Nagoya Math. J., **251** (2023), 715–733
DOI 10.1017/nmj.2023.10

ON A BERNSTEIN–SATO POLYNOMIAL OF A MEROMORPHIC FUNCTION

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To the memory of Professor Hikosaburo Komatsu

Abstract. We define Bernstein–Sato polynomials for meromorphic functions and study their basic properties. In particular, we prove a Kashiwara–Malgrange-type theorem on their geometric monodromies, which would also be useful in relation with the monodromy conjecture. A new feature in the meromorphic setting is that we have several b-functions whose roots yield the same set of the eigenvalues of the Milnor monodromies. We also introduce multiplier ideal sheaves for meromorphic functions and show that their jumping numbers are related to our b-functions.

§1. Introduction

The theory of b-functions initiated by Bernstein and Sato independently is certainly on a crossroad of various branches of mathematics, such as generalized functions, singularity theory, prehomogeneous vector spaces, D-modules, number theory, algebraic geometry, and computer algebra. We often call them Bernstein–Sato polynomials. To see the breadth of their influence to mathematics, we can now consult, for example, the excellent survey articles by [2] and [6].

Let us briefly recall the definitions of classical Bernstein–Sato polynomials and some related results. For this purpose, let X be a complex manifold, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions on it. Denote by $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients on X. Let $f \in \mathcal{O}_X$ be a holomorphic function defined on a neighborhood of a point $x_0 \in X$ such that $f(x_0) = 0$. Then the (local) Bernstein–Sato polynomial $b_f(s) \in \mathbb{C}[s]$ of f (at $x_0 \in X$) is the nonzero polynomial $b(s) \neq 0$ of the lowest degree satisfying the equation

$$b(s)f^s = P(s)f^{s+1} \tag{1.1}$$

for some $P(s) \in \mathcal{D}_X[s]$. In the algebraic and analytic cases, the existence of such $b(s) \neq 0$ was proved by Bernstein and Björk, respectively. Then Kashiwara [18] proved that the roots of the Bernstein–Sato polynomial $b_f(s)$ are negative rational numbers. Later Oaku found an algorithm to calculate them. See [30] for the details. One of the most striking results on $b_f(x)$ is the Kashiwara–Malgrange theorem in [19] and [23], which asserts that the set of the eigenvalues of the local (Milnor) monodromies of f at various points $x \in f^{-1}(0)$ close to $x_0 \in f^{-1}(0)$ is equal to the one $\{\exp(2\pi i\alpha) \mid \alpha \in (b_f)^{-1}(0)\}$. Motivated by it, Denef and Loeser formulated their celebrated monodromy conjecture in [11]. Later in [33] and [34], Sabbah developed a theory of b-functions of several variables. More precisely, he considered several holomorphic functions $f_1, f_2, \dots, f_k \in \mathcal{O}_X$ $(k \geq 1)$ and proved the existence of a

Received June 9, 2022. Revised March 19, 2023. Accepted May 4, 2023.
2020 Mathematics subject classification: Primary 14F10, 14F18, 32C38, 32S40.

nonzero polynomial $b(s) \in \mathbb{C}[s] = \mathbb{C}[s_1, s_2, \dots, s_k]$ of k variables $s = (s_1, s_2, \dots, s_k)$ satisfying the equation

$$b(s)\left(\prod_{i=1}^{k} f_i^{s_i}\right) = P(s)\left(\prod_{i=1}^{k} f_i^{s_i+1}\right) \tag{1.2}$$

for some $P(s) \in \mathcal{D}_X[s] = \mathcal{D}_X[s_1, s_2, \dots, s_k]$ (see also [16] for a different proof and some additional results). The nonzero ideal $I \subset \mathbb{C}[s]$, thus, obtained is now called the Bernstein–Sato ideal of $f = (f_1, f_2, \dots, f_k)$. The geometric meaning of this I was clarified only recently in [10]. Moreover by Budur–Mustata–Saito [8], the theory of b-functions has been also generalized to higher-codimensional subvarieties, that is, to arbitrary ideals $\mathcal{J} \subset \mathcal{O}_X$ of $\mathcal{O}_X$. Their b-functions are related to the monodromies of the Verdier specializations along $\mathcal{J}$ (see [6] and [8] for details).

