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CENTRALISER NEAR-RING REPRESENTATIONS

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1. Introduction

Let V be a group, written additively but not necessarily abelian, and let S be a semigroup of endomorphisms of V. The set $C(S; V) = \{f: V \to V \mid f\sigma = \sigma f$ for all $\sigma \in S$ and $f(0) = 0\}$ forms a zero-symmetric near-ring with identity under the operations of function addition and composition, called the *centraliser near-ring* determined by S and V. Centraliser near-rings are very general, for if N is any zero-symmetric near-ring with 1 then there exists a group V and a semigroup S of endomorphisms of V such that $N \simeq C(S; V)$.

In this paper all near-rings will be finite, zero-symmetric and have an identity element. For definitions and results concerning near-rings see Pilz [**11**].

The first centraliser near-ring representation result was given by Wielandt [**13**]. Here he announced the characterisation of finite simple near-rings as centraliser near-rings $C(\mathscr{A}; V)$ where $\mathscr{A}$ is a group of fixed point free automorphisms of the group V. In 1973, Betsch [**2**] extended Wielandt's work to a class of infinite near-rings. Recently, there have been several investigations into the structure of centraliser near-rings. (See [**6**], [**7**], [**8**] and [**9**].) In [**7**] we established the following result.

Theorem 1.1. *Let V be a finite group and $\mathscr{A}$ a group of automorphisms of V. Then $C(\mathscr{A}; V)$ is simple if and only if the stabiliser subgroups,* $\text{stab}_{\mathscr{A}}(v) \equiv \{\alpha \in \mathscr{A} \mid \alpha v = v\}$, *are conjugate for all $v \in V^* \equiv V - \{0\}$.*

Our investigations in this paper are concerned with the following representation question. If N is a simple subnear-ring of $C(\{1\}); V)$, when is $N = C(\mathscr{A}; V)$ for some $\mathscr{A} \subseteq \text{Aut}\, V$? Equivalently, let V be a near-ring module over the simple near-ring N and for $a \in N$ define $\lambda_a: V \to V$ by $\lambda_a v = av$, $v \in V$. Then N is isomorphic to $\bar{N} \equiv \{\lambda_a \mid a \in N\} \subseteq C(\{1\}; V)$, and we interpret a representation of N as a $C(\mathscr{A}; V)$ to mean a representation of $\bar{N}$.

This centraliser representation problem is the non-linear analogue of the following ring theory problem. Let V be an abelian group and let S be a simple subring of End V. When does there exist a ring R such that $S = \text{End}_R\, V$? A partial solution to this problem is a consequence of the Noether–Skolem Theorem [**4**], page 104, in the setting where End V is simple.

We now give a short summary of our results. In the next section we consider the general representation problem giving necessary and sufficient conditions for a centraliser near-ring representation of a simple near-ring N. In Section 3 we apply these results to near-fields and fields where the representation problem is discussed under various situations.

2. Characterisation Theorems

In this section we give necessary and sufficient conditions on a simple near-ring N, $N \subseteq C(\{1\}; V)$, in order that $N = C(\mathscr{A}; V)$ for some $\mathscr{A} \subseteq \operatorname{Aut} V$.

Lemma 2.1. *Let N be a simple subnear-ring of $C(\{1\}; V)$ and let $B = \operatorname{Aut}_N V$. If there exists $v_1, \ldots, v_t \in V$ such that $\{v_1, \ldots, v_t\} \subseteq \theta_B(v_1)$ where $\theta_B(v_1)$ is the orbit of V containing v_1 determined by the action of B on V, and $V = Nv_1 \dot{\cup} \ldots \dot{\cup} Nv_t$ (disjoint union) where each Nv_i is a faithful N-simple, N-subgroup of V, then $C(B; V)$ is simple.*

Proof. We show first that elements of Nv_i have the same B-stabiliser. Let v be a nonzero element in Nv_i. Since Nv_i is N-simple, $Nv = Nv_i$. If $\alpha \in \operatorname{stab}_B(v)$ then $\alpha(v) = v$ and $\alpha(Nv_i) = Nv_i$. Thus α restricted to Nv_i is an N-automorphism of Nv_i fixing v. Since Nv_i is N-simple α must be the identity map on Nv_i. From this, we conclude that $\operatorname{stab}_B v = \operatorname{stab}_B v_i$ for all $v \in Nv_i$.

