A NEW PROOF OF AN INEQUALITY OF HEINZ

P. S. Bullen[1]

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In a recent paper, [1], Dixmier has proved Heinz' inequality by deducing it from a lemma due to Thorin. In this note it is proved directly from a convexity theorem.

Let $(M^{(k)}, \mathcal{M}^{(k)}, \mu^{(k)})$, $k = 0, \ldots, n$, be measure spaces and $L^{q^{(k)}}(M^{(k)}, \mathcal{M}^{(k)}, \mu^{(k)})$ be all the functions on $M^{(k)}$ such that $\|f\|_{q^{(k)}, \mu^{(k)}} = \left(\int_{M^{(k)}} |f|^{q^{(k)}} d\mu^{(k)}\right)^{\frac{1}{q^{(k)}}} < \infty$.

Let $\underline{M} = M^{(1)} \times \ldots \times M^{(n)}$ with elements $\underline{f} = (f_1, \ldots, f_n)$; also write $\underline{q} = (q^{(1)}, \ldots, q^{(n)})$, $\underline{\mu} = (\mu^{(1)}, \ldots, \mu^{(n)})$ and $L^{\underline{q}}(\underline{M}, \underline{\mathcal{M}}, \underline{\mu})$ for $L^{q^{(1)}}(M^{(1)}, \mathcal{M}^{(1)}, \mu^{(1)}) \times \ldots$ $\times L^{q^{(n)}}(M^{(n)}, \mathcal{M}^{(n)}, \mu^{(n)})$.

An operator T on $\underline{M}$ to $M^{(o)}$ is called sublinear if

(i) $T\underline{f}_j$, $j = 1, 2$, being defined implies that $T(\underline{f}_1 + \underline{f}_2)$ is defined,

(ii) $|T(\underline{f}_1 + \underline{f}_2)| \leq |T\underline{f}_1| + |T\underline{f}_2|$,

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(iii) $T(\lambda \underline{f}) = |\lambda| \, T\underline{f}$.

Such an operator is said to be of type $[(q^{(o)}, \mu^{(o)}), (\underline{q}, \underline{\mu})]$ with constant K if it satisfies $\|T\underline{f}\|_{q^{(o)}, \mu^{(o)}} < K \prod_{k=1}^{n} \left\{ \|f^{(k)}\|_{q^{(k)}, \mu^{(k)}} \right\}$ for all $\underline{f} \varepsilon L^{\underline{q}}(\underline{M}, \underline{\mathcal{M}}, \underline{\mu})$.

Given $\underline{q}_1, \ldots, \underline{q}_m$ and $\underline{i} = (i_1, \ldots, i_m)$, $i_j \geq 0$, $i_1 + \ldots + i_m = 1$ we define $\underline{q}$ (or more precisely $\underline{q}(i)$) by

$$\frac{1}{q^{(k)}} = \frac{i_1}{q_i^{(k)}} + \ldots + \frac{i_m}{q_m^{(k)}}, \quad k = 1, \ldots, n.$$

Again, if we are given $\underline{\mu}_1, \ldots, \underline{\mu}_m$ and if $\underline{\mu}_\sigma = \underline{\mu}_1 + \ldots + \underline{\mu}_m$ we define $\underline{\alpha}_j = (\alpha_j^1, \ldots, \alpha_j^n)$ by $\mu_j^{(k)} = \int_E \alpha_j^{(k)} \, d\mu_\sigma^{(k)}$, $k = 1, \ldots, n$, $j = 1, \ldots, m$. Now define $\underline{\mu}$ (or more precisely $\underline{\mu}(i)$) by

$$\mu^{(k)} = \int_E \left\{ \prod_{j=1}^{m} (\alpha_j^{(k)})^{i_j/q_j^{(k)}} \right\}^{q^{(k)}} d\mu_\sigma^{(k)}, \quad k = 1, \ldots, n.$$

In a similar way given $q_j^{(o)}$, $\mu_j^{(o)}$, $j = 1, \ldots, m$ we can define $q^{(o)}$ and $\mu^{(o)}$.

