A NOTE ON APPROXIMATION BY AUTOMORPHIC IMAGES OF FUNCTIONS IN L^1 AND L^∞

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0. Introduction

In [1], it was shown that if $f \in L^p(R^n)$, where $1 < p < \infty$, then the closed subspace of $L^p(R^n)$ spanned by functions of the form

$$(x_1, \cdots, x_n) \to f(a_1x_1+b_1, \cdots, a_nx_n+b_n)$$

[where $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ are real numbers; $a_k \neq 0$; $k = 1, \cdots, n$] coincides with the whole of $L^p(R^n)$. In the present note, analogous results are derived for the spaces of integrable functions, essentially bounded measurable functions, bounded continuous functions, and continuous functions vanishing at infinity.

1. Notation

Throughout, G will denote a locally compact Abelian Hausdorff group, and X its Pontryagin character group. The Haar measure on G will be denoted by dx. It will be assumed that G is a direct product of non-compact subgroups $G_1, \cdots, G_n$, with (non-discrete) character groups $X_1, \cdots, X_n$. Thus X is a direct product of $X_1, \cdots, X_n$. If $x \in G$, and k is an integer such that $1 \leqq k \leqq n$, then x_k will denote the k-th component of x.

Write N for the set $\{1, \cdots, n\}$. If $I \subseteqq N$, we define the sets $E_I(G)$ and $F_I(G)$ as follows:

$$E_I(G) = \{x \in G : x_k = 0 \text{ for all } k \in N \setminus I\},$$

$$F_I(G) = \{x \in G : x_k = 0 \text{ if } k \in N \setminus I;\, x_k \neq 0 \text{ if } k \in I\}.$$

Clearly,

$$\overline{F_I(G)} = E_I(G) \text{ for all } I \subset N. \tag{1.1}$$

The sets $F_I(X)$ and $E_I(X)$ are similarly defined.

$L^1(G)$ and $L^\infty(G)$ will be written for the spaces of integrable functions, and essentially bounded measurable functions, respectively, on G. For the precise definition of these spaces, especially the latter, see Hewitt and Ross

[9], Definitions **12.1** and **12.11**. $L^1(G)$ will be given the usual norm topology and $L^\infty(G)$ the weak topology $\sigma(L^\infty, L^1)$ generated by $L^1(G)$. We shall also consider the spaces $C_0(G)$ — the space of functions which are continuous and vanish at infinity — and $BC(G)$ — the space of bounded continuous functions on G. The former will be equipped with the uniform norm topology, and the latter with the strict topology (Herz [2]). The strict topology is defined by the semi-norms

$$f \to \|kf\|, \text{ for all } f \in BC(G),$$

where $\| \ \|$ is the uniform norm, and k ranges over $C_0(G)$. The dual of $BC(G)$ with this topology can be identified with the space of bounded Radon measures on G: every continuous linear functional on $BC(G)$ has the form

$$f \to \int_G f d\mu, \text{ for all } f \in BC(G),$$

where μ is some bounded Radon measure on G. This is easily verified.

Let α be a bicontinuous automorphism of G; that is, α is continuous and has a continuous inverse. If f is a function on G, we define the function f^α by

$$f^\alpha(x) = f(\alpha(x)) \text{ for all } x \in G.$$

It is simple to verify that if $f \in L^1(G)$, then $f^\alpha \in L^1(G)$; and similarly for each of the other spaces mentioned above. Now consider a group H of continuous automorphisms of G, and a function f in one of the above spaces. The closed translation-invariant subspace (of the relevant function space) generated by $\{f^\alpha : \alpha \in H\}$ is denoted by $T_H[f]$.