Let $\theta_B(w)$ be any B-orbit. Since $w \in Nv_j$ for some j, then $w = nv_j$ for some $n \in N$. But each v_i belongs to the same B-orbit so $\alpha v_j = v_i$ for some $\alpha \in B$. Hence $\alpha w = \alpha n v_j = n \alpha v_j = nv_i$. This means $\theta_B(w) \cap Nv_i \neq \emptyset$ for each i. Since all B-stabilisers of elements in $\theta_B(w)$ are conjugate and since $\theta_B(w)$ intersects every Nv_i then any two B-stabilisers are conjugate which implies that $C(B; V)$ is simple.

This leads to the main characterisation result.

Theorem 2.1. *Let N be a simple subnear-ring of $C(\{1\}; V)$, and let $B = \operatorname{Aut}_N V$. Then the following are equivalent.*

(1) *$N = C(\mathscr{A}; V)$ for some $\mathscr{A} \subseteq \operatorname{Aut} V$.*

(2) *$N = C(B; V)$.*

(3) i. *$V = Nv_1 \dot{\cup} \ldots \dot{\cup} Nv_t$ where each Nv_i is a faithful N-simple N-subgroup of V and each $v_i \in \theta_B(v_1)$.*
ii. *Let $S_1 = \operatorname{stab}_B(v_1)$, then $\operatorname{Fix} S_1 \equiv \{v \in V \mid \alpha v = v \text{ for all } \alpha \in S_1\}$ is a subset of Nv_1.*

Part 3ii may be replaced by 3ii′: Fix S_1 is N-simple.

Proof. If part (1) is true then $\mathscr{A} \subseteq B$ and so $C(B; V) \subseteq C(\mathscr{A}; V)$. But by definition of B, $N \subseteq C(B; V)$ and hence (1) implies (2).

If (2) is true then $N = C(B; V)$ and N is simple. Select a nonzero $v \in V$ then $Nv = \{w \in V \mid \operatorname{stab}_B w = \operatorname{stab}_B v\} \cup \{0\}$. Because $N = C(B; V)$ then N acts transitively on the nonzero elements of Nv (see [7]). Hence Nv is N-simple. Moreover there exist elements $v_1, \ldots, v_t$ all in $\theta_B(v_1)$ such that $V = Nv_1 \dot{\cup} \ldots \dot{\cup} Nv_t$ and $Nv_1 = \operatorname{Fix} S_1$ (see [7]).

Assume (3) is true. Then Lemma 2.1 implies $C(B; V)$ is simple. Hence $\operatorname{Fix} S_1 = C(B; V)v_1$. We have $Nv_1 \subseteq C(B; V)v_1 = \operatorname{Fix} S_1$, and by (3)ii (or (3)ii′), $Nv_1 = C(B; V)v_1$. But $C(B; V)v_1$ is the set of all possible images of v_1 under functions in $C(B; V)$ and Nv_1

is the set of all possible images of v_1 under functions from N, and by assumption $v_1, \ldots, v_t$ belong to the same B-orbit. So $N = C(B; V)$ as desired. Since (2) implies (1) is obvious, the proof is complete.

In the following theorem we establish the existence of near-rings $C(\{1\}; V)$ which contain simple subnear-rings that are not centraliser representable on V.

Theorem 2.2. *Let N be a simple near-ring. Then there exists an N-module V such that N has no representation as a centraliser near-ring on V.*

Proof. It is shown in [5] that if V is a group and $\mathscr{A}$ a group of automorphisms of V then $C(\mathscr{A}; V)$ is a simple ring if and only if $C(\mathscr{A}; V)$ is a field. Hence a simple ring which is not a field has no centraliser representation. If N is a field then Theorem 3.2 at the end of this paper applies. So we may assume N is a simple nonring.

From [2] we have the representation $N = C(\mathscr{A}; W)$ where $\mathscr{A}$ is a group of automorphisms acting fixed point free on W. Let $V = W \dot{+} W$, and for each $f \in N$ extend f to all of V by defining $f\binom{x}{y} = \binom{f(x)}{f(y)}$. In this way V is an N-module and we may regard N as a subnear-ring of $C(\{1\}; V)$. We will show that N has no centraliser representation on V.