THEOREM 1. <u>If</u> T <u>is simultaneously of types</u> $[(q_j^{(o)}, \mu_j^{(o)}, \underline{q}_j, \underline{\mu}_j)]$ <u>with constants</u> K_j, $j = 1, \ldots, m$, <u>then</u> T <u>is of type</u> $[(q^{(o)}, \mu^{(o)}) (\underline{q}, \underline{\mu})]$ <u>with constant</u> $\prod_{j=1}^{m} (K_j^{i_j})$.

The notation on this theorem is that of Stein and Weiss, [2], and the theorem is a multilinear analogue of their theorem 2.11. The proof in the case $m = 2$ follows theirs and the general result is obtained by induction on m.

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THEOREM 2. <u>Let</u> $H^{(1)}, \ldots, H^{(n)}$ <u>be complex Hilbert spaces,</u> A_p <u>a positive semi-definite self-adjoint operator in</u> $H^{(p)}$, $1 \leq p \leq n$. <u>Let</u> $F(x^{(1)}, \ldots, x^{(n)})$ <u>be a multilinear form in</u> $H^{(1)} \times \ldots \times H^{(n)}$ <u>such that</u>

$$|F(x^{(1)}, \ldots, x^{(n)})| \leq ||A_1 x^{(1)}|| \; ||x^{(2)}|| \ldots ||x^{(n)}|| ,$$

$$|F(x^{(1)}, \ldots, x^{(n)})| \leq ||x^{(1)}|| \; ||A_2 x^{(2)}|| \ldots ||x^{(n)}|| ,$$

$$|F(x^{(1)}, \ldots, x^{(n)})| \leq ||x^{(1)}|| \; ||x^{(2)}|| \ldots ||A_n x^{(n)}|| .$$

Then if $\gamma_p \geq 0$, $1 \leq p \leq n$, and $\gamma_1 + \ldots + \gamma_n = 1$, we have

$$|F(x^{(1)}, \ldots, x^{(n)})| \leq ||A_1^{\gamma_1} x^{(1)}|| \; ||A_2^{\gamma_2} x^{(2)}|| \ldots ||A_n^{\gamma_n} x^{(n)}|| .$$

This result is due to Dixmier, [1]. It is sufficient to prove it in the finite dimensional case, (see [1]). Heinz' inequality is an immediate corollary. We prove it by recognizing it as a special case of Theorem 1. In fact the following specializations are seen to effect the reduction.

(a) $m = n$

(b) $M^{(o)} = \{1\}$, $q_j^{(o)} = 1$, $\mu_j^{(o)} = 1$ on each point of $M^{(o)}$, $j = 1, 2, \ldots, n$.

(c) Choose an orthonormal basis $\{e_\nu^{(p)}\}$ in each $H^{(p)}$ such that

$$A_p e_\nu^{(p)} = \lambda_\nu^{(p)} e_\nu^{(p)}$$

Then if
$$x^{(p)} = \sum_\nu \xi_\nu^{(p)} e_\nu^{(p)}$$

we have

$$||A_p^\tau x^{(p)}|| = \left\{ \sum_\nu |(\lambda_\nu^{(p)})^\tau \xi_\nu^{(p)}|^2 \right\}^{1/2} , \quad 0 \leq \tau \leq 1 .$$

So take $M^{(k)}$ to be the set of positive integers, $k = 1, \ldots, n$. Let $q_j^{(k)} = 2$, $k, j = 1, \ldots, n$ and if $k \neq j$ take a measure of one on each integer. If however $k = j$ take a measure of $(\lambda_\nu^{(k)})^2$ on the νth integer.

COROLLARY. *Let* H *be a complex Hilbert space,* A *and* B *two self adjoint semi-definite operators in* H, Q *a linear operator in* H *such that* $||Qx|| < ||Bx||$ *and* $||Q^* y|| \leq ||A y||$ *for all* x *and* y *in* H. *Then, for all* x *and* y *in* H *and all* τ, $0 \leq \tau \leq 1$,

$$| < Qx, y > | \leq ||B^p x|| \, ||A^{1-p} y|| .$$

REFERENCES

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2. E. M. Stein and G. Weiss, Interpolation of Operators with Change of Measures. Trans. of the Amer. Math. Soc., 87(1958), pp. 159-172.

University of British Columbia