

HIGH ORDER DIVERGENCE-FORM ELLIPTIC OPERATORS ON LIE GROUPS

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We give a straightforward proof that divergence-form elliptic operators of order m on a d -dimensional Lie group with $m \geq d$ have Hölder continuous kernels satisfying Gaussian bounds.

1. INTRODUCTION

Consider the operator

$$H = \sum_{|\alpha|, |\beta| \leq n} (A^\alpha)^* c_{\alpha, \beta} A^\beta$$

of order $m = 2n$ acting on $L_2 = L_2(G; dg)$ where G is a d -dimensional Lie group with left Haar measure dg , the $c_{\alpha, \beta}$ are operators of multiplication by L_∞ -functions, the A^α, A^β are products of the skew-adjoint generators $A_i = dL(a_i)$, $i \in \{1, \dots, d\}$ of left translations on L_2 corresponding to the multi-indices α, β and a_1, \dots, a_d is a vector space basis of the Lie algebra \mathfrak{g} of G . (Definitions and background information can be found in [17].) We assume

$$(1) \quad \operatorname{Re} \sum_{|\alpha|=|\beta|=n} (\psi_\alpha, c_{\alpha, \beta} \psi_\beta) \geq \mu \sum_{|\alpha|=n} \|\psi_\alpha\|_2^2$$

for some $\mu > 0$, uniformly for all $\psi_\alpha \in L_2$, and define the ellipticity constant μ_C as the least upper bound of the μ for which this condition is satisfied. Then H can be precisely defined as the closed maximal accretive operators associated with the sectorial forms

$$h(\psi, \varphi) = \sum_{|\alpha|, |\beta| \leq n} (A^\alpha \psi, c_{\alpha, \beta} A^\beta \varphi)$$

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with domain $D(h) = L_{2;n}$, the functions which are n -times differentiable in the L_2 -sense. It follows that each such H generates a strongly continuous, holomorphic, semi-group S on L_2 (see, for example, [14, Chapter VI]) with a kernel K_t defined as a distribution such that

$$(\psi, S_t \varphi) = \int_G dg \overline{\psi(g)} \int_G d\hat{h} K_t(g; h) \varphi(h)$$

for all $\varphi, \psi \in C_c^\infty(G)$ and $t > 0$ where $d\hat{h}$ denotes right Haar measure. Our aim is to give an elementary analysis of K_t when the dimension d is small relative to m . Specifically we assume $d \leq m$ and establish that K satisfies Gaussian bounds and is Hölder continuous with an order related to the relative size of d and m .

If $G = \mathbb{R}^d$ and $n = 1$ then H corresponds to a second-order strongly elliptic partial differential operator in divergence form. Such operators are of fundamental importance in the calculus of variations and have been studied in enormous detail. It is not difficult to establish for $d = 1$ that the kernels are bounded and Hölder continuous for all $t > 0$. A similar result for $d = 2$ follows from the early work of Morrey [15]. If $d \geq 3 > m$, then the situation is more complicated and it is useful to distinguish two distinct cases, real coefficients and complex coefficients. For real coefficients the Hölder continuity of the kernel is a famous result established independently by De Giorgi [13] and Nash [16]. But for complex coefficients, or for systems with real coefficients, this property no longer holds if $d \geq 5$. The situation for $d \in \{3, 4\}$ is unclear. (A discussion of counterexamples for systems, with references, is given in [12], [2] and [8].) Recently, Davies [7] has examined higher-order operators and has obtained Gaussian bounds whenever $d < m$. Our proofs are rather different and are an extension of the perturbation theory and embedding arguments given in [3, Section 3].

All subsequent estimates are in terms of the ellipticity constant μ_G and the norms of the coefficients. Let $M_G = \sup\{\|c_{\alpha,\beta}\|_\infty : |\alpha|, |\beta| \leq n\}$ where $\|\cdot\|_\infty$ denotes the L_∞ -norm and let $\mathcal{E}_m(\mu, M)$ denote the set of H for which $\mu_G \geq \mu$ and $M_G \leq M$. Moreover, for $g \in G$ let $|g|$ be the Riemannian distance from g to the identity element of G , canonically associated with the vector spaces a_1, \dots, a_d . Finally the smoothness of the kernels as functions over $G \times G$ is expressed in terms of the left derivatives A_i , and B_i , with respect to the first and second variable of the kernel, respectively.

THEOREM 1.1. *Assume $d \leq m$. For each $M \geq \mu > 0$ there is a $\nu \in (0, 1)$ such that for all α, β and $\sigma \in (0, 1)$ with $|\alpha| + \sigma \leq 2^{-1}(m - d) + \nu$ and $|\beta| + \sigma \leq 2^{-1}(m - d) + \nu$ and all $\kappa > 0$ there exist $a, b > 0$ and $\omega \geq 0$ such that for all $H \in \mathcal{E}_m(\mu, M)$ the corresponding kernels K satisfy*

$$(2) \quad |(A^\alpha B^\beta K_t)(g; h)| \leq a t^{-(d+|\alpha|+|\beta|)/m} e^{\omega t} e^{-b(|gh^{-1}|^m t^{-1})^{1/(m-1)}}$$

and

$$\begin{aligned} & |(A^\alpha B^\beta K_t)(k^{-1}g; l^{-1}h) - (A^\alpha B^\beta K_t)(g; h)| \\ & \leq a t^{-(d+|\alpha|+|\beta|)/m} e^{\omega t} \left(\frac{|k| + |l|}{t^{1/m} + |gh^{-1}|} \right)^\sigma e^{-b(|gh^{-1}|^m t^{-1})^{1/(m-1)}} \end{aligned}$$

uniformly for all $g, h, k, l \in G$ and $t > 0$ with $|k| + |l| \leq \kappa t^{1/m} + 2^{-1} |gh^{-1}|$.