Assume first that N is not a near-field. Then under the action of $\mathscr{A}$, W has at least two nonzero orbits. Let $w_1, w_2 \in W$ be nonzero elements belonging to different orbits. We have, as sets, $N\binom{w_1}{0} = \binom{Nw_1}{0}$ and $N\binom{w_1}{w_2} = N\binom{w_1}{0} + N\binom{0}{w_2} = \binom{Nw_1}{0} + \binom{0}{Nw_2} = \binom{Nw_1}{Nw_2}$. The cardinality $\left|\binom{Nw_1}{Nw_2}\right|$ of $N\binom{w_1}{w_2}$ is greater than the cardinality $\left|\binom{Nw_1}{0}\right|$ of $N\binom{w_1}{0}$, so N cannot be centraliser representable on V since part (3)i of Theorem 2.1 implies that $|Nv| = |Nw|$ for nonzero $v, w \in V$.

Now assume N is a near-field. Then $W = (N, +)$ and $\mathscr{A}$ consists of the right multiplication maps by elements of N on W. The linear maps in N acting on $V = W \dot{+} W$ are $N_L = \{\left(\begin{smallmatrix} f & 0 \\ 0 & f \end{smallmatrix}\right) \mid f \in N$ is linear on $W\}$. A calculation shows that $\mathrm{End}_{N_L} V = \{\left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\right) \mid \alpha_i \in \mathscr{A} \cup \{0\}\}$. We show now that $\mathrm{Aut}_N V = \{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) \mid \alpha, \beta \in \mathscr{A}\} \cup \{\left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right) \mid \alpha, \beta \in \mathscr{A}\}$. For suppose $B = \left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\right) \in \mathrm{Aut}_N V$ with say $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. Then for every $f \in N$, $\binom{x}{y} \in V$,

$$fB\binom{x}{y} = Bf\binom{x}{y};$$

or

$$f(\alpha_1 x + \alpha_2 y) = \alpha_1 f(x) + \alpha_2 f(y)$$

$$= f(\alpha_1 x + \alpha_2 y)$$

for every $x, y \in W$. But since α_1, α_2 are invertible this implies f acts linearly on W and $C(\mathscr{A}; W)$ is a field. Hence one of α_1, α_2 must be 0. Similarly for α_3, α_4 and B has the desired form.

It remains to show that $N \neq C(\mathrm{Aut}_N V; V)$. This is done by showing the latter is not simple. Let $\bar{\mathscr{A}} = \mathrm{Aut}_N V$ and let w be a nonzero element of W. Then $\mathrm{stab}_{\bar{\mathscr{A}}}\binom{w}{0} = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & \beta \end{smallmatrix}\right) \mid \beta \in \mathscr{A}\}$ and $\mathrm{stab}_{\bar{\mathscr{A}}}\binom{w}{w} = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$. Since these two stabiliser subgroups are not conjugates in $\bar{\mathscr{A}}$, $C(\bar{\mathscr{A}}; V)$ is not simple.

Turning now to near-fields N in $C(\{1\}; V)$, the characterisations as to when N is a centraliser near-ring on V appear tighter. We again fix some notation. As above let B

$=\mathrm{Aut}_N V$ and let $S_0=\mathrm{stab}_B(v_0)$ for fixed nonzero v_0 in V. Also let $\mathrm{Fix}\, S_0=\{v\in V \mid \alpha v=v$ for each $\alpha\in S_0\}$ and $\mathcal{N}_0=$normaliser of S_0 in B.

Corollary 2.1. *Let N be a near-field in $C(\{1\}, V)$. The following are equivalent.*

(1) $N=C(\mathscr{A}; V)$ *for some* $\mathscr{A}\subseteq \mathrm{Aut}\, V$.

(2) $N=C(B; V)$.

(3) *B is transitive on V and* $\mathrm{Fix}\, S_0\subseteq Nv_0$.

(4) *B is transitive on V and* $\mathrm{Fix}\, S_0$ is N-simple.

(5) *B is transitive on V and $\mathcal{N}_0/S_0$ is isomorphic to N^*.*

Proof. That (1) is equivalent to (2) is clear. Assume $N=C(B; V)$. Since N is a near-field, B is transitive on V (see [7]), and $\mathrm{Fix}\, S_0\subseteq Nv_0$ from Theorem 2.1. So (2) implies (3).

