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Jordan–Chevalley Decomposition in Lie Algebras

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Abstract. We prove that if $\mathfrak{s}$ is a solvable Lie algebra of matrices over a field of characteristic 0 and $A \in \mathfrak{s}$, then the semisimple and nilpotent summands of the Jordan–Chevalley decomposition of A belong to $\mathfrak{s}$ if and only if there exist $S, N \in \mathfrak{s}$, S is semisimple, N is nilpotent (not necessarily $[S, N] = 0$) such that $A = S + N$.

1 Introduction

All Lie algebras and representations considered in this paper are finite dimensional over a field $\mathbb{F}$ of characteristic 0. An important question concerning a given representation $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{g}$ is (*cf.* [B2, Ch. VII, §5])

(∗) Does $\pi(\mathfrak{g})$ contain the semisimple and nilpotent parts of the Jordan–Chevalley decomposition (JCD) in $\mathfrak{gl}(V)$ of $\pi(x)$ for a given $x \in \mathfrak{g}$?

For semisimple Lie algebras, this is true for any representation and this classic result is a cornerstone of the representation theory of semisimple Lie algebras (see [Hu, §6.4 and Ch. VI] or [FH, §9.3 and Ch. 14]). We are interested in the classification of indecomposable representations of certain families of non semisimple Lie algebras (see [CS2, CS3]), and an extension of the classical result to more general Lie algebras will prove useful in this endeavour. In a different direction, the recent article [Ki], studies the existence of a Jordan–Chevalley–Seligman decomposition in prime characteristic.

The question (∗) led us to study the existence and uniqueness of abstract JCD's in arbitrary Lie algebras [CS]. Recall that an element x of a Lie algebra $\mathfrak{g}$ is said to have an *abstract JCD* if there exist unique $s, n \in \mathfrak{g}$ such that $x = s + n$, $[s, n] = 0$ and given any finite dimensional representation $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$ the JCD of $\pi(x)$ in $\mathfrak{gl}(V)$ is $\pi(x) = \pi(s) + \pi(n)$. The Lie algebra $\mathfrak{g}$ itself is said to have an abstract JCD if every one of its elements does. The main results of [CS] are Theorems 1 and 2, and they respectively state that *a Lie algebra has an abstract JCD if and only if it is perfect*, and *an element of a Lie algebra* $\mathfrak{g}$ *has an abstract JCD if and only if it belongs to* $[\mathfrak{g}, \mathfrak{g}]$. These theorems, though related to question (∗), do not provide a satisfactory answer to it.

The purpose of this note is two-fold: on one hand we prove Theorem 1.1 below, which directly addresses question (∗) and allows us to derive from it [CS, Theorems 1 and 2]. On the other hand, we recently realized that there is a gap in the original

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proof of [CS, Theorems 1 and 2], since [CS, Lemma 2.1] is not true. Therefore, we leave [CS, Theorems 1 and 2] in good standing by giving a correct proof derived from Theorem 1.1.

Theorem 1.1 *Let $\mathfrak{s}$ be a solvable Lie algebra of matrices, let $A \in \mathfrak{s}$, and assume that $A = S + N$ with $S, N \in \mathfrak{s}$, S semisimple, N nilpotent (we are not assuming $[S, N] = 0$). Then the semisimple and nilpotent summands of the JCD of A belong to $\mathfrak{s}$.*

This theorem is a consequence of the following result.

Theorem 1.2 *Let $\mathbb{F}$ be algebraically closed. Given a square matrix $A = S + N$ with S semisimple and N nilpotent, let $\{S_n\}$ and $\{N_n\}$ be sequences defined inductively by*

$$S_0 = S \quad \text{and} \quad N_0 = N,$$

and, if $[S_n, N_n] \neq 0$, let $(N_n)_{\lambda_n}$ be a non-zero eigenmatrix of $\operatorname{ad}(S_n)$ corresponding to a non-zero eigenvalue λ_n appearing in the $\operatorname{ad}(S_n)$-decomposition of N_n, and let

$$S_{n+1} = S_n + (N_n)_{\lambda_n} \quad \text{and} \quad N_{n+1} = N_n - (N_n)_{\lambda_n}.$$

(The sequences depend on the choice of the non-zero eigenvalues.)