2. Preliminaries

The Fourier transform $\hat{f}$ of a function $f \in L^\infty(G)$ is defined as a pseudomeasure on X. For the definition of pseudomeasures, and some of their properties, see Edwards [3], Gaudry [5] and [6]. If $f \in L^\infty(G)$, we denote the support of the pseudomeasure $\hat{f}$ by supp $\hat{f}$; and if $f \in L^1(G)$, the set $Z(f)$ is defined by

$$Z(f) = \{\chi \in X : \hat{f}(\chi) = 0\}.$$

Write $C_c(G)$ for the space of continuous functions on G which have compact support; and consider a bicontinuous automorphism α of G. It can be verified that the mapping

$$f \to \int_G f^\alpha dx, \text{ for all } f \in C_c(G)$$

defines a non-zero, translation-invariant, positive Radon measure on G. Hence, by the uniqueness of Haar measure, there exists a constant $c_\alpha \geqq 0$ such that

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$$\int_G f^\alpha dx = c_\alpha \cdot \int_G f dx, \text{ for all } f \in L^1(G). \tag{2.1}$$

The map $\alpha^* : \chi \to \chi^{\alpha^{-1}}$ defines a bicontinuous automorphism of X; and it follows from (2.1) that if f is in $L^1(G)$ or $L^\infty(G)$, then

$$\operatorname{supp} \widehat{f^\alpha} = \alpha^* (\operatorname{supp} \hat{f}). \tag{2.2}$$

If H is a group of continuous automorphisms of G, we write

$$H^* = \{\alpha^* : \alpha \in H\}.$$

We say that a group K of continuous automorphisms of X is *thick* (or, more briefly, a *thick automorphism group* of X) if, for each subset $I \subset N$, and $\chi \in F_I(X)$, we have

$$\overline{\{\beta(\chi) : \beta \in K\}} = E_I(X). \tag{2.3}$$

We now prove several lemmas.

LEMMA 1. *Suppose that K is a thick automorphism group of X, and S is a closed subset of X. If $I \subset N$, then $F_I(X)$ is either contained in $\bigcap \{\beta(S) : \beta \in K\}$ or $F_I(X)$ is disjoint from $\bigcap \{\beta(S) : \beta \in K\}$.*

PROOF. Suppose that $F_I(X)$ is not disjoint from $\bigcap \{\beta(S) : \beta \in K\}$. Then there exists $\chi \in F_I(X)$ such that

$$\chi \in \bigcap \{\beta(S) : \beta \in K\}.$$

Hence, since K is a group, $\gamma(\chi) \in \bigcap \{\beta(S) : \beta \in K\}$ for each $\gamma \in K$. Now, $\bigcap \{\beta(S) : \beta \in K\}$ is closed because S is closed; and K is a thick automorphism group of X. Thus

$$\bigcap \{\beta(S) : \beta \in K\} \supset \overline{\{\gamma(\chi) : \gamma \in K\}} \supset F_I(X).$$

This completes the proof of the lemma.

Let K be a thick automorphism group of X. Consider a closed subset $S \subsetneqq X$, and the class

$$\boldsymbol{S} = \{I \subset N : F_I(X) \subset \bigcap \{\beta(S) : \beta \in K\}\}.$$

Suppose that $I_1, \cdots, I_s$ are the maximal elements of $\boldsymbol{S}$, under the partial order induced by set inclusion. Then we have

LEMMA 2. *Let K be a thick automorphism group of X. If S is a closed subset of X, and $I_1, \cdots, I_s$ are as above, then*

$$\bigcap \{\beta(S) : \beta \in K\} = \bigcup_{k=1}^{s} E_{I_k}(X).$$

PROOF. By (1.1), and the definition of $\boldsymbol{S}$

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$$\bigcup_{k=1}^{s} E_{I_k}(X) = \{\bigcup_{k=1}^{n} F_{I_k}(X)\} \subset \bigcap \{\beta(S) : \beta \in K\}$$

since S is closed.

Conversely, suppose that $\chi \in \bigcap \{\beta(S) : \beta \in K\}$. We have $\chi \in F_I(X)$, for some $I \subset N$. Thus

$$F_I(X) \cap (\bigcap \{\beta(S) : \beta \in K\}) \neq 0$$

and so, by Lemma 1,

$$F_I(X) \subset \bigcap \{\beta(S) : \beta \in K\}.$$

Hence $I \in \mathbf{S}$, so that $I \subset I_k$, for some positive integer $k \leqq s$.