To show (3) implies (4), we note that the transitivity of B implies that $C(B; V)$ is a near-field and so $\mathrm{Fix}\, S_0=C(B; V)v_0$. But then $\mathrm{Fix}\, S_0=Nv_0$. If H is a nonzero N-subgroup of $\mathrm{Fix}\, S_0$ then for some $n\in N$, $nv_0=h\in H$. But then $v_0=n^{-1}h$ is in H and so $H=Nv_0=\mathrm{Fix}\, S_0$. Hence $\mathrm{Fix}\, S_0$ is N-simple.

Assume (4) is true. Then by Theorem 2.1 $N=C(B; V)$, and from [7] Theorem 3, $N\cong C(\mathcal{N}_0/S_0;\ \mathrm{Fix}\, S_0)$ where $\mathcal{N}_0/S_0$ acts fixed point free on $\mathrm{Fix}\, S_0$. But also $N\cong C(N^*; N)$ and by the isomorphism result of Ramakotaiah [12], $\mathcal{N}_0/S_0\cong N^*$. So (4) implies (5).

If (5) holds then $N\subseteq C(B; V)$ with $C(B; V)$ a near-field. If $K=C(B; V)$ then $K^*\cong \mathcal{N}_0/S_0$ and thus $|N^*|\leqq|K^*|\leqq|\mathcal{N}_0/S_0|=|N^*|$. So $N=K$ as desired.

If $N\subseteq C(\{1\}; V)$ is a near-field and if $\mathrm{Aut}_N V$ is transitive on V then $C(\mathrm{Aut}_N V; V)$ is a near-field containing N. The following example shows that N need not equal $C(\mathrm{Aut}_N V; V)$.

Example 2.2. Let $N=D(5,2,3)$ be a Dickson near-field of order 5^6 with centre of order 5^2. Then the field $N_1=D(5,3,1)$ is a subnear-field of N of order 5^3 (see Dancs-Grove [3]). Let $V=\langle N, +\rangle$, the additive group of N (or of the Galois field $GF(5^6)$). The field N_1 acts on V by left multiplication so we may regard N_1 (and N) as subnear-rings of $C(\{1\}, V)$. Since $\mathrm{Aut}_{N_1} V$ contains $\{\rho_n: V\to V \mid \rho_n(v)=vn,\ n\in N,\ v\in V\}$, $\mathrm{Aut}_{N_1} V$ is transitive on V. We have $N_1\subseteq C(\mathrm{Aut}_{N_1} V; V)\subseteq C(\mathrm{Aut}_N V; V)=N$ and since N_1 is a maximal subnear-field of N (Dancs-Grove [3]) then either $N_1=C(\mathrm{Aut}_{N_1} V; V)$ or $N=C(\mathrm{Aut}_{N_1} V; V)$. We will show that the latter is true.

If $N_1=C(\mathrm{Aut}_{N_1} V; V)$ then $\mathrm{Aut}_{N_1} V$ is not fixed point free on V since $|\mathrm{Aut}_{N_1} V|>|N_1|$. Thus there exists a $\Phi\in \mathrm{Aut}_{N_1} V$ such that $\Phi\neq 1$ and $\Phi(1)=1$. We will show this is impossible.

Using the notation of Pilz [11], page 244, let g be a generator of the multiplicative cyclic group $GF(5^6)^*$ used in the construction of N. Let H be the subgroup of $GF(5^6)^*$ of index 3 generated by g^3 and let σ be the Galois automorphism of $GF(5^6)$ defined by $x\to x^{5^2}$. The cosets of H in $GF(5^6)^*$ are H, Hg, Hg^2 and the multiplication in N is defined in terms of the multiplication in $GF(5^6)$ by $a\circ b=a^{\sigma^i}b$ if $b\in Hg^i$ and $a\circ 0=0$.

If $\Phi \in \mathrm{Aut}_{N_1} V$ is as described above we will show that Φ is $GF(5^3)$-linear on V as a vector space over $GF(5^3)$ and that $\Phi(Hg^i)=Hg^i$, $i=0,1,2$.