If $\{S, N\}$ generates a solvable Lie algebra $\mathfrak{s}$, then (independently of the choice of the eigenvalues)

(i) *S_n is semisimple, N_n is nilpotent, and $S_n, N_n \in \mathfrak{s}$ for all n,*
(ii) *there is n_0 such that $[S_{n_0}, N_{n_0}] = 0$.*

In particular, $A = S_{n_0} + N_{n_0}$ is the Jordan–Chevalley decomposition of A with both components $S_{n_0}, N_{n_0} \in \mathfrak{s}$. Moreover, if $\pi\colon \mathfrak{s} \to \mathfrak{gl}(V)$ is a representation such that $\pi(S)$ is semisimple and $\pi(N)$ is nilpotent, then $\pi(A) = \pi(S_{n_0}) + \pi(N_{n_0})$ is the Jordan–Chevalley decomposition of $\pi(A)$.

2 Jordan–Chevalley Decomposition of Upper Triangular Matrices

This section is devoted to proving Theorem 1.2, and thus we assume that $\mathbb{F}$ is algebraically closed. Let $\mathfrak{t}$ denote the Lie algebra of upper triangular $n \times n$ matrices over $\mathbb{F}$, let $\mathfrak{t}' = [\mathfrak{t}, \mathfrak{t}]$, and let $\mathfrak{s}$ be a Lie subalgebra of $\mathfrak{t}$.

Lemma 2.1 *Let $S, X, N \in \mathfrak{s}$ and assume that $\operatorname{ad}_{\mathfrak{s}}(S)(N) = \lambda N$, with $\lambda \in \mathbb{F}$, and $\operatorname{ad}_{\mathfrak{s}}(S)(X) = \mu X$, with $0 \neq \mu \in \mathbb{F}$ (in particular, $X \in \mathfrak{t}'$). Then*

$$\exp\big(-\mu^{-1} \operatorname{ad}_{\mathfrak{s}}(X)\big)(N) = \sum_{j=0}^{n-1} \frac{(-\mu)^{-j}}{j!} \operatorname{ad}_{\mathfrak{s}}(X)^j(N)$$

is an eigenmatrix of $\operatorname{ad}_{\mathfrak{s}}(S + X)$ of eigenvalue λ, and it belongs to $\mathfrak{s}$. In particular, S is semisimple if and only if $S + X$ is semisimple.

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Proof Since $X \in \mathfrak{t}'$, we see that $-\mu^{-1}\operatorname{ad}_{\mathfrak{s}}(X)$ is a nilpotent derivation of $\mathfrak{s}$, so $\exp(-\mu^{-1}\operatorname{ad}_{\mathfrak{s}}(X)) \in \operatorname{Aut}(\mathfrak{s})$. In particular, $\exp(-\mu^{-1}\operatorname{ad}_{\mathfrak{s}}(X))(N) \in \mathfrak{s}$ and

$$\begin{aligned}\Big[\exp\big(-\mu^{-1}\operatorname{ad}_{\mathfrak{s}}(X)\big)(S), \exp\big(-\mu^{-1}\operatorname{ad}_{\mathfrak{s}}(X)\big)(N)\Big] &= \exp\big(-\mu^{-1}\operatorname{ad}_{\mathfrak{s}}(X)\big)\big([S,N]\big)\\ &= \lambda\exp\big(-\mu^{-1}\operatorname{ad}_{\mathfrak{s}}(X)\big)(N).\end{aligned}$$

But $[S,X] = \mu X$ yields $\exp(-\mu^{-1}\operatorname{ad}_{\mathfrak{s}}(X))(S) = S + X$, so $\exp(-\mu^{-1}\operatorname{ad}_{\mathfrak{s}}(X))(N)$ is an eigenmatrix of $\operatorname{ad}_{\mathfrak{s}}(S+X)$ of eigenvalue λ. Consequently, if $\operatorname{ad}_{\mathfrak{t}}(S)$ is semisimple then $\exp(-\mu^{-1}\operatorname{ad}_{\mathfrak{t}}(X))$ transforms a basis of eigenmatrices of $\operatorname{ad}_{\mathfrak{t}}(S)$ into a basis of eigenmatrices of $\operatorname{ad}_{\mathfrak{t}}(S+X)$.