In particular if $d = m$ the kernel is Hölder continuous and satisfies Gaussian bounds but it is not necessarily differentiable. The degree of smoothness is an increasing function of the difference between m and d . The degree of regularity is, however, not optimal. For example, if $d = 1 = n$ then the Hölder continuity bounds are valid for $\sigma \in (0, 1]$, and $|\alpha| = 0 = |\beta|$ (see [3, 11]).

Note that if $H \in \mathcal{E}_m(\mu, M)$ and $|\theta| < \arctan(\mu/M)$ then $e^{i\theta}H \in \mathcal{E}_m(\mu^\theta, M)$ with $\mu^\theta = \mu \cos \theta - M |\sin \theta|$. Therefore the kernel K_z of S_z satisfies bounds similar to the above, with t replaced by $|z|$, uniformly for $z \in \mathbb{C} \setminus \{0\}$ with $|\arg z| \leq \theta$.

2. PROOFS

The proof of Theorem 1.1 is based on a combination of perturbation arguments and Sobolev embeddings which extends ideas of [3] but avoids any form of scaling. Since the proof for non-unimodular G introduces some extraneous difficulties we first examine unimodular G and subsequently explain the extension to non-unimodular groups. The proof for unimodular G is in two steps. The first step establishes uniform bounds. The second step extends the uniform bounds to Gaussian bounds by Davies' exponential perturbation method [7].

STEP 1. (Uniform bounds: unimodular G) The proof starts with the L_2 -estimates

$$(3) \quad \|S_t\|_{2 \rightarrow 2} \leq e^{\omega t}, \quad \|HS_t\|_{2 \rightarrow 2} \leq a t^{-1} e^{\omega t}$$

which yield bounds

$$(4) \quad \|A^\alpha S_t\|_{2 \rightarrow 2} \leq a t^{-|\alpha|/m} e^{\omega t},$$

whenever $|\alpha| \leq n$, for all $t > 0$. (Here and in the sequel we use a and ω to denote parameters whose values may vary line by line. The a are strictly positive, the ω are non-negative and they are all decreasing functions of μ_C and increasing functions of M_C .) These estimates follow from the ellipticity condition (1) applied to $e^{i\theta}H$, the Cauchy integral formula and $\varepsilon, \varepsilon^{-1}$ arguments by standard reasoning. (Further details are given in the derivation of (15).)

Next, since $d \leq m$, one has the Sobolev inequalities

$$\|A^\alpha \varphi\|_p \leq \varepsilon^{n-|\alpha|-ds} \sup_{|\beta|=n} \|A^\beta \varphi\|_2 + c \varepsilon^{-|\alpha|-ds} \|\varphi\|_2$$

with $s = 1/2 - 1/p$ for all $\varepsilon \in (0, 1]$ and all α with $|\alpha| < n$ whenever $s \in [0, 1/d)$ (see [17, Appendix B.2]). Thus if $d = 1$, or $d = 2$, the inequalities are valid for all $p \in [2, \infty)$ and if $d \geq 3$ they are valid for $p \in [2, 2d/(d - 2))$. Therefore, replacing φ by $S_t \varphi$, setting $\varepsilon = t^{1/m}$ and using the foregoing L_2 -bounds, one finds

$$(5) \quad \|A^\alpha S_t\|_{2 \rightarrow p} \leq a t^{-ds/m} t^{-|\alpha|/m} e^{\omega t}$$

for all α with $|\alpha| < n$ and all $t > 0$. (Initially these estimates are valid for $t \in (0, 1]$ and then, by use of the semigroup property, for all $t > 0$.) The main technical difficulty in this first step is to establish that the bounds (5) remain valid for $|\alpha| = n$ and some $p > 2$.

The proof uses the positive self-adjoint operator Δ_m determined by the quadratic form

$$\varphi \mapsto \delta_m(\varphi) = \sum_{|\alpha|=n} \|A^\alpha \varphi\|_2^2$$

with domain $L_{2;n}$. Then Δ_m is a strongly elliptic operator with constant coefficients which generates a continuous semigroup T with a smooth Gaussian kernel (see [17, Chapters 1 and 3], or, for a short proof, see [10]). Consequently, for each α one has bounds

$$\|A^\alpha T_t\|_{p \rightarrow q} \leq a t^{-|\alpha|/m} t^{-ds/m} e^{\omega t}$$

uniformly for all $t > 0$ and all $p, q \in [1, \infty]$ with $q \geq p$, where $s = 1/p - 1/q$. Hence if $\lambda_0 = \omega + 1$, $|\alpha| < n$ and $p, q \in [1, \infty]$, with $n - |\alpha| - ds > 0$, one obtains bounds

$$(6) \quad \|A^\alpha Q_\lambda\|_{p \rightarrow q} \leq a \lambda^{-(n-|\alpha|-ds)/m}$$

for all $\lambda \geq \lambda_0$, where $Q_\lambda = (\lambda I + \Delta_m)^{-1/2}$, by the usual Laplace transform arguments.