Now

$$E_{I_k}(X) \supset E_I(X) \supset F_I(X),$$

so that $\chi \in E_{I_k}$. Since $\chi \in \bigcap \{\beta(S) : \beta \in K\}$ was arbitrary, we infer that

$$\bigcap \{\beta(S) : \beta \in K\} \subset \bigcup_{k=1}^{s} E_{I_k}.$$

This completes the proof of Lemma 2.

Corollary. *Let K be a thick automorphism group of X. If S is a closed subset of X, then $\bigcap \{\beta(S) : \beta \in K\}$ is an S-set.*

Proof. By Lemma 2, $\bigcap \{\beta(S) : \beta \in K\}$ is a union of closed subgroups of X. Thus $\bigcap \{\beta(S) : \beta \in K\}$ is a C-set (Rudin [7], Theorem 7.5.2 (b) and (d)), and hence an S-set.

Lemma 3. *Let K be a thick autmorphism group of X, and S a subset of X. Then*

$$\overline{\bigcup \{\beta(S) : \beta \in K\}} = \bigcup \{E_I(X) : F_I(X) \cap S \neq \emptyset\}.$$

Proof. Suppose that $I \subseteqq N$ is such that $F_I(X) \cap S \neq \emptyset$. Let $\chi \in F_I(X) \cap S$. Then (by 2.3)

$$E_I(X) = \overline{\{\beta(\chi) : \beta \in K\}} \subset \overline{\bigcup \{\beta(S) : \beta \in K\}}.$$

Thus we infer that

$$\bigcup \{E_I(X) : F_I(X) \cap S \neq \emptyset\} \subset \overline{\bigcup \{\beta(S) : \beta \in K\}}.$$

The converse inclusion is obvious.

Corollary. *If K is a thick automorphism group of X, and S is a subset of X, then $\overline{\bigcup \{\beta(S) : \beta \in K\}}$ is an S-set.*

The proof of this assertion is similar to the proof of the corollary to Lemma 2.

3. Main results

We now proceed to the statement and proof of the main results contained in this note. Throughout, whenever we talk about an 'automorphism group', we shall mean a group of *continuous* automorphisms.

THEOREM 1. *Let H be an automorphism group of G, such that H^* is a thick automorphism group of X.*

(a) *If $f \in L^1(G)$ and $g \in L^1(G)$, then $g \in \boldsymbol{T}_H[f]$ if and only if*

$$\bigcap \{\alpha^*(Z(f)) : \alpha \in H\} \subset Z(g). \tag{3.1}$$

(b) *If $f \in L^\infty(G)$ and $g \in L^\infty(G)$, then $g \in \boldsymbol{T}_H[f]$ if and only if*

$$\operatorname{supp} \hat{g} \subset \overline{\bigcup \{\alpha^*(\operatorname{supp} \hat{f}) : \alpha \in H\}}. \tag{3.2}$$

PROOF OF (a). In view of (2.2), condition (3.1) is certainly necessary for $g \in \boldsymbol{T}_H[f]$.

Conversely, suppose that (3.1) is satisfied. According to the Hahn-Banach theorem, we have to show that if $\varphi \in L^\infty(G)$ is such that

$$\varphi * f^\alpha = 0, \text{ for all } \alpha \in H, \tag{3.3}$$

then

$$\varphi * g(0) = 0.$$

Thus, suppose that $\varphi \in L^\infty(G)$ is such that (3.3) holds. Then, because of (2.2)

$$\operatorname{supp} \hat{\varphi} \subset \bigcap \{\alpha^*(Z(f)) : \alpha \in H\}. \tag{3.4}$$

Now, H^* is a thick automorphism group of X. Thus $\cap \{\alpha^*(Z(f)) : \alpha \in H\}$ is an S-set (by the corollary to Lemma 2) and (by (3.1)) $\hat{g}$ vanishes on $\bigcap \{\alpha^*(Z(f)) : \alpha \in H\}$. These facts, together with (3.4), entail that $\varphi * g = 0$.