To complete the proof it is sufficient to show that a matrix $A \in \mathfrak{t}$ is semisimple if and only if $\operatorname{ad}_{\mathfrak{t}}(A)$ is semisimple. The 'only if' part is clear. Conversely, if $\operatorname{ad}_{\mathfrak{t}}(A)$ is semisimple and $A = A_s + A_n$ is the JCD of A, then $A_s, A_n \in \mathfrak{t}$ (both are polynomials in A), and it follows that $\operatorname{ad}_{\mathfrak{t}}(A) = \operatorname{ad}_{\mathfrak{t}}(A_s) + \operatorname{ad}_{\mathfrak{t}}(A_n)$ is the JCD of $\operatorname{ad}_{\mathfrak{t}}(A)$. By uniqueness, $\operatorname{ad}_{\mathfrak{t}}(A_n) = 0$, and this implies that $A_n = 0$, since $A_n \in \mathfrak{t}'$ and the centralizer of $\mathfrak{t}$ in $\mathfrak{t}'$ is 0. ■

Let $S \in \mathfrak{s}$ be semisimple. Let Λ be the set of eigenvalues of $\operatorname{ad}_{\mathfrak{s}}(S)$, and for each $\lambda \in \Lambda$, let $\mathfrak{s}_\lambda \subset \mathfrak{s}$ be the corresponding eigenspace. Given $N \in \mathfrak{s}$, let

$$N = \sum_{\lambda\in\Lambda} N_\lambda,$$

where each $N_\lambda \in \mathfrak{s}_\lambda$. We refer to the above as the $\operatorname{ad}_{\mathfrak{s}}(S)$-decomposition of N.

For $k = 0,\dots,n-1$, let $\mathfrak{t}_k$ be the subspace of $\mathfrak{t}$ consisting of those matrices whose non-zero entries lay only on the diagonal (i,j) such that $j - i = k$. Given $N \in \mathfrak{t}$, let $d_k(N) \in \mathfrak{t}_k$ be defined so that $N = \sum_{k=0}^{n-1} d_k(N)$. We now introduce a function that will used to measure how close two matrices are to commuting with each other.

Definition 2.2 Let $S, N \in \mathfrak{s}$, with S semisimple, and let $N = \sum_{\lambda\in\Lambda} N_\lambda$ be the decomposition of N as a sum of eigenmatrices of $\operatorname{ad}_{\mathfrak{s}}(S)$. For $k = 0,\dots,n-1$, let

$$C_{S,k}(N) = \big\{\lambda \in \Lambda : \lambda \neq 0 \text{ and } d_k(N_\lambda) \neq 0\big\},$$

let $c_{S,k}(N)$ be the number of elements in $C_{S,k}(N)$ ($c_{S,0}(N) = 0$, since $\lambda \neq 0$ implies that $N_\lambda \in \mathfrak{t}'$), and let

$$\gamma_S(N) = \big(c_{S,1}(N),\dots,c_{S,n-1}(N)\big) \in \mathbb{Z}_{\geq 0}^{n-1}.$$

It is clear that $c_{S,k}(N) \leq \dim\mathfrak{s}$ for all k and $[S,N] = 0$ if and only if $\gamma_S(N) = (0,\dots,0)$.

Lemma 2.3 *Let $S, X, N \in \mathfrak{s}$ with S semisimple and $\operatorname{ad}_{\mathfrak{s}}(S)(X) = \mu X$, with $0 \neq \mu \in \mathbb{F}$. Let $k_0 \geq 1$ be the lowest k such that $d_k(X) \neq 0$ ($\mu \neq 0$ implies $X \in \mathfrak{t}'$ and hence $k_0 \geq 1$). Then $C_{S+X,k}(N) = C_{S,k}(N)$ for all $k \leq k_0$.*

Proof We first point out that it follows from Lemma 2.1 that $S + X$ is semisimple, and thus it makes sense to consider $C_{S+X,k}(N)$.