The aim of this short note is to define Bernstein–Sato polynomials for meromorphic functions and study their basic properties. For two holomorphic functions $F, G \in \mathcal{O}_X$ such that $F \neq 0$, $G \neq 0$ defined on a neighborhood of a point $x_0 \in X$ and coprime to each other such that $F(x_0) = 0$, let us consider the meromorphic function

$$f(x) = \frac{F(x)}{G(x)} \tag{1.3}$$

associated with them. Let $D = F^{-1}(0) \cup G^{-1}(0) \subset X$ be the divisor defined by $F \cdot G \in \mathcal{O}_X$, and let

$$\mathcal{O}_X\left[\frac{1}{FG}\right] = \left\{\frac{h}{(FG)^l} \mid h \in \mathcal{O}_X, l \geq 0\right\} \tag{1.4}$$

be the localization of $\mathcal{O}_X$ along $D \subset X$. Recall that this sheaf is endowed with the structure of a left $\mathcal{D}_X$-module. Then the polynomial ring $(\mathcal{O}_X[\frac{1}{FG}])[s]$ over it is naturally a left $\mathcal{D}_X[s]$-module. As in the classical case where $G = 1$ and f is holomorphic, on the rank-one free module

$$\mathcal{L} := \left(\mathcal{O}_X\left[\frac{1}{FG}\right]\right)[s] f^s \simeq \left(\mathcal{O}_X\left[\frac{1}{FG}\right]\right)[s] \tag{1.5}$$

over it, we define naturally a structure of a left $\mathcal{D}_X[s]$-module and can consider its $\mathcal{D}_X[s]$-submodule $\mathcal{D}_X[s] f^s \subset \mathcal{L}$ generated by $f^s \in \mathcal{L}$. However, in order to prove a Kashiwara–Malgrange-type theorem (see Theorem 1.4) for b-functions on the geometric monodromies of f in our meromorphic setting, we have to consider also other types of $\mathcal{D}_X[s]$-submodules of $\mathcal{L}$. Considering

$$\frac{1}{G^m} f^{s+k} = \frac{f^k}{G^m} \cdot f^s \in \mathcal{L} \tag{1.6}$$

for various integers $m \geq 0$ and $k \geq 0$, we obtain the following result.

THEOREM 1.1. *Let $m \geq 0$ be a nonnegative integer. Then there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that*

$$b(s)\left(\frac{1}{G^m} f^s\right) \in \sum_{k=1}^{+\infty} \mathcal{D}_X[s]\left(\frac{1}{G^m} f^{s+k}\right), \tag{1.7}$$

that is, there exist $P_1(s), P_2(s), \ldots, P_N(s) \in \mathcal{D}_X[s]$ *for which we have*

$$b(s)\left(\frac{1}{G^m} f^s\right) = \sum_{k=1}^{N} P_k(s)\left(\frac{1}{G^m} f^{s+k}\right). \tag{1.8}$$

Although the proof of Theorem 1.1 relies on the classical theory of Kashiwara and Malgrange, we need some new ideas to formulate and prove it. See §2 for details. This could be the reason why Bernstein–Sato polynomials for meromorphic functions were not defined nor studied before.

Definition 1.2. For $m \geq 0$, we denote by $b^{\mathrm{mero}}_{f,m}(s) \in \mathbb{C}[s]$, the minimal polynomial (i.e., the nonzero polynomial of the lowest degree) satisfying the equation in Theorem 1.1 and call it the Bernstein–Sato polynomial or the b-function of f of order m.