Since V has dimension 2 over $GF(5^3)$ and since $g^3 \notin GF(5^3)$ then $\{1,g^3\}$ forms a basis for V. So every element in V has the form $\alpha 1+\beta g^3$, $\alpha,\beta \in GF(5^3)$. But $\alpha 1+\beta g^3=\alpha \circ 1+\beta \circ g^3$, $\alpha,\beta \in N_1$ since $\{1,g^3\}\subseteq H$. If $\delta \in N_1$ then

$$\delta \circ \Phi(\alpha \circ 1+\beta \circ g^3)=\delta \circ(\alpha+\beta \circ \Phi(g^3))=\delta^{\sigma^i}(\alpha+\beta \circ \Phi(g^3))=\delta^{\sigma^i}\alpha+\delta^{\sigma^i}(\beta \circ \Phi g^3))$$

where $\alpha+\beta \circ \Phi(g^3)\in Hg^i$. On the other hand

$$\begin{aligned}\delta \circ \Phi(\alpha \circ 1+\beta \circ g^3)&=\Phi(\delta \circ(\alpha \circ 1+\beta \circ g^3))=\Phi(\delta^{\sigma^j}(\alpha \circ 1+\beta \circ g^3))\\&=\Phi(\delta^{\sigma^j}\alpha 1+\delta^{\sigma^j}(\beta \circ g^3))=\delta^{\sigma^j}\alpha+\Phi(\delta^{\sigma^j}\beta g^3)\\&=\delta^{\sigma^j}\alpha+\Phi((\delta^{\sigma^j}\beta)\circ g^3)=\delta^{\sigma^j}\alpha+\delta^{\sigma^j}\beta \circ \Phi(g^3)\end{aligned}$$

where $\alpha \circ 1+\beta \circ g^3 \in Hg^j$. Comparing the two results and using the fact that $\Phi(g^3)\notin N_1$ we conclude that if $\alpha \neq 0$ then $\delta^{\sigma^i}=\delta^{\sigma^j}$, and so $\sigma^i=\sigma^j$. This means Φ preserves Hg, Hg^2 and thus H. If $a \in N_1$, $v \in V$ then since Φ preserves cosets $\Phi(a \circ v)=\Phi(a^{\sigma^i}v)$ while $a \circ \Phi(v)=a^{\sigma^i}\Phi(v)$. So $a^{\sigma^i}\Phi(v)=\Phi(a^{\sigma^i}v)$ for all $a \in GF(5^3)$. Hence Φ is $GF(5^3)$-linear on V.

To finish the example it suffices to show that the two dimensional vector space V (the additive group of $GF(F^6)$) over $GF(5^3)$ has no nontrivial one-to-one $GF(5^3)$-linear maps Φ which preserve the cosets of H in $GF(5^3)^*$ and fix 1. This is done in the following lemma due to Martin R. Pettet.

Lemma 2.2. (M. R. Pettet) *Let H be the subgroup of $GF(5^6)^*$ of index 3. If $\Phi: GF(5^6)\to GF(5^6)$ is a $GF(5^3)$-linear group automorphism which preserves the cosets of H in $GF(5^6)^*$ and such that $\Phi(1)=1$, then $\Phi=1$.*

Proof. Assume such a Φ exists with $\Phi \neq 1$. Then there exists such a Φ whose order is a prime p, i.e. $\Phi^p=1$. Since $GF(5^3)^*\subseteq H$, $H \cup \{0\}$ is the union of 42 one dimensional $GF(5^3)$-subspaces, one of which is $GF(5^3)$. Since Φ leaves $GF(5^3)$ fixed it permutes the other 41 subspaces in H. If Φ does not leave another subspace fixed then, since 41 is a prime, Φ permutes the 41 subspaces cyclically which means that $p=41$. But $p=41$ is impossible since 41 does not divide the order of $GL(V/GF(5^3))$, i.e. there is no $GF(5^3)$-automorphism of V of order 41. So Φ leaves $GF(5^3)$ and at least one other subspace in $H \cup \{0\}$ invariant. Hence Φ has two linearly independent eigenvectors, say $\{1,\alpha\}$ and the matrix of Φ with respect to this basis for V is $\left(\begin{smallmatrix}1&0\\0&c\end{smallmatrix}\right)$, $c \in GF(5^3)^*$. Since $\Phi^p=1$ then $c^p=1$ and so p divides $|GF(5^3)^*|=2^2.31$, hence $p=2$ or $p=31$. If $p=31$ then there would be $42-31=11$ fixed subspaces in $H \cup \{0\}$ resulting in too many distinct eigenvectors. Thus $p=2$ and the matrix of Φ is $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$.