Let

$$N = \sum_{\lambda \in \Lambda} N_\lambda, \qquad N_\lambda \in \mathfrak{s},$$

be the $\mathrm{ad}_{\mathfrak{s}}(S)$-decomposition of N. Let

$$\widetilde{N}_{\lambda,0} = \exp\big(-\mu^{-1}\,\mathrm{ad}_{\mathfrak{s}}(X)\big)(N_\lambda),$$

and, for $j \geq 1$, let $\widetilde{N}_{\lambda,j} = \frac{\mu^{-j}}{j!}\,\mathrm{ad}_{\mathfrak{s}}(X)^j(\widetilde{N}_{\lambda,0})$.

It follows from Lemma 2.1 that $\widetilde{N}_{\lambda,j}$ is an eigenmatrix of $\mathrm{ad}_{\mathfrak{s}}(S+X)$ of eigenvalue $\lambda + j\mu$. Since

$$N_\lambda = \exp\big(\mu^{-1}\,\mathrm{ad}_{\mathfrak{s}}(X)\big)(\widetilde{N}_{\lambda,0}) = \widetilde{N}_{\lambda,0} + \widetilde{N}_{\lambda,1} + \widetilde{N}_{\lambda,2} + \cdots,$$

it follows that

$$N = \sum_{\lambda \in \Lambda}\sum_{j \geq 0} \widetilde{N}_{\lambda,j} = \sum_{\lambda \in \Lambda} \widetilde{N}_{\lambda,0} + \sum_{\lambda \in \Lambda}\sum_{j \geq 1} \widetilde{N}_{\lambda,j}$$

and this leads to the decomposition of N as a sum of eigenmatrices of $\mathrm{ad}_{\mathfrak{s}}(S+X)$ (after adding up those $\widetilde{N}_{\lambda,j}$ with the same eigenvalue).

Let $k \leq k_0$ (recall that k_0 is the lowest k such that $d_k(X) \neq 0$). Since $k_0 \geq 1$, it follows that

$$d_k(\widetilde{N}_{\lambda,j}) = \begin{cases} d_k(N_\lambda) & \text{if } j = 0, \\ 0 & \text{if } j \geq 1. \end{cases}$$

This implies $C_{S+X,k}(N) = C_{S,k}(N)$. ■

Lemma 2.4 *Let $S, N \in \mathfrak{s}$, with S semisimple, and let $N = \sum_{\lambda \in \Lambda} N_\lambda$ be the $\mathrm{ad}_{\mathfrak{s}}(S)$-decomposition of N. Assume that there is $\lambda_0 \in \Lambda$ with $\lambda_0 \neq 0$ such that $N_{\lambda_0} \in \mathfrak{s}_{\lambda_0}$ is non-zero. Then*

$$\gamma_{S+N_{\lambda_0}}(N - N_{\lambda_0}) < \gamma_S(N)$$

in the lexicographical order. (The pair $(S + N_{\lambda_0}, N - N_{\lambda_0})$ is closer to commuting than the pair (S, N).)

Proof Let k_0 be the lowest k such that $d_k(N_{\lambda_0}) \neq 0$ ($k_0 \geq 1$, since $N_{\lambda_0} \in \mathfrak{t}'$). It is clear that

(2.1) $$c_{S,k}(N - N_{\lambda_0}) = \begin{cases} c_{S,k}(N) & \text{if } k < k_0, \\ c_{S,k_0}(N) - 1 & \text{if } k = k_0, \end{cases}$$

and thus $\gamma_S(N - N_{\lambda_0}) < \gamma_S(N)$.

It follows from Lemma 2.3 that, for $k \leq k_0$,

$$c_{S+N_{\lambda_0},k}(N - N_{\lambda_0}) = c_{S,k}(N - N_{\lambda_0}),$$

and this, combined with (2.1), implies $\gamma_{S+N_{\lambda_0}}(N - N_{\lambda_0}) < \gamma_S(N)$ in the lexicographical order. ■

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2 Since $\{S, N\}$ generates a solvable Lie algebra $\mathfrak{s}$, and $\mathbb{F}$ is algebraically closed, it follows from Lie's Theorem that we can assume $S, N \in \mathfrak{s} \subset \mathfrak{t}$, and since N is nilpotent, $N \in \mathfrak{t}'$.