By a theorem of Sabbah [33], there exists a nonzero polynomial $b(s_1, s_2) \neq 0$ of two variables s_1, s_2 such that

$$b(s_1, s_2) F^{s_1} G^{s_2} = P(s_1, s_2) F^{s_1+1} G^{s_2+1} \tag{1.9}$$

for some $P(s_1, s_2) \in \mathcal{D}_X[s_1, s_2]$. Then, by setting $s_1 = s$ and $s_2 = -s-m-2$, we obtain the desired condition

$$b(s, -s-m-2)\left(\frac{1}{G^m} f^s\right) = G^2 P(s, -s-m-2)\left(\frac{1}{G^m} f^{s+1}\right). \tag{1.10}$$

This important remark is due to Oaku. However, for the given $F(x), G(x) \in \mathcal{O}_X$, it would not be so easy to verify that the polynomial $b(s, -s-m-2) \in \mathbb{C}[s]$ of s thus obtained is nonzero. Recall that by Bahloul [3], [4], Bahloul–Oaku [5], Oaku–Takayama [31], and Ucha–Castro [37], we have algorithms to compute the Bernstein–Sato ideal $I \subset \mathbb{C}[s_1, s_2]$ at least when F and G are polynomials. Motivated by this observation, instead of the equation (1.7), one may also consider the simpler one

$$b(s)\left(\frac{1}{G^m} f^s\right) \in \mathcal{D}_X[s]\left(\frac{1}{G^m} f^{s+1}\right). \tag{1.11}$$

Then, of course, the minimal polynomial $b(s) \neq 0$ satisfying that it is divided by our b-function $b^{\mathrm{mero}}_{f,m}(s)$, but from the proof of Theorem 1.4, it looks that we do not have a Kashiwara–Malgrange-type result as in it by this simpler definition of b-functions. This explains the reason why the right-hand side of the equation (1.7) is not so simple. Note also that if $G = 1$ and $f = \frac{F}{G} = F$ is holomorphic, we have $f \in \mathcal{O}_X \subset \mathcal{D}_X$, and for any $m \geq 0$, our b-function $b^{\mathrm{mero}}_{f,m}(s)$ coincides with the classical one $b_f(s) \in \mathbb{C}[s]$ introduced by Bernstein and Sato. But in the meromorphic case, the relation among $b^{\mathrm{mero}}_{f,m}(s)$ for various $m \geq 0$ is not very clear so far. See Lemma 3.3 for a weak relation among their roots. Nevertheless, we can prove a Kashiwara–Malgrange-type result as follows. First, recall the following theorem due to [15].

Theorem 1.3. *(Gusein-Zade, Luengo, and Melle-Hernández [15]) For any point* $x \in F^{-1}(0)$ *close to the point* x_0*, there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *and the open ball* $B(x;\varepsilon) \subset X$ *of radius* $\varepsilon > 0$ *with center at* x *(in a local chart of* X*) the restriction*

$$B(x;\varepsilon) \setminus G^{-1}(0) \longrightarrow \mathbb{C} \tag{1.12}$$

https://doi.org/10.1017/nmj.2023.10 Published online by Cambridge University Press

of $f : X \setminus G^{-1}(0) \longrightarrow \mathbb{C}$ *is a locally trivial fibration over a sufficiently small punctured disk in* $\mathbb{C}$ *with center at the origin* $0 \in \mathbb{C}$.

We call the fiber in this theorem the Milnor fiber of the meromorphic function $f(x) = \frac{F(x)}{G(x)}$ at $x \in F^{-1}(0)$ and denote it by M_x. As in the holomorphic case (see [26]), we obtain also its Milnor monodromy operators

$$\Phi_{j,x} : H^j(M_x;\mathbb{C}) \xrightarrow{\sim} H^j(M_x;\mathbb{C}) \qquad (j \geq 0). \tag{1.13}$$

Then we have the following result. Let $\mathrm{E}_{f,x_0} \subset \mathbb{C}^*$ be the set of the eigenvalues of the monodromies $\Phi_{j,x}$ of f at the points $x \in F^{-1}(0)$ close to x_0 and $j \geq 0$.