PROOF OF (b). The 'only if' part is obvious, if we bear in mind relation (2.2).

On the other hand, suppose that (3.2) holds. Then if $\varphi \in L^1(G)$ is such that

$$\varphi * f^\alpha = 0, \text{ for all } \alpha \in H,$$

it follows (again using (2.2)) that

$$\hat{\varphi} = 0 \text{ on } \overline{\bigcup \{\alpha^*(\operatorname{supp} \hat{f}) : \alpha \in H\}}. \tag{3.5}$$

Now, $\overline{\bigcup \{\alpha^*(\operatorname{supp} \hat{f}) : \alpha \in H\}}$ is an S-set (by the corollary to Lemma 3); and (by 3.2))

$$\operatorname{supp} \hat{g} \subset \overline{\bigcup \{\alpha^*(\operatorname{supp} \hat{f}) : \alpha \in H\}}.$$

Therefore (3.5) entails that $\varphi * g = 0$.

We now invoke the Hahn-Banach theorem to deduce that $g \in \boldsymbol{T}_H[f]$.

THEOREM 2. *Let H be an automorphism group of G, such that H^* is a thick automorphism group of X. If $f \in BC(G)$ and $g \in BC(G)$, then $g \in \boldsymbol{T}_H[f]$ if and only if*

$$\operatorname{supp} \hat{g} \subset \overline{\bigcup \{\alpha^*(\operatorname{supp} \hat{f}) : \alpha \in H\}}.$$

PROOF. Suppose that μ is a bounded Radon measure on G such that

$$\mu * f^\alpha = 0, \text{ for all } \alpha \in H.$$

Then, if φ is any element in $L^1(G)$, it follows that

$$(\varphi * \mu) * f^\alpha = 0, \text{ for all } \alpha \in H.$$

Since $\varphi * \mu \in L^1(G)$, we can argue as in the proof of Theorem 1(b) and infer that $\varphi * \mu * g = 0$. The fact that $\varphi \in L^1(G)$ was arbitrary now leads us to the conclusion that $\mu * g = 0$.

An application of the Hahn-Banach theorem completes the proof, if we bear in mind the remark about the dual of $BC(G)$ made in § 1.

THEOREM 3. *Let H be an automorphism group of G, such that H^* is a thick automorphism group of X. If $f \in C_0(G)$, $f \neq 0$, then $\boldsymbol{T}_H[f] = C_0(G)$.*

PROOF. The same argument as is used in the proof of Theorem 4.2 Edwards [4] shows that

$$E_{N\backslash\{1\}}(X) \cup \cdots \cup E_{N\backslash\{n\}}(X)$$

is 1-thin. (We are here keeping in mind the fact that each G_k is non-compact). Thus, if f is a non-zero element of $C_0(G)$, then

$$\operatorname{supp} \hat{f} \cap F_N(X) \neq \emptyset,$$

and so (by Lemma 3)

$$\overline{\bigcup \{\alpha^*(\operatorname{supp} \hat{f}) : \alpha \in H\}} = X. \tag{3.6}$$

Therefore, if μ is a bounded Radon measure such that

$$\mu * f^\alpha = 0, \text{ for all } \alpha \in H,$$

it follows (from 3.6) that $\mu = 0$. Now again use the Hahn-Banach theorem to get the required result.

4. The case $G = R^n$

In this section, we restate some of the results of § 3 for the case when G is the n-dimensional Euclidean space R^n. The results will be phrased in a somewhat more manageable, and apparently different, form than their counterparts in § 3.

For each positive integer $k \leqq n$, J_k will denote the function defined by $J_k(x) = x_k$, for all $x \in R^n$.