It is also necessary to have precise estimates on the n -th derivatives of the square root of the resolvent of Δ_m . It follows from [5] that one has bounds

$$(7) \quad \left(\sum_{|\alpha|=n} \|A^\alpha Q_{\lambda_0} \varphi\|_p^2 \right)^{1/2} \leq c_p(\lambda_0) \|\varphi\|_p$$

for all $\varphi \in L_p$ and all $p \in (1, \infty)$. Now we argue that one can improve these estimates uniformly in λ if p is sufficiently close to 2. □

LEMMA 2.1. *For each $\varepsilon > 0$ there exist $\delta \in (0, 1/2)$ and $\lambda_0 > 0$ such that*

$$\left(\sum_{|\alpha|=n} \|A^\alpha Q_\lambda \varphi\|_p^p \right)^{1/p} \leq (1 + \varepsilon) \|\varphi\|_p$$

uniformly for $\varphi \in L_p$, $\lambda \geq \lambda_0$ and $p \in (1, \infty)$ with $|1/2 - 1/p| \leq \delta$.

PROOF: First note that

$$(8) \quad \left(\sum_{|\alpha|=n} \|A^\alpha Q_\lambda \varphi\|_2^2 \right)^{1/2} = \|\Delta_m^{1/2} (\lambda I + \Delta_m)^{-1/2} \varphi\|_2 \leq \|\varphi\|_2$$

for all $\varphi \in L_2$ and all $\lambda > 0$.

Secondly, if λ_0 is large enough then the semigroup generated by $\lambda_0 I + \Delta_m$ on L_p is exponentially decreasing and holomorphic in the right half-plane. Therefore, for all $p \in (1, \infty)$ the operator $\lambda_0 I + \Delta_m$ has a bounded $H_\infty(\Lambda(\theta))$ -functional calculus, in the sense of [6], for any $\theta \in (0, \pi/2)$, by [9, Theorem 3.1]. So there exists $c_p > 0$ such that

$$\|f(\lambda_0 I + \Delta_m)\|_{p \rightarrow p} \leq c_p \sup\{|f(z)| : z \in \Lambda(\theta)\}$$

for all $f \in H_\infty(\Lambda(\theta))$. Hence application of this estimate to the holomorphic function $f(z) = z^{1/2}(\lambda - \lambda_0 + z)^{-1/2}$ yields

$$(9) \quad \|(\lambda_0 I + \Delta_m)^{1/2} (\lambda I + \Delta_m)^{-1/2}\|_{p \rightarrow p} = \|f(\lambda_0 I + \Delta_m)\|_{p \rightarrow p} \leq c_p$$

for all $\lambda \geq \lambda_0$. One then concludes from combination of (7) and (9) that

$$\left(\sum_{|\alpha|=n} \|A^\alpha Q_\lambda \varphi\|_p^p \right)^{1/p} \leq c_p(\lambda_0) \|(\lambda_0 I + \Delta_m)^{1/2} (\lambda I + \Delta_m)^{-1/2} \varphi\|_p \leq c_p c_p(\lambda_0) \|\varphi\|_p$$

for all $p \in (1, \infty)$ and $\varphi \in L_p$, uniformly for $\lambda \geq \lambda_0 > 0$.

Now let $Y = G \times \{1, 2, \dots, d^n\}$ with the natural sum measure, using left Haar measure on each copy of G . Then for all $\lambda \geq \lambda_0$ the operator $T_\lambda: L_p(G) \rightarrow L_p(Y)$ defined by $T_\lambda \varphi = (A^{\alpha_1} Q_\lambda \varphi, \dots, A^{\alpha_{d^n}} Q_\lambda \varphi)$, where α_i is an enumeration of the d^n multi-indices with $|\alpha_i| = n$, satisfies $\|T_\lambda\|_{L_p(G) \rightarrow L_p(Y)} \leq c_p c_p(\lambda_0)$ by the preceding estimate and $\|T_\lambda\|_{L_2(G) \rightarrow L_2(Y)} \leq 1$ by (8). Hence by interpolation for all $\varepsilon > 0$ there exists $\delta \in (0, 1/2)$ such that $\|T_\lambda\|_{L_p(G) \rightarrow L_p(Y)} \leq 1 + \varepsilon$ uniformly for all $\lambda \geq \lambda_0$ and those p such that $|1/2 - 1/p| \leq \delta$. But this is equivalent to the statement of the lemma. \square

The next lemma provides the crucial estimates on the resolvent of H by expressing it as a perturbation of a multiple of Δ_m following a technique of [4, Chapter 1, Section 4], which also occurs in [3, Proposition 3.1]. Let

$$P = \sum_{|\alpha|=|\beta|=n} (A^\alpha)^* c_{\alpha,\beta} A^\beta$$

denote the principal part of H .