THEOREM 1.4. *Let* $m \geq 0$ *be a nonnegative integer. Then we have*

$$\{\exp(2\pi i\alpha) \mid \alpha \in (b^{\mathrm{mero}}_{f,m})^{-1}(0)\} \subset \mathrm{E}_{f,x_0}. \tag{1.14}$$

If we assume, moreover, that $m \geq 2\dim X$, *then we have an equality*

$$\{\exp(2\pi i\alpha) \mid \alpha \in (b^{\mathrm{mero}}_{f,m})^{-1}(0)\} = \mathrm{E}_{f,x_0}. \tag{1.15}$$

Combining Theorem 1.4 with the results in [29] and [32], one may formulate a monodromy conjecture for rational functions, like the original one in [11]. For previous works in this direction, see, for example, [14] and [38]. Note that if in a coordinate system F and G depend on separated variables, we can easily see that our $b^{\mathrm{mero}}_{f,m}(s)$ coincides with the b-function $b_F(s)$ of the holomorphic function F. At the moment, except for such trivial cases, we cannot calculate $b^{\mathrm{mero}}_{f,m}(s)$ explicitly. Instead, by [29, Th. 3.3 and Cor. 3.4], for many $f = \frac{F}{G}$, we can calculate E_{f,x_0} completely. Namely, for $m \geq 2\dim X$, the roots of $b^{\mathrm{mero}}_{f,m}(s)$ of such f can be determined up to some shifts of integers and multiplicities. Moreover, in §4, we also give an upper bound

$$(b^{\mathrm{mero}}_{f,m})^{-1}(0) \subset B^{\pi}_{f,m} \subset \mathbb{Q} \qquad (m \geq 0) \tag{1.16}$$

for the roots of $b^{\mathrm{mero}}_{f,m}(s)$ described in terms of resolutions of singularities $\pi : Y \longrightarrow X$ of the divisor $D \subset X$ such that $\pi^{-1}(D) \subset Y$ is normal crossing. If $G = 1$ and f is holomorphic, this corresponds to the negativity of the roots of b-functions proved by Kashiwara [18]. Indeed, in particular, for $m = 0$, our upper bound means that the roots of $b^{\mathrm{mero}}_{f,0}(s)$ are negative rational numbers. Moreover, by defining a reduced b-function $\tilde{b}^{\mathrm{mero}}_{f}(s)$ of f, we obtain also a lower bound

$$(\tilde{b}^{\mathrm{mero}}_{f})^{-1}(0) \subset (b^{\mathrm{mero}}_{f,m})^{-1}(0) \subset \mathbb{Q} \qquad (m \geq 0). \tag{1.17}$$

This $\tilde{b}^{\mathrm{mero}}_{f}(s)$ could be a candidate for the b-function of the meromorphic function f. However to our regret, as we shall see in Proposition 4.5, it has much less information on the singularities of f than $b^{\mathrm{mero}}_{f,m}(s)$. See §4 for details. Finally, in §5, we introduce multiplier ideal sheaves for the meromorphic function $f = \frac{F}{G}$ and show that their jumping numbers are contained in the set

$$\bigcup_{i=0,1,2,\dots} \left\{-(b^{\mathrm{mero}}_{f,0})^{-1}(0) + i\right\} \subset \mathbb{Q}_{>0}. \tag{1.18}$$

This is an analog for meromorphic functions of the main theorem of Ein–Lazarsfeld–Smith–Varolin [13]. See Corollary 5.4 for details.

After we posted this paper to the arXiv, we were informed from the authors Àlvarez Montaner, González Villa, León-Cardenal, and Núñez-Betancourt of [1] that they were also developing a theory of b-functions for meromorphic functions similar to but different from ours. Among other things, for the meromorphic function $f=\frac{F}{G}$, they define their b-function $b_{F/G}(s)\in\mathbb{C}[s]$ to be the minimal polynomial $b(s)\neq 0$ satisfying the equation