Since Φ preserves the cosets of H this means that $a+b\alpha \equiv a-b\alpha \bmod H$ for every $a,b \in GF(5^3)$, not both zero. The above is clearly true if $b=0$ so if $b \neq 0$ we have $(a+\alpha)(a-\alpha)^{-1}\in H$ for every $a \in GF(5^3)$. The map $a \to (a+\alpha)(a-\alpha)^{-1}$ from $GF(5^3)$ into H is one-to-one. Since there are 125 such elements $(a+\alpha)(a-\alpha)^{-1}$ in H and 42 $GF(5^3)$-subspaces

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in $H \cup \{0\}$ there is at least one subspace containing three elements of the form $(a+\alpha)(a-\alpha)^{-1}$, say $(a+\alpha)(a-\alpha)^{-1}$, $(b+\alpha)(b-\alpha)^{-1}$, $(c+\alpha)(c-\alpha)^{-1}$ where

$$(a+\alpha)(a-\alpha)^{-1} = f_1(b+\alpha)(b-\alpha)^{-1} = f_2(c+\alpha)(c-\alpha)^{-1},$$

a, b, c, f_1, $f_2 \in GF(5^3)$, all distinct and $f_1 \neq 1 \neq f_2$. From the above we have

$$(f_1-1)\alpha^2 = (f_1+1)(a-b)\alpha + (f_1-1)ab$$

and

$$(f_2-1)\alpha^2 = (f_2+1)(a-c)\alpha + (f_2-1)ac.$$

This gives two quadratic polynomials over $GF(5^3)$ having α as a root. Since $\alpha \notin GF(5^3)$ these two polynomials give rise to the same minimal polynomial. From this we have $ab = ac$, so $b = c$ or $a = 0$. The latter is impossible since the polynomials are irreducible. But $b = c$ is also impossible since $(b+\alpha)(b-\alpha)^{-1} \neq (c+\alpha)(c-\alpha)^{-1}$. This contradiction shows $\Phi = 1$ as desired.

3. Specialised actions and further examples

We now apply the characterisation theorem of the previous section to obtain results on specified actions of a near-ring N on an N-module V. Recall that when N is a near-field, V is a near-vector space over N if $V = V_1 \oplus \cdots \oplus V_t$ where each V_i is an N-submodule of V and $V_i \cong N$ (see [**1**]).

Theorem 3.1. *Let N be a near-field and V a near-vector space over N. Then $N = C(\mathscr{A}; V)$ for some $\mathscr{A} \subseteq \mathrm{Aut}\, V$.*

Proof. From the results of Beidleman [**1**] it is easy to see that $\mathrm{Aut}_N V$ is transitive on V. Also if $v_1 \in V_1$, then for $S_1 = \mathrm{stab}(v_1)$ we have $\mathrm{Fix}\, S_1 = V_1 = Nv_1$. Thus by Theorem 2.1, $N = C(\mathscr{A}; V)$ as desired.

Corollary 3.1. *Let F be a field, $F \subseteq C(\{1\}, V)$. If F acts linearly on V then $F = C(\mathscr{A}; V)$ for some $\mathscr{A} \subseteq \mathrm{Aut}\, V$.*

Proof. If F acts linearly on V then V is a vector space over F and the theorem applies.

Corollary 3.2. *Let F be a field and let V be a monogenic near-ring module over F. Then $F = C(\mathscr{A}; V)$ for some $\mathscr{A} \subseteq \mathrm{Aut}\, V$.*

Proof. Let $V = Fv_0$. Then for $h \in F, w, u \in V$ we have $w = fv_0$, $u = gv_0$ for some $f, g \in F$ and $h(w+u) = h(fv_0 + gv_0) = h(f+g)v_0 = (hf+hg)v_0 = hw + hu$. Thus F acts linearly on V and Corollary 3.1 applies.