LEMMA 2.2. *There exist $\lambda_1 > 0$, $\delta \in (0, (2d)^{-1})$ and $a > 0$ such that*

$$\|A^\alpha(\lambda I + P)^{-1}A^\beta\|_{p \rightarrow p} \leq a \lambda^{-(m-|\alpha|-|\beta|)/m}$$

for all α, β with $|\alpha| \leq n$ and $|\beta| \leq n$ and

$$\|A^\alpha(\lambda I + P)^{-1}A^\beta\|_{2 \rightarrow p} \leq a \lambda^{-(m-|\alpha|-|\beta|-ds)/m}$$

for all α, β with $|\alpha| \leq n$ and $|\beta| < n$ uniformly for all $\lambda \geq \lambda_1$ and $p \in [2, \infty)$ with $s = 1/2 - 1/p \leq \delta$.

PROOF: It suffices, by interpolation, to prove bounds with a dependent on p .

Let $C = (c_{\alpha,\beta})$ denote the $d^n \times d^n$ -matrix formed by the coefficients with $|\alpha| = n = |\beta|$ and set $\|C\|_p = \sup_{g \in G} \|C(g)\|_{p \rightarrow p}$, where $\|C(g)\|_{p \rightarrow p}$ denotes the l_p -norm of the matrix $C(g)$. Set $N = \mu_C^{-1} \|C\|_2^2$, $\sigma = \mu_C \|C\|_2^{-1}$ and $\tilde{C} = I - N^{-1}C$. Then

$$\|\tilde{C}\xi\|^2 = |\xi|^2 - 2N^{-1} \operatorname{Re}(\xi, C\xi) + N^{-2} |C\xi|^2 \leq (1 - \sigma^2) |\xi|^2$$

for all $\xi \in \mathbf{C}^{d^n}$. So $\|\tilde{C}\|_2^2 \leq 1 - \sigma^2 < 1$. But $\|\tilde{C}\|_\infty \leq d^n \|\tilde{C}\|_2$ and $\|\tilde{C}\|_1 \leq d^n \|\tilde{C}\|_2$. Hence, by interpolation,

$$\|\tilde{C}\|_p \leq d^{n(1-2/p)} \|\tilde{C}\|_2 \leq d^{n(1-2/p)} (1 - \sigma^2)^{1/2} = d^{n|1/q-1/p|} (1 - \sigma^2)^{1/2}$$

for all $p \in [2, \infty]$, with q conjugate to p , and then, by a similar argument, for all $p \in [1, 2]$. Therefore one may choose $\delta_0 \in (0, 1/2)$ such that

$$(10) \quad \|\tilde{C}\|_p \leq 1 - 4^{-1}\sigma^2 < 1$$

for all $p \in [1, \infty]$ satisfying $|1/2 - 1/p| \leq \delta_0$.

Since $P = N(\Delta_m - \tilde{P})$ with $\tilde{P} = \sum_{|\alpha|=|\beta|=n} (A^\alpha)^* \tilde{c}_{\alpha,\beta} A^\beta$ one has a formal representation

$$(11) \quad N(\lambda NI + P)^{-1} = Q_\lambda(I - B_\lambda)^{-1}Q_\lambda$$

where

$$B_\lambda = Q_\lambda \tilde{P} Q_\lambda = \sum_{|\alpha|=|\beta|=n} (A^\alpha Q_\lambda)^* \tilde{c}_{\alpha,\beta} (A^\beta Q_\lambda)$$

which we argue is well-defined on the L_p -spaces with p sufficiently close to 2.

Let $\varepsilon \in (0, 1]$ and fix δ such that the estimates of Lemma 2.1 are valid. Then for all $p \in (1, \infty)$ with $|1/2 - 1/p| \leq \delta$, $\varphi \in L_p \cap L_2$ and $\chi \in L_q \cap L_2$, where q is again conjugate to p , one has

$$\begin{aligned} |(\chi, B_\lambda \varphi)| &\leq \|\tilde{C}\|_p \left(\sum_{|\alpha|=|\beta|=n} \|A^\alpha Q_\lambda \chi\|_q^q \right)^{1/q} \left(\sum_{|\alpha|=|\beta|=n} \|A^\beta Q_\lambda \varphi\|_p^p \right)^{1/p} \\ &\leq (1 - 4^{-1} \sigma^2) (1 + \varepsilon)^2 \|\chi\|_q \|\varphi\|_p, \end{aligned}$$

uniformly for all $\lambda \geq \lambda_0$ whenever $|1/q - 1/2| = |1/2 - 1/p| \leq \delta \wedge \delta_0$, where the last estimate uses (10). Hence by choosing ε sufficiently small one may ensure that

$$\|B_\lambda\|_{p \rightarrow p} < 1 - 8^{-1} \sigma^2$$

for all p with $|1/2 - 1/p| \leq \delta$, possibly by reducing the value of δ . Therefore the $(I - B_\lambda)^{-1}$ are defined as bounded operators, by a norm convergent power series, on each of the L_p -spaces with $|1/2 - 1/p| \leq \delta$ and their norms are uniformly bounded for $\lambda \geq \lambda_0$. Then, however, the representation (11) is well-defined for this range of p and λ and can be used to obtain the estimates of the lemma.

