cf. also Kuba's thesis [5].) However, the details of the analysis and the results are much different from the situation considered in the present paper.

ADDED IN AUGUST 1991. M.N. Huxley has meanwhile anounced an improvement of the error term in the classical divisor problem to $O(x^{23/73}(\log x)^{315/146})$ (Lecture at Oberwolfach, March 1991, and preprint "Exponential sums and lattice points, II".) Using the corresponding refined version of his method, one can readily sharpen the result of the present article to $O(x^{23/73k}(\log x)^{315/146})$.

REFERENCES

1. Fricker, F., *Einführung in die Gitterpunktlehre,* Birkhäuser, Basel-Boston-Stuttgart, 1982.

2. Huxley, M.N., *Exponential sums and lattice points,* Proc. London Math. Soc. (3) **60** (1990), 471-502.

3. Iwaniec, H., and Mozzochi, C.J., *On the divisor and circle problems,* J. Number Th. **29** (1988), 60-93.

4. Krätzel, E., *Lattice points,* Kluwer Acad. Publ., Dordrecht-Boston-London, 1988.

5. Kuba, G., *Neuere Methoden der Gitterpunktlehre und spezielle zahlentheoretische Funktionen,* Thesis, Vienna University, 1990.

6. Kuba, G., and Nowak, W.G., *On representations of positive integers as a sum of two polynomials,* Arch. Math. (Basel), to appear.

7. Müller, W., and Nowak, W.G., *Lattice points in planar domains: Applications of Huxley's "Discrete Hardy-Littlewood method",* in: "Number theoretic analysis", Vienna 1988-89, Springer Lecture Notes **1452** (eds. E. Hlawka and R. F. Tichy), 1990, pp. 139-164.

8. Nowak, W.G., *On a result of Smith and Subbarao concerning a divisor problem,* Can. Math. Bull. **27** (1984), 501-504.

9. Nowak, W.G., *On a divisor problem in arithmetic progressions. J. Number Theory* **31** *(1989),* 174-182.

10. Smith, R.A., and Subbarao, M.V., *The average number of divisors in an arithmetic progression,* Can. Math. Bull. **24** (1981), 37-41.

11. Varbanec, P.D., and Zarzycki, P., *Divisors of integers in arithmetic progressions,* Can. Math. Bull. **33** (1990), 129-134.

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