Corollary 3.3. *Let F be a field acting on V. Then $F \subseteq C(B; V)$ where $B = \mathrm{Aut}_F V$. If B is a p-group acting transitively on V then $F = C(B; V)$.*

Proof. From Passman [10], page 34, either B is a cyclic group or else $|V|=3^2$. If B is cyclic, then from Maxson and Smith [6], B acts fixed point free on V and $C(B;V)$ is a field. Since V is $C(B;V)$-monogenic, $C(B;V)$ acts linearly on V and thus so does F. Hence $F=C(B;V)$.

Suppose $|V|=3^2$, and let $v_0\in V^*$. If $S=\operatorname{stab} v_0$, we have $\operatorname{Fix} S=C(B;V)v_0$ and since $|\operatorname{Fix} S|$ divides $|V^*|=3^2-1$ then $|\operatorname{Fix} S|=3^l-1$ where $l=1$ or 2. If $l=1$ then $|C(B;V)|=3$ and $C(B;V)=GF(3)$ acts linearly on V. If $l=2$, V is C(B; V)-monogenic and again F acts linearly on V so $F=C(B;V)$ as desired.

From the above corollaries it is natural to conjecture that for a field F acting on V, $F=C(\mathscr{A};V)$ implies V is a vector space over F. This is false, however, as the next example, due to S. Gagola, provides a field acting non-linearly on V but $F=C(\mathscr{A};V)$.

Example 3.1. Let $V=GF(p^4)$, where p is a prime different from 3 and let σ be the Galois automorphism $x\to x^p$ of $GF(p^4)$. For $a\in GF(p^4)^*$ and $i=0,1,2,3$ define the maps $T_{a,\sigma^i}:V\to V$ by $T_{a,\sigma^i}v=av^{\sigma^i}$. It is easy to verify that $\mathscr{F}=\{T_{a,\sigma^i}\,|\,a\in GF(p^4)^*,\ i=0,1,2,3\}$ is a group of automorphisms of V. Let

$$\mathscr{A}=\{T_{a,\sigma^i}\,|\,\text{if } a \text{ is a square in } GF(p^4)^*$$

$$\text{then } i=0,2 \text{ while if } a \text{ is not a square then } i=1,3\},$$

a subgroup of $\mathscr{F}$. Since $\mathscr{A}$ is a transitive automorphism group then $C(\mathscr{A};V)$ is a near-field. Also $S\equiv\operatorname{stab}(1)=\{T_{1,\sigma^i}\,|\,i=0,2\}$ and $\operatorname{Fix} S=GF(p^2)\subset V=GF(p^4)$. If $N(S)$ is the normaliser of S in $\mathscr{A}$ then it is easy to verify that $N(S)/S\cong GF(p^2)^*$. Thus $C(\mathscr{A};V)\cong GF(p^2)$.

We now show that the field $C(\mathscr{A};V)$ does not act linearly on V. Suppose $f\in C(\mathscr{A};V)$. Then $fT_{a,\sigma^i}=T_{a,\sigma^i}f$ implies $f(a)=af(1)^{\sigma^i}$. If a is a square then $f(a)=af(1)$, while if a is not a square then $f(a)=af(1)^\sigma$. Thus f is completely determined by its action on 1. Since $1\in\operatorname{Fix} S$ we have $f(1)\in\operatorname{Fix} S$. Now suppose $f\in C(\mathscr{A};V)$ acts linearly on V. Suppose $b\in V$ is not a square. Consider $1+b$. If $1+b$ is a square then $f(1+b)=(1+b)f(1)=f(1)+bf(1)$, while $f(1)+f(b)=f(1)+bf(1)^\sigma$. Comparing the two results gives $f(1)^\sigma=f(1)$, or $f(1)\in GF(p)$. If $1+b$ is not a square then $f(1+b)=(1+b)f(1)^\sigma=f(1)^\sigma+bf(1)^\sigma$, while $f(1)+f(b)=f(1)+bf(1)^\sigma$. Again $f(1)\in GF(p)$. Hence $f\in C(\mathscr{A};V)$ is linear on V if and only if $f(1)\in GF(p)$. Therefore the field $C(\mathscr{A};V)$ does not act linearly on V.

We conclude by defining a class of actions of a field F on a vector space V that cannot give rise to centraliser near-rings. But first a lemma from linear algebra.