First, for all $|\alpha| \leq n$, $|\beta| \leq n$ and p with $|1/2 - 1/p| \leq \delta \wedge (1/d)$ one has

$$\begin{aligned} \|A^\alpha (\lambda N I + P)^{-1} A^\beta\|_{p \rightarrow p} &\leq N^{-1} \|A^\alpha Q_\lambda\|_{p \rightarrow p} \left(1 - \|B_\lambda\|_{p \rightarrow p}\right)^{-1} \|A^{\beta_*} Q_\lambda\|_{q \rightarrow q} \\ &\leq N^{-1} \lambda^{-(n-|\alpha|)/m} 8 \sigma^{-2} \lambda^{-(n-|\beta|)/m} = a \lambda^{-(n-|\alpha|-|\beta|)/m}, \end{aligned}$$

by (6) and (7), uniformly for $\lambda \geq \lambda_0$. Here β_* denotes the multi-index obtained from β by reversing its order. Finally,

$$\begin{aligned} \|A^\alpha (\lambda N I + P)^{-1} A^\beta\|_{2 \rightarrow p} &\leq N^{-1} \|A^\alpha Q_\lambda\|_{p \rightarrow p} \left(1 - \|B_\lambda\|_{p \rightarrow p}\right)^{-1} \|A^{\beta_*} Q_\lambda\|_{q \rightarrow 2} \\ &\leq N^{-1} a \lambda^{-(n-|\alpha|)/m} 8 \sigma^{-2} a \lambda^{-(n-|\beta|-d(1/q-1/2))/m} \\ &= a \lambda^{-(m-|\alpha|-|\beta|-ds)/m} \end{aligned}$$

if δ is small enough. Thus the statements of the lemma are valid with $\lambda_1 = N \lambda_0$. □

Now we can extend (5) to the α with $|\alpha| = n$ for p close to 2.

LEMMA 2.3. *Fix δ as in Lemma 2.2. Then there exist $a > 0$ and $\omega \geq 0$ such that*

$$(12) \quad \|A^\alpha S_t\|_{2 \rightarrow p} \leq a t^{-(n+ds)/m} e^{\omega t}$$

uniformly for all α with $|\alpha| = n$, $t > 0$ and $p \in [2, \infty)$ with $s = 1/2 - 1/p \leq \delta$.

PROOF: It follows from the representation

$$A^\alpha S_t = A^\alpha(\lambda I + P)^{-1}(\lambda I + H)S_t + A^\alpha(\lambda I + P)^{-1}(P - H)S_t$$

that

$$\|A^\alpha S_t\|_{2 \rightarrow p} \leq \|A^\alpha(\lambda I + P)^{-1}\|_{2 \rightarrow p} \|(\lambda I + H)S_t\|_{2 \rightarrow 2} + \|A^\alpha(\lambda I + P)^{-1}(P - H)S_t\|_{2 \rightarrow p}.$$

There are two terms on the right hand side which we denote by $L(t)$ and $R(t)$, respectively. We set $\lambda = t^{-1}$. The leading term $L(t)$ can be bounded by the second estimate of Lemma 2.2, with $|\alpha| = n$ and $|\beta| = 0$, together with (3). One finds

$$L(t) \leq a \lambda^{-(n-ds)/m} (\lambda + t^{-1}) e^{\omega t} \leq a t^{-(n+ds)/m} e^{\omega t}$$

first for $t \leq \lambda_1^{-1}$ and then, by the semigroup property, for all $t > 0$ with possibly increased values of a and ω . Alternatively, the remainder $R(t)$ is bounded by

$$R(t) \leq \sum'_{\beta, \gamma} \|A^\alpha(\lambda I + P)^{-1} A^\beta\|_{p \rightarrow p} \|c_{\beta, \gamma}\|_\infty \|A^\gamma S_t\|_{2 \rightarrow p} + \sum''_{\beta, \gamma} \|A^\alpha(\lambda I + P)^{-1} A^\beta\|_{2 \rightarrow p} \|c_{\beta, \gamma}\|_\infty \|A^\gamma S_t\|_{2 \rightarrow 2}$$

where the first sum is over the β, γ with $|\beta| = n$ and $|\gamma| \leq n - 1$ and the second over $|\beta| \leq n - 1$ and $|\gamma| \leq n$. But these terms are bounded by the first estimate of Lemma 2.2 and (5), and the second estimate of Lemma 2.2 and (4), respectively. One finds

$$R(t) \leq a t^{-(n-ds)/m} e^{\omega t} \left(\sum_{|\beta|+|\gamma| < m} \|c_{\beta, \gamma}\|_\infty t^{(m-|\beta|-|\gamma|)/m} \right)$$

first for small t and then, by increasing a and ω , for all $t > 0$. Combination of these estimates gives the desired result and in fact the final estimate can be written in the form

$$(13) \quad \|A^\alpha S_t\|_{2 \rightarrow p} \leq a t^{-(n-ds)/m} e^{\omega t} \left(\sum_{|\beta|+|\gamma| \leq m} \|c_{\beta, \gamma}\|_\infty t^{(m-|\beta|-|\gamma|)/m} \right).$$

This will be useful in the next step of the proof. □

REMARK 2.4. It is worth noting that the estimates of Lemma 2.3 combined with the usual perturbation expansion of the resolvent allow one to convert the bounds on the derivatives $A^\alpha(\lambda I + P)^{-1}A^\beta$ into analogous bounds on the derivatives $A^\alpha(\lambda I + H)^{-1}A^\beta$ at the cost of increasing the value of λ_1 .