We will prove (i) by induction. Assume (i) is true for S_n and N_n and let us suppose that $[S_n, N_n] \neq 0$. Since $\lambda_n \neq 0$, we have $(N_n)_{\lambda_n} \in \mathfrak{t}'$, and hence N_{n+1} is nilpotent. It follows from Lemma 2.1 that S_{n+1} is semisimple and $S_{n+1}, N_{n+1} \in \mathfrak{s}$. This proves (i).

It follows from Lemma 2.4 that

$$\gamma_{S_{n+1}}(N_{n+1}) = \gamma_{S_n+(N_n)_{\lambda_n}}\big(N_n - (N_n)_{\lambda_n}\big) < \gamma_{S_n}(N_n)$$

in the lexicographical order. This implies that there exists n_0 such that $\gamma_{S_{n_0}}(N_{n_0}) = 0$ and hence $[S_{n_0}, N_{n_0}] = 0$. This proves (ii), and it is clear $A = S_{n_0} + N_{n_0}$ is the Jordan–Chevalley decomposition of A.

Finally, let $\pi\colon \mathfrak{s} \to \mathfrak{gl}(V)$ be a representation such that $\pi(S) = \pi(S_0)$ is semisimple and $\pi(N) = \pi(N_0)$ is nilpotent. Since π is a representation, if $N_n = \sum_{\lambda\in\Lambda_n}(N_n)_\lambda$ is the $\operatorname{ad}_{\mathfrak{s}}(S_n)$-decomposition of N_n, then

$$\pi(N_n) = \sum_{\lambda\in\Lambda_n} \pi\big((N_n)_\lambda\big)$$

is the $\operatorname{ad}_{\pi(\mathfrak{s})}(\pi(S_n))$-decomposition of $\pi(N_n)$. Therefore, assuming that $\pi(S_n)$ is semisimple and $\pi(N_n)$ is nilpotent, it follows, just as above, that $\pi(S_{n+1})$ is semisimple and $\pi(N_{n+1})$ is nilpotent. This implies that $\pi(A) = \pi(S_{n_0}) + \pi(N_{n_0})$ is the Jordan–Chevalley decomposition of $\pi(A)$. ■

Proof of Theorem 1.1 Theorem 1.2 shows that Theorem 1.1 is true when $\mathbb{F}$ is algebraically closed, since in this case, Lie's Theorem allows us to assume that $\mathfrak{s} \subset \mathfrak{t}$.

In general, let $\bar{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$. Suppose $A, S, N \in \mathfrak{s}$, where $A = S + N$, S is semisimple, and N is nilpotent. Let $A = S' + N'$ be the JCD of A in $\mathfrak{gl}(n, \mathbb{F})$, as ensured in [HK, §7.5]. The minimal polynomial of S', say p, is a product of distinct monic irreducible polynomials over $\mathbb{F}$ [HK, §7.5]. Since $\mathbb{F}$ has characteristic 0, it follows that p has distinct roots in $\bar{\mathbb{F}}$, whence S' is diagonalizable over $\bar{\mathbb{F}}$. Therefore, $A = S' + N'$ is the JCD of A in $\mathfrak{gl}(n, \bar{\mathbb{F}})$. Let $\bar{\mathfrak{s}}$ be the $\bar{\mathbb{F}}$-linear span of $\mathfrak{s}$ in $\mathfrak{gl}(n, \bar{\mathbb{F}})$. Then $\bar{\mathfrak{s}}$ is a solvable subalgebra of $\mathfrak{gl}(n, \bar{\mathbb{F}})$. As the theorem is true over $\bar{\mathbb{F}}$, we infer $S', N' \in \bar{\mathfrak{s}}$. Thus, $S', N' \in \mathfrak{gl}(n, \mathbb{F}) \cap \bar{\mathfrak{s}} = \mathfrak{s}$. This completes the proof of Theorem 1.1. ■

3 Jordan–Chevalley Decomposition in a Lie Algebra

Theorem 3.1 *An element x of a Lie algebra $\mathfrak{g}$ has an abstract JCD if and only if x belongs to the derived algebra $[\mathfrak{g}, \mathfrak{g}]$, in which case the semisimple and nilpotent parts of x also belong to $[\mathfrak{g}, \mathfrak{g}]$.*

Necessity This is clear, since any linear map from $\mathfrak{g}$ to $\mathfrak{gl}(V)$ such that $\dim \pi(\mathfrak{g}) = 1$, and $\pi([\mathfrak{g}, \mathfrak{g}]) = 0$ is a representation.