We identify the character group of R^n with R^n. The continuous character of R^n corresponding to an element $(\chi_1, \cdots, \chi_n) \in R^n$ is the function

$$(x_1, \cdots, x_n) \to \exp [i(x_1\chi_1 + \cdots + x_n\chi_n)].$$

We note that if K is a thick automorphism group of R^n, then every element $\alpha \in K$ has the form

$$\alpha(x_1, \cdots, x_n) = (\alpha_1 x_1, \cdots, \alpha_n x_n), \tag{4.1}$$

when $\alpha_1, \cdots, \alpha_n$ are non-zero real numbers; and that the mapping $\alpha \to (\alpha_1, \cdots, \alpha_n)$ is an algebraic isomorphism of H onto a dense multiplicative subgroup of $F_N(R^n)$. Conversely, each dense multiplicative subgroup of $F_N(R^n)$ determines a thick automorphism group of R^n via (4.1). In what follows, we shall *identify* a thick automorphism group of R^n with its isomorphic image in $F_N(R^n)$, the identification being expressed by (4.1).

If W is any open set in R^n, we shall write $\boldsymbol{D}(W)$ for the space of indefinitely differentiable functions with compact supports contained in W. For details of these spaces, see, for example, Schwartz [8].

Finally, note that if $f \in L^\infty(R^n)$, then the pseudomeasure $\hat{f}$ can easily be identified with the distributional Fourier transform of f.

THEOREM 4. *Let H be a thick automorphism group of R^n.*

(a) *Suppose that $f \in L^1(R^n)$ and $g \in L^1(R^n)$. Then $g \in \boldsymbol{T}_H[f]$ if and only if the following statement is true: for every subset $\{n_1, \cdots, n_k\}$ of $\{1, \cdots, n\}$ for which*

$$\int_R \cdots \int_R f(x_1, \cdots, x_n)dx_{n_1} \cdots dx_{n_k} = 0 \text{ (almost everywhere)}, \tag{4.2}$$

it is also true that

$$\int_R \cdots \int_R g(x_1, \cdots, x_n)dx_{n_1} \cdots dx_{n_k} = 0 \text{ (almost everywhere)}. \tag{4.3}$$

(b) *Suppose that $f \in L^\infty(R^n)$ and $g \in L^\infty(R^n)$. Then $g \in \boldsymbol{T}_H[f]$ if and only if the following statement is true: for every multi-index β such that $\beta_k \leqq 1$ for each $k \leqq n$ and*

$$D^\beta f = 0, \tag{4.4}$$

it is also true that

$$D^\beta g = 0. \tag{4.5}$$

PROOF OF (a). Suppose that $g \in \boldsymbol{T}_H[f]$, and

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$$\int_R \cdots \int_R f(x_1, \cdots, x_n) dx_{n_1} \cdots dx_{n_k} = 0 \text{ a.e.}$$

Without loss of generality, assume that $\{n_1, \cdots, n_k\} = \{1, \cdots, k\}$. Then we also have that, for each $\alpha \in H$,

$$\int_R \cdots \int_R \exp [i(x_{k+1}\chi_{k+1} + \cdots + x_n\chi_n)] f(\alpha_1 x_1, \cdots, \alpha_n x_n) dx_1 \cdots dx_n = 0,$$
$$\text{for all } \psi_{k+1}, \cdots, \chi_n \in R.$$

Since $g \in \boldsymbol{T}_H[f]$, this entails that

$$\int_R \cdots \int_R \exp [i(x_{k+1}\chi_{k+1} + \cdots + x_n\chi_n)] g(x_1, \cdots, x_n) dx_1 \cdots dx_n = 0,$$
$$\text{for all } \chi_{k+1}, \cdots, \chi_n \in R,$$

and so

$$\int_R \cdots \int_R g(x_1, \cdots, x_n) dx_1 \cdots dx_k = 0 \text{ a.e.}$$

Conversely, suppose that (4.2) implies (4.3). Note that for the case $G = R^n$, $H^* = H$. Thus, according to Theorem 1 (a), we have to show that