We next use the bounds of Lemma 2.3 together with the Sobolev inequalities

$$\|A^\alpha\varphi\|_\infty \leq \varepsilon^{n-|\alpha|-d/p} \sup_{|\beta|=n} \|A^\beta\varphi\|_p + c_p \varepsilon^{-|\alpha|-d/p} \|\varphi\|_p$$

which are valid for all $\varepsilon \in (0, 1]$ whenever $n - |\alpha| - d/p > 0$, in order to obtain bounds from L_2 into L_∞ . Setting $\varepsilon = t^{1/m}$ and $1/p = 1/2 - \delta$, with δ as in Lemmas 2.2 and 2.3 gives

$$\|A^\alpha S_t\|_{2 \rightarrow \infty} \leq t^{(n-|\alpha|-d/p)/m} \sup_{|\beta|=n} \|A^\beta S_t\|_{2 \rightarrow p} + c_p t^{-(|\alpha|+d/p)/m} \|S_t\|_{2 \rightarrow p}$$

for $t \in (0, 1]$ and $|\alpha| < 2^{-1}(m - d) + d\delta$. Note that $2^{-1}(m - d) + d\delta$ is not an integer. But then the estimates of Lemma 2.3 together with (5) give bounds

$$(14) \quad \|A^\alpha S_t\|_{2 \rightarrow \infty} \leq a t^{-(d/2+|\alpha|)/m} e^{\omega t}$$

for $t \in (0, 1]$ and then, by enlarging a and ω , for all $t > 0$. Since similar bounds are valid for the adjoint semigroup one concludes that

$$\|A^\alpha S_t A^{\beta*}\|_{1 \rightarrow \infty} \leq \|A^\beta S_{t/2}^*\|_{2 \rightarrow \infty} \|A^\alpha S_{t/2}\|_{2 \rightarrow \infty} \leq a t^{-d/m} t^{-(|\alpha|+|\beta|)/m} e^{\omega t}$$

for all $t > 0$ and α, β with $|\alpha| < 2^{-1}(m - d) + d\delta$ and $|\beta| < (m - d)/2 + d\delta$. Then by the Dunford–Pettis theorem [1, Theorem 1.3] the operator $A^\alpha S_t A^{\beta*}$ has a kernel $K_t^{(\alpha, \beta)} \in L_\infty(G \times G)$ satisfying $\|K_t^{(\alpha, \beta)}\|_\infty \leq a t^{-d/m} t^{-(|\alpha|+|\beta|)/m} e^{\omega t}$. Set $K_t = K_t^{(\alpha, \beta)}$ if $|\alpha| = |\beta| = 0$.

Next, the Hölder space C^τ is defined for $\tau = N + \sigma$ with $N \in \mathbb{N}_0$ and $\sigma \in (0, 1)$ as the subspace of $L_{\infty; N}$ for which the norm

$$\|\varphi\|_{C^\tau} = \sup_{0 < |k| \leq 1} \sup_{|\alpha| \leq N} |k|^{-\sigma} \|(I - L(k))A^\alpha\varphi\|_\infty$$

is finite. Then one has the Sobolev inequalities

$$\|\varphi\|_{C^\tau} \leq \varepsilon^{n-\tau-d/p} \sup_{|\alpha|=n} \|A^\alpha\varphi\|_p + c_p \varepsilon^{-d/p} \|\varphi\|_p$$

which are valid for all $\varepsilon \in (0, 1]$ whenever $n - \tau > d/p$. In particular if $1/p = 1/2 - \delta$ these inequalities are valid for all $\tau \in (0, 2^{-1}(m - d) + d\delta) \setminus \mathbb{N}$. Now the arguments in the

above proof can be repeated and one deduces that for all $\tau = |\alpha| + \sigma < 2^{-1}(m - d) + d\delta$ there are a and ω such that

$$|k|^{-\sigma} \|(I - L(k))A^\alpha S_t\|_{2 \rightarrow \infty} \leq \|S_t\|_{L_2 \rightarrow C^\tau} \leq a t^{-(d/2+\tau)/m} e^{\omega t}$$

for all $t > 0$ first for all $|k| \in (0, 1]$ and then, by the bounds (14), for all $k \in G$. Similar bounds are valid on the adjoint semigroup. Therefore combination with (14) gives

$$\begin{aligned} & \|L(k)A^\alpha S_t A^{\beta*} L(l^{-1}) - A^\alpha S_t A^{\beta*}\|_{1 \rightarrow \infty} \\ & \leq a(|k|^\sigma |l|^\sigma t^{-2\sigma/m} + |k|^\sigma t^{-\sigma/m} + |l|^\sigma t^{-\sigma/m}) t^{-d/m} t^{-(|\alpha|+|\beta|)/m} e^{\omega t} \end{aligned}$$

for all α, β and $\sigma \in (0, 1)$ with $|\alpha| + \sigma < 2^{-1}(m - d) + d\delta$ and $|\beta| + \sigma < 2^{-1}(m - d) + d\delta$ and for all $t > 0, k, l \in G$. So

$$\begin{aligned} & \|(I - \tilde{L}(k, l))K_t^{(\alpha, \beta)}\|_\infty \\ & \leq a(|k|^\sigma |l|^\sigma t^{-2\sigma/m} + |k|^\sigma t^{-\sigma/m} + |l|^\sigma t^{-\sigma/m}) t^{-d/m} t^{-(|\alpha|+|\beta|)/m} e^{\omega t} \end{aligned}$$

where \tilde{L} denotes the left translations on $G \times G$. Hence the measurable function $K_t^{(\alpha, \beta)}$ is continuous and the lemma of Du Bois-Reymond implies that K_t is $|\alpha|$ -times differentiable in the first variable and the derivatives are $|\beta|$ -times differentiable in the second variable.