$$b(s)f^s\in\mathcal{D}_X[s]f^{s+1} \tag{1.19}$$

and apply it to the studies of the analytic continuations of archimedean local zeta functions and multiplier ideals associated with $f=\frac{F}{G}$. Moreover, in [1, Th. 6.7], they obtain a result on the jumping numbers of multiplier ideals similar to Corollary 5.4. Since our $b^{\mathrm{mero}}_{f,0}(s)$ divides their $b_{F/G}(s)$, it is not clear if Corollary 5.4 follows from [1, Th. 6.7]. In addition, our b-function $b^{\mathrm{mero}}_{f,0}(s)$ satisfies a nice relationship with the V-filtration of a holonomic D-module (see Theorem 5.3). From this, we see also that the minimal jumping number $\alpha>0$ is equal to the negative of the largest root of $b^{\mathrm{mero}}_{f,0}(s)$ (see Corollary 5.4). Altogether, the results in [1] look very useful and complementary to ours. Especially for some basic properties of the multiplier ideals, we refer to [1, §§6 and 7].

§2. Proof of Theorem 1.1

We follow the classical arguments of Gyoja [18], Kashiwara [23], Malgrange [16], and Sabbah [25]. For the theory of D-modules, we refer to [12], [17], [20], [21], and [25] and use freely the notions and the terminologies in them. Let $\mathbb{C}[s,t]$ be the $\mathbb{C}$-algebra generated by the two elements s,t satisfying the relation $ts=(s+1)t$, that is, $[t,s]=t$. Similarly, we define $\mathbb{C}[s,t^{\pm}]$, $\mathcal{D}_X[s,t]$ and $\mathcal{D}_X[s,t^{\pm}]$. Then there exists a natural isomorphism

$$\mathbb{C}[s,t^{\pm}]\xrightarrow{\sim}\mathbb{C}[t,\partial_t]\left[\frac{1}{t}\right]\qquad(s\longmapsto-\partial_t t) \tag{2.1}$$

of $\mathbb{C}$-algebras (see [16]) and the one

$$\mathcal{D}_X[s,t^{\pm}]\xrightarrow{\sim}(\mathcal{D}_X\otimes_{\mathbb{C}_X}\mathbb{C}_X[t,\partial_t])\left[\frac{1}{t}\right] \tag{2.2}$$

of $\mathcal{D}_X$-algebras associated with it. In the product space $X\times\mathbb{C}$, we define a hypersurface $Z\subset X\times\mathbb{C}$ by

$$Z=\{(x,t)\in X\times\mathbb{C}\mid tG(x)-F(x)=0\}. \tag{2.3}$$

Note that Z is the closure of the graph of the meromorphic function $f=\frac{F}{G}:X\setminus G^{-1}(0)\longrightarrow\mathbb{C}$ in $X\times\mathbb{C}$. Let

$$\mathcal{H}^1_{[Z]}(\mathcal{O}_{X\times\mathbb{C}})\simeq\frac{\mathcal{O}_{X\times\mathbb{C}}[\frac{1}{tG-F}]}{\mathcal{O}_{X\times\mathbb{C}}} \tag{2.4}$$

be the first local cohomology sheaf of $\mathcal{O}_{X\times\mathbb{C}}$ along $Z\subset X\times\mathbb{C}$ and define a regular holonomic $\mathcal{D}_{X\times\mathbb{C}}$-module $\mathcal{M}$ by

$$\mathcal{M}:=\{\mathcal{H}^1_{[Z]}(\mathcal{O}_{X\times\mathbb{C}})\}\left[\frac{1}{G}\right]\simeq\frac{\mathcal{O}_{X\times\mathbb{C}}[\frac{1}{(tG-F)G}]}{\mathcal{O}_{X\times\mathbb{C}}[\frac{1}{G}]}, \tag{2.5}$$

which is endowed with the canonical section

$$\delta(t-f(x)) := \left[\frac{1}{t-f(x)}\right] = \left[\frac{G(x)}{tG(x)-F(x)}\right] \in \mathcal{M}. \tag{2.6}$$

Unlike the classical case where f is holomorphic, this section does not necessarily generate $\mathcal{M}$ over $\mathcal{D}_{X\times\mathbb{C}}$ (see Lemma 3.2). Nevertheless, as in [23] and [16], for any nonnegative integer $m \geq 0$, there exists an isomorphism

$$\mathcal{D}_X[s,t^{\pm}]\left(\frac{1}{G^m}f^s\right) \xrightarrow{\sim} (\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t])\left[\frac{1}{t}\right]\left(\frac{1}{G^m}\delta(t-f(x))\right) \tag{2.7}$$