$$Z(g) \supset \bigcap \{\alpha(Z(f)) : \alpha \in H\}.$$

Let $\chi \in \bigcap \{\alpha(Z(f)) : \alpha \in H\}$. Then $\chi \in F_I(R^n)$, for some $I \subset N$. By Lemma 2, $F_I(R^n) \subset Z(f)$. Thus if $N \setminus I = \{n_1, \cdots, n_k\}$, we have

$$\int_R \cdots \int_R f(x_1, \cdots, x_n) dx_{n_1} \cdots dx_{n_k} = 0 \text{ a.e.},$$

and hence also

$$\int_R \cdots \int_R g(x_1, \cdots, x_n) dx_{n_1} \cdots dx_{n_k} = 0 \text{ a.e.}$$

From this, it follows trivially that $\chi \in Z(g)$.

Proof of (b). First suppose that $g \in \boldsymbol{T}_H[f]$, and β is a multi-index such that $D^\beta f = 0$. Then $D^\beta f^\alpha = 0$, for all $\alpha \in H$, and so

$$f^\alpha * (D^\beta \varphi) = 0, \text{ for all } \alpha \in H \text{ and all } \varphi \in \boldsymbol{D}(R^n).$$

Thus, since $g \in \boldsymbol{T}_H[f]$, it follows that

$$D^\beta g(\varphi) = g * D^\beta \varphi(0) = 0, \text{ for all } \varphi \in \boldsymbol{D}(R^n).$$

On the other hand, suppose that (4.4) implies (4.5), and let us assume that, contrary to Theorem 1 (b),

$$\operatorname{supp} \hat{g} \not\subset \overline{\bigcup \{\alpha(\operatorname{supp} \hat{f}) : \alpha \in H\}}.$$

Choose $\chi \in \operatorname{supp} \hat{g}$ such that $\chi \notin \overline{\bigcup \{\alpha(\operatorname{supp} \hat{f}) : \alpha \in H\}}$. Then, by Lemma 3, $\chi \neq 0$. Suppose that $\chi \in F_I(R^n)$, where $I = \{n_1, \cdots, n_k\}$. Choose a neighbourhood W of χ such that

$$W \cap (E_{N \setminus \{n_1\}}(R^n) \cup \cdots \cup E_{N \setminus \{n_1\}}(R^n)) = \emptyset.$$

Since $\chi \in \operatorname{supp} \hat{g}$, there exists $\varphi \in \boldsymbol{D}(W)$ such that $\hat{g}(\varphi) \neq 0$. By the choice of W, there exists $\psi \in \boldsymbol{D}(W)$ such that $J_{n_1} \cdots J_{n_k}\psi = \varphi$; so that

$$\begin{aligned} D_{n_1} \cdots D_{n_k} g(\hat{\psi}) &= \hat{g}(J_{n_1} \cdots J_{n_k}\psi) \\ &= \hat{g}(\varphi) \\ &\neq 0. \end{aligned}$$

Thus $D_{n_1} \cdots D_{n_k} g \neq 0$.

On the other hand, Lemma 3 and the fact that $\chi \notin \overline{\cup \{\alpha(\operatorname{supp} \hat{f}) : \alpha \in H\}}$ imply that

$$\operatorname{supp} \hat{f} \subseteqq E_{N\setminus\{n_1\}}(R^n) \cup \cdots \cup E_{N\setminus\{n_k\}}(R^n).$$

Furthermore, for each $\varphi \in \boldsymbol{D}(R^n)$, $J_{n_1} \cdots J_{n_k}\varphi$ vanishes on the S-set

$$E_{N\setminus\{n_1\}}(R^n) \cup \cdots \cup E_{N\setminus\{n_k\}}(R^n).$$

Thus $D_{n_1} \cdots D_{n_k} f(\hat{\varphi}) = \hat{f}(J_{n_1} \cdots J_{n_k}\varphi) = 0$, for all $\varphi \in \boldsymbol{D}(R^n)$. This leads to a contradiction.

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