Sufficiency By Ado's theorem, we can assume that $\mathfrak{g}$ is a Lie algebra of matrices. Fix a Levi decomposition $\mathfrak{g} = \mathfrak{g}_s \ltimes \mathfrak{r}$ and let $\mathfrak{n} = [\mathfrak{g}, \mathfrak{r}]$. We know that $\mathfrak{n}$ is nilpotent (see [FH, Lemma C.20]). If $x \in [\mathfrak{g}, \mathfrak{g}]$, then $x = a + r$ for unique $a \in \mathfrak{g}_s$ and $r \in \mathfrak{n}$. If $a = a_s + a_n$ is the JCD of the matrix a, since $\mathfrak{g}_s$ is semisimple, it follows that $a_s, a_n \in \mathfrak{g}_s = [\mathfrak{g}_s, \mathfrak{g}_s]$ (see, for instance, [Hu, §6.4]). Let $\mathfrak{s} = \mathbb{F}a_s \oplus \mathbb{F}a_n \oplus \mathfrak{n} \subset [\mathfrak{g}, \mathfrak{g}]$. Since $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{n}$ and $\mathfrak{n}$

is nilpotent, we obtain that $\mathfrak{s}$ is a solvable Lie algebra. We now apply Theorem 1.1 to the Lie algebra $\mathfrak{s}$ with $S = a_s$, $N = a_n + r$. We obtain that if $x = S' + N'$ is the JCD of x, then $S', N' \in \mathfrak{s} \subset [\mathfrak{g}, \mathfrak{g}]$.

Finally, let $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Since $r \in \mathfrak{n}$, it follows that $\pi(r)$ is nilpotent (see [FH, Lemma C.19] or [B1, Ch.1, §5]). Since $\mathfrak{g}_s$ is semisimple, $\pi(S) = \pi(a_s)$ is semisimple and $\pi(a_n)$ is nilpotent. Since $\mathfrak{s}$ is solvable, it follows from Lie's Theorem that $\pi(N) = \pi(a_n + r)$ is nilpotent. It follows from Theorem 1.2 (applied over a field extension of $\mathbb{F}$) that $\pi(x) = \pi(S') + \pi(N')$ is the JCD of $\pi(x)$. ■

References

[B1] N. Bourbaki, *Lie Groups and Lie Algebras: chapters 1-3*. Springer-Verlag, Berlin, 1989.

[B2] ———, *Lie Groups and Lie Algebras: chapters 7-9*. Springer-Verlag, Berlin, 2005.

[CS] L. Cagliero and F. Szechtman, *Jordan–Chevalley decomposition in finite dimensional Lie algebras*. Proc. Amer. Math. Soc. **139**(2011), no. 11, 3909–3913. http://dx.doi.org/10.1090/S0002-9939-2011-10827-X

[CS2] ———, *The classification of uniserial $\mathfrak{sl}(2) \ltimes V(m)$-modules and a new interpretation of the Racah–Wigner $6j$-symbol*. J. Algebra **386**(2013), 142–175. http://dx.doi.org/10.1016/j.jalgebra.2013.03.022

[CS3] ———, *On the theorem of the primitive element with applications to the representation theory of associative and Lie Algebras*. Canad. Math. Bull. **57**(2014), no. 4, 735–748. http://dx.doi.org/10.4153/CMB-2013-046-9

[FH] W. Fulton and J. Harris, *Representation theory: A first course*. Graduate Texts in Mathematics, 129, Readings in Mathematics, Springer-Verlag, New York, 1991. http://dx.doi.org/10.1007/978-1-4612-0979-9

[HK] K. Hoffman and R. Kunze, *Linear algebra.* Second ed., Prentice-Hall, New Jersey, 1971.

[Hu] J. E. Humphreys, *Introduction to Lie algebras and representation theory*. Graduate Texts in Mathematics, 9, Springer-Verlag, New York-Berlin, 1978.

[Ki] K.-T. Kim, *Criteria for the existence of a Jordan–Chevalley-Seligman decomposition*. J. Algebra **424**(2015), 376–389. http://dx.doi.org/10.1016/j.jalgebra.2014.09.028

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