$(\frac{1}{G^m}f^s \longmapsto \frac{1}{G^m}\delta(t-f(x)))$ on a neighborhood of $F^{-1}(0) \subset X$ which is linear over $\mathcal{D}_X[s,t^{\pm}] \simeq (\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t])[\frac{1}{t}]$. Since there is no nonzero section of $\mathcal{M}$ supported in $G^{-1}(0)\times\mathbb{C} \subset X\times\mathbb{C}$ by Hilbert's nullstellensatz, to show (2.7), it suffices to compare the annihilators of the generators of its both sides on $X \setminus G^{-1}(0)$. Here, the right-hand side of (2.7) is understood to be a subsheaf of $(\mathcal{M}|_{\{t=0\}})[\frac{1}{t}]$ and the multiplication by t on it corresponds to the action $s \longmapsto s+1$ on the left-hand side (see, e.g., [16] for details). Restricting the isomorphism (2.7) to a subsheaf, we obtain an isomorphism

$$\mathcal{D}_X[s]\left(\frac{1}{G^m}f^s\right) \xrightarrow{\sim} \mathcal{D}_X[-\partial_t t]\left(\frac{1}{G^m}\delta(t-f(x))\right). \tag{2.8}$$

Now, let us consider the V-filtration $\{V_j(\mathcal{D}_{X\times\mathbb{C}})\}_{j\in\mathbb{Z}}$ of $\mathcal{D}_{X\times\mathbb{C}}$ along the hypersurface $\{t = 0\} = X\times\{0\} \subset X\times\mathbb{C}$. Similarly, we define a filtration $\{V_j(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t])\}_{j\in\mathbb{Z}}$ of $\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t] \subset \mathcal{D}_{X\times\mathbb{C}}|_{\{t=0\}}$. Denote the section

$$\frac{1}{G^m}\delta(t-f(x)) \in \mathcal{M}|_{\{t=0\}} \tag{2.9}$$

of $\mathcal{M}|_{\{t=0\}}$ simply by σ_m. Then, by $t\cdot\delta(t-f) = f\cdot\delta(t-f)$, we obtain isomorphisms

$$V_0(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t])\sigma_m \simeq \sum_{k=0}^{+\infty}\mathcal{D}_X[s]\left(\frac{1}{G^m}f^{s+k}\right), \tag{2.10}$$

$$V_{-1}(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t])\sigma_m \simeq \sum_{k=1}^{+\infty}\mathcal{D}_X[s]\left(\frac{1}{G^m}f^{s+k}\right). \tag{2.11}$$

This implies that the $V_0(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t])$-module

$$\mathcal{K} := \frac{V_0(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t])\sigma_m}{V_{-1}(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t])\sigma_m} \tag{2.12}$$

is isomorphic to

$$\frac{\sum_{k=0}^{+\infty}\mathcal{D}_X[s](\frac{1}{G^m}f^{s+k})}{\sum_{k=1}^{+\infty}\mathcal{D}_X[s](\frac{1}{G^m}f^{s+k})}. \tag{2.13}$$

Here, we used the identification

$$V_0(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t,\partial_t]) \simeq \mathcal{D}_X[s,t] \tag{2.14}$$

https://doi.org/10.1017/nmj.2023.10 Published online by Cambridge University Press

given by $-\partial_t t \longmapsto s$. Moreover, by Lemma 2.1, there also exists an isomorphism

$$(\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}}) \otimes_{\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]} \mathcal{K} \simeq \mathcal{K}^\infty := \frac{V_0(\mathcal{D}_{X\times\mathbb{C}})\sigma_m}{V_{-1}(\mathcal{D}_{X\times\mathbb{C}})\sigma_m}. \tag{2.15}$$