STEP 2. (Gaussian bounds) The Gaussian bounds on the kernel now follow by repetition of the foregoing arguments for the perturbed semigroup

$$S_t^\rho = U_\rho S_t U_\rho^{-1} = e^{-tH_\rho}$$

where $\rho \mapsto U_\rho$ is the unitary group of multiplication operators defined by $U_\rho \varphi = e^{-\rho\psi} \varphi$, with real-valued $\psi \in C_c^\infty(G)$ and H_ρ is the operator obtained from H by the replacement A_i by $A_i + \rho(A_i\psi)$. All subsequent bounds depend on derivatives $A^\alpha \psi$ of ψ and are uniform over the set D_n of real-valued $\psi \in C_c^\infty(G)$ with $\|A^\alpha \psi\|_\infty \leq 1$ for all α with $1 \leq |\alpha| \leq n$. First the bounds (3) are replaced by

$$\|S_t^\rho\|_{2 \rightarrow 2} \leq e^{\omega(1+\rho^m)t}, \quad \|H_\rho S_t^\rho\|_{2 \rightarrow 2} \leq a t^{-1} e^{\omega(1+\rho^m)t}$$

with a and ω independent of ρ and $\psi \in D_n$ (see [17, Lemma III.4.4]). Then

$$\sup_{|\alpha|=n} \|A^\alpha S_t^\rho \varphi\|_2^2 \leq \mu_C^{-1} \operatorname{Re}(S_t^\rho \varphi, P S_t^\rho \varphi)$$

by ellipticity. But

$$\operatorname{Re}(\chi, P\chi) \leq 2 |(\chi, H_\rho \chi)| + a(1 + \rho^m) \|\chi\|_2^2$$

as a form bound by a calculation similar to the proof [17, Lemma II.4.5]. Thus

$$\sup_{|\alpha|=n} \|A^\alpha S_t^\rho\|_{2 \rightarrow 2}^2 \leq 2\mu_G^{-1} \|S_t^\rho\|_{2 \rightarrow 2} \|H_\rho S_t^\rho\|_{2 \rightarrow 2} + a(1 + \rho^m) \|S_t^\rho\|_{2 \rightarrow 2}^2 \leq a t^{-1} e^{\omega(1+\rho^m)t}.$$

Then, by a standard $\varepsilon, \varepsilon^{-1}$ argument one obtains the analogue

$$(15) \quad \|A^\alpha S_t^\rho\|_{2 \rightarrow 2} \leq a t^{-|\alpha|/m} e^{\omega(1+\rho^m)t},$$

of (4) for all α with $|\alpha| \leq n$ and all t, ρ and ψ . The arguments leading from (4) to (5) applied to (15) now yield

$$(16) \quad \|A^\alpha S_t^\rho\|_{2 \rightarrow p} \leq a t^{-d s/m} t^{-|\alpha|/m} e^{\omega(1+\rho^m)t}$$

for all $t > 0, \rho \in \mathbf{R}, \psi \in D_n$ and all α with $|\alpha| < n$ and all $p \in [2, \infty)$ with $s = 1/2 - 1/p < 1/d$. But P is the principal part of H_ρ and

$$P - H_\rho = \sum_{|\beta|+|\gamma|<m} A^\beta c_{\beta,\gamma}(\rho) A^\gamma$$

where $\rho \mapsto c_{\beta,\gamma}(\rho)$ is a polynomial of order at most $m - |\beta| - |\gamma|$ and the coefficients are uniformly bounded for $\psi \in D_n$. Moreover, the $c_{\beta,\gamma}(\rho)$ are linear in the $c_{\beta,\gamma}$. Therefore the reasoning of Lemmas 2.1, 2.2 and 2.3 can be applied to H_ρ and S_t^ρ to obtain the bounds

$$\|A^\alpha S_t^\rho\|_{2 \rightarrow p} \leq a t^{-(n+d s)/m} e^{\omega(1+\rho^m)t} \left(\sum_{|\beta|+|\gamma| \leq m} \|c_{\beta,\gamma}\|_\infty (1 + \rho^{(m-|\beta|-|\gamma|)}) t^{(m-|\beta|-|\gamma|)/m} \right)$$

for all α with $|\alpha| = n$ in place of (13). But by increasing the values of a and ω one has

$$(17) \quad \|A^\alpha S_t^\rho\|_{2 \rightarrow p} \leq a t^{-(n+d s)/m} e^{\omega(1+\rho^m)t}$$

for all $t > 0$ uniformly for $\rho \in \mathbf{R}, \psi \in D_n$ and $p \in [2, \infty)$ with $s = 1/2 - 1/p \leq \delta$.