Lemma 3.1. (S. Gagola) *Let V be a finite dimensional vector space over a finite field F, and let W,Y be proper subspaces of V. If $F=GF(2)$, assume one of W and Y is not a maximal subspace. Then there is a basis B of V such that $B\subseteq V-(W\cup Y)$.*

Proof. If $F=GF(2)$ we may assume one of W and Y is maximal, while if $F\neq GF(2)$ we may assume that W and Y are both maximal. If $W=Y$ (or if $Y\subset W$ in the case $F=GF(2)$) let $v\in V-(W\cup Y)$ and let $w_1,\dots,w_{n-1}$ be a basis for W. Then $B=\{v,v+w_1,\dots,v+w_{n-1}\}$ is a basis for V contained in $V-(W\cup Y)$ as desired.

If $W \neq Y$, then $\dim V + \dim(W \cap Y) = \dim W + \dim Y$. If $F \neq GF(2)$ then $n + \dim(W \cap Y) = 2(n-1)$ or $\dim(W \cap Y) = n-2$. Let $w_1, \ldots, w_{n-2}$ be a basis for $W \cap Y$. Select $w_{n-1} \in W$, $y \in Y$ such that $\{w_1, \ldots, w_{n-2}, w_{n-1}\}$ is a basis for W and $\{w_1, \ldots, w_{n-2}, y\}$ is a basis for Y. Let $v = w_{n-1} + y$, an element of $V - (W \cap Y)$. Let $a \in F^*$, $a \neq 1$, then $B = \{v + w_1, \ldots, v + w_{n-2}, w_{n-1} + y, aw_{n-1} + y\}$ is a basis for V of the desired type.

If $F = GF(2)$ then we may assume $\dim W = n-1$, $\dim Y = n-2$ and $Y \not\subset W$. Let $\{w_1, \ldots, w_{n-2}\}$ be a basis for $W \cap Y$, $\{w_1, \ldots, w_{n-2}, w_{n-1}\}$ be a basis for W, and $\{w_1, \ldots, w_{n-3}, y\}$ be a basis for Y. If $v = w_{n-1} + y$, then $B = \{v + w_1, \ldots, v + w_{n-3}, v + w_{n-2}, w_{n-2} + y, w_{n-1} + y\}$ is a basis of the desired type.

As an application of this lemma, let V be a vector space over the field F and suppose the function $f: V \to V$ is linear off a proper subspace W of V, i.e. $f(v_1 + v_2) = f(v_1) + f(v_2)$ whenever $v_1, v_2 \in V - W$. If $f\alpha = \alpha f$ for some $\alpha \in \operatorname{Aut} V$ let $Y = \alpha^{-1} W$. From the above lemma there is a basis B for V outside of $W \cup Y$, say $B = \{v_1, \ldots, v_n\}$. Let $\beta \in \operatorname{Aut} V$ be such that $\beta(x) = f(x)$ for each $x \in V - W$. For $i = 1, 2, \ldots, n$ we have

$$\alpha\beta(v_i) = \alpha f(v_i) = f\alpha(v_i) = \beta\alpha(v_i)$$

since $\alpha(v_i) \notin W \cup Y$. This means $\alpha\beta = \beta\alpha$.

To fix the setting for the next theorem let V be a vector space over a nonprime field F with scalar multiplication given by $(a, v) = av$, $a \in F$, $v \in V$. Let W be a nonzero proper subspace of V and let σ be an automorphism of F, $\sigma \neq 1$. We define another action $*: F \times V \to V$ by

$$a * v = \begin{cases} av, & v \in W \\ a^{\sigma} v, & v \in W. \end{cases}$$

This gives rise to a subfield $\bar{F}$ of $C(\{1\}; V)$ where $\bar{F} = \{f_a: V \to V \mid f_a v = a * v\}$. Each $f_a \in \bar{F}$ is linear off W and by the above remarks each $\alpha \in \operatorname{Aut}_F V$ commutes with the linear maps $\{\lambda_a: V \to V \mid \lambda_a v = av,\ a \in F,\ v \in V\}$. So each $\alpha \in \operatorname{Aut}_F V$ is F-linear, meaning $C(\operatorname{Aut}_F V; V)$ contains $\{\lambda_a \mid a \in F\}$. This establishes the following theorem.

Theorem 3.2. *Let F, $\bar{F}$, and V be as in the above discussion. Then $\bar{F} \neq C(\mathscr{A}; V)$ for any $\mathscr{A} \subseteq \operatorname{Aut} V$.*

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