At this point one can repeat the arguments following the proof of Lemma 2.3 with S_t replaced by S_t^ρ and using (15), (16) and (17) in place of (4), (5) and (12) to conclude that

$$\|A^\alpha S_t^\rho A^{\beta \circ}\|_{1 \rightarrow \infty} \leq a t^{-d/m} t^{-(|\alpha|+|\beta|)/m} e^{\omega(1+\rho^m)t}$$

for all $t > 0, \rho \in \mathbf{R}, \psi \in D_n$ and α, β with $|\alpha| < 2^{-1}(m - d) + d\delta$ and $|\beta| < 2^{-1}(m - d) + d\delta$. The a and ω are independent of ρ, ψ and t . Then

$$\|U_\rho A^\alpha S_t A^{\beta \circ} U_\rho^{-1}\|_{1 \rightarrow \infty} \leq a t^{-d/m} t^{-(|\alpha|+|\beta|)/m} e^{\omega(1+\rho^m)t}.$$

As the kernel of $U_\rho A^\alpha S_t A^{\beta*} U_\rho^{-1}$ is given by $(g, h) \mapsto (-1)^{|\beta|} e^{\rho(\psi(g)-\psi(h))} (A^\alpha B^\beta K_t)(g; h)$ it then follows that

$$|(A^\alpha B^\beta K_t)(g; h)| \leq a t^{-d/m} t^{-(|\alpha|+|\beta|)/m} e^{\omega t} e^{\omega \rho^m t - |\rho| |\psi(g)-\psi(h)|}$$

and the Gaussian bounds (2) follow by minimising over $\rho \in \mathbb{R}$ and $\psi \in D_n$. Note that the distance which naturally enters these estimates is given by

$$(18) \quad d_n(g; h) = \sup_{\psi \in D_n} |\psi(g) - \psi(h)|.$$

But this distance is equivalent to the normal distance $(g, h) \mapsto |gh^{-1}|$ (see [17, pp.201–202]).

Finally, similar estimates can be deduced for the Hölder continuity of the derivatives of the kernel. Specifically, for all α, β and $\sigma \in (0, 1)$ with $|\alpha| + \sigma < 2^{-1}(m - d) + d\delta$ and $|\beta| + \sigma < 2^{-1}(m - d) + d\delta$ one obtains

$$\begin{aligned} &|e^{-\rho(\psi(k^{-1}g)-\psi(l^{-1}h))} (A^\alpha B^\beta K_t)(k^{-1}g; l^{-1}h) - e^{-\rho(\psi(g)-\psi(h))} (A^\alpha B^\beta K_t)(g; h)| \\ &\leq a (|k|^\sigma |l|^\sigma t^{-2\sigma/m} + |k|^\sigma t^{-\sigma/m} + |l|^\sigma t^{-\sigma/m}) t^{-d/m} t^{-(|\alpha|+|\beta|)/m} e^{\omega(1+\rho^m)t} \end{aligned}$$

uniformly for all $g, h, k, l \in G, t > 0, \rho \in \mathbb{R}$ and $\psi \in D_n$. Using (2) one can minimise the above bounds and obtain the second bounds of Theorem 1.1 by a slightly more complicated argument which we omit (see [11, proof of Proposition 4.5]).

STEP 3. (Non-unimodular G) If G is non-unimodular it is necessary to analyse the left differential operator H on the spaces $L_p^\wedge = L_p(G; d\hat{g})$ formed with respect to right Haar measure. In fact the kernel K_t is bounded if and only if the semigroup S_t extends to a bounded operator from L_1^\wedge to L_∞ .

The left derivatives A_i are no longer skew-adjoint on L_2^\wedge but the L_2^\wedge -adjoint is given by $A_i^\wedge = -A_i + b_i I$ where $b_i = (A_i \Delta)(e)$ is the derivative of the modular function Δ at the identity e . Nevertheless H is still defined on L_2^\wedge as a maximal accretive operator associated with a sectorial form

$$\hat{h}(\varphi, \psi) = \sum_{|\alpha|, |\beta| \leq n} \langle A^\alpha \varphi, \hat{c}_{\alpha, \beta} A^\beta \psi \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on L_2^\wedge and the $\hat{c}_{\alpha, \beta}$ are linear in the $c_{\alpha, \beta}$ and polynomial in the b_i , but the principal coefficients are unchanged. Hence H generates a semigroup \hat{S} on L_2^\wedge with kernel \hat{K} and now one has

$$\sup_{g, h \in G} |\hat{K}_t(g; h)| = \|\hat{S}_t\|_{\hat{1} \rightarrow \infty}$$

et cetera. Once one proves that \widehat{K} satisfies Gaussian bounds it is then straightforward to verify that $\widehat{S}_t(L_2 \cap L_2) \subseteq (L_2 \cap L_2)$, its extension to L_2 is generated by the L_2 -version of H and $\widehat{K}_t = K_t$. Therefore one must now repeat the proofs of Steps 1 and 2 but relative to the L_p -spaces. This presents no difficulty.

First, the Sobolev embeddings inequalities used in the proofs remain valid for left derivatives on the L_p -spaces (see [17, Appendix B.2]) and, in fact, they fail for the L_p -spaces. Secondly, the principal part P is now defined relative to L_2 as the operator associated with the form

$$p(\varphi, \psi) = \sum_{|\alpha|, |\beta|=n} \langle A^\alpha \varphi, c_{\alpha, \beta} A^\beta \psi \rangle$$

and the approximant Δ_m is introduced by the form $\varphi \mapsto \sum_{|\alpha|=n} \|A^\alpha \varphi\|_2^2$. After these changes the proofs proceed as previously. We omit further details. \square

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