By the classical result on the specializability of $\mathcal{M}$ along $\{t=0\}$, there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that

$$b(-\partial_t t)\sigma_m \in V_{-1}(\mathcal{D}_{X\times\mathbb{C}})\sigma_m. \tag{2.16}$$

This condition is equivalent to the one that the image

$$\mathcal{G} := \mathrm{Im}[b(-\partial_t t) : \mathcal{K}^\infty \longrightarrow \mathcal{K}^\infty] \tag{2.17}$$

is zero. Note that the sheaf homomorphism

$$b(-\partial_t t) : \mathcal{K} \longrightarrow \mathcal{K} \tag{2.18}$$

is $\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]$-linear and the above one in (2.17) is obtained by applying the tensor product $(\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}}) \otimes_{\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]} (\cdot)$ to it. Since $\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}}$ is flat over $\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]$, we thus obtain an isomorphism

$$\mathcal{G} \simeq (\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}}) \otimes_{\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]} \mathrm{Im}[b(-\partial_t t) : \mathcal{K} \longrightarrow \mathcal{K}]. \tag{2.19}$$

By $\mathcal{G} \simeq 0$ and the faithfully flatness of $\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}}$ over $\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]$, we obtain also

$$\mathrm{Im}[b(-\partial_t t) : \mathcal{K} \longrightarrow \mathcal{K}] \simeq 0. \tag{2.20}$$

It follows from the previous description (2.13) of $\mathcal{K}$ that we have the desired condition

$$b(s)\left(\frac{1}{G^m} f^s\right) \in \sum_{k=1}^{+\infty} \mathcal{D}_X[s]\left(\frac{1}{G^m} f^{s+k}\right). \tag{2.21}$$

This completes the proof. □

LEMMA 2.1. *There exists an isomorphism* $(\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}}) \otimes_{\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]} \mathcal{K} \simeq \mathcal{K}^\infty$.

Proof. By our construction of the regular holonomic $\mathcal{D}_{X\times\mathbb{C}}$-module $\mathcal{M}$ in the proof of Theorem 1.1, there exists a natural morphism

$$\Phi : \mathcal{M}_0 := \frac{(\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t])[\frac{1}{(tG-F)G}]}{(\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t])[\frac{1}{G}]} \longrightarrow \mathcal{M}|_{\{t=0\}} \tag{2.22}$$

of $\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]$-modules. Since $F, G \in \mathcal{O}_X$ are coprime each other, the same is true also for $tG - F, G \in \mathcal{O}_{X\times\mathbb{C}}$, and hence the morphism Φ is injective. Therefore, for $j = 0, -1$, the $V_0(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t, \partial_t])$-module $V_j(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t, \partial_t])\sigma_m \subset \mathcal{M}|_{\{t=0\}}$ is isomorphic to the image of the morphism

$$V_j(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t, \partial_t]) \longrightarrow \mathcal{M}_0 \qquad (P \longmapsto P\sigma_m). \tag{2.23}$$

By the isomorphisms

$$(\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}}) \otimes_{\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]} V_j(\mathcal{D}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t, \partial_t]) \simeq V_j(\mathcal{D}_{X\times\mathbb{C}})|_{\{t=0\}} \qquad (j = 0, -1), \tag{2.24}$$

$$(\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}}) \otimes_{\mathcal{O}_X \otimes_{\mathbb{C}_X} \mathbb{C}_X[t]} \mathcal{M}_0 \simeq \mathcal{M}|_{\{t=0\}} \tag{2.25}$$

https://doi.org/10.1017/nmj.2023.10 Published online by Cambridge University Press

and the flatness of $\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}}$ over $\mathcal{O}_X\otimes_{\mathbb{C}_X}\mathbb{C}_X[t]$, we obtain isomorphisms

$$\begin{aligned} V_j(\mathcal{D}_{X\times\mathbb{C}})\sigma_m =&\mathrm{Im}[V_j(\mathcal{D}_{X\times\mathbb{C}})|_{\{t=0\}}\longrightarrow \mathcal{M}|_{\{t=0\}}]\\ \simeq&(\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}})\otimes_{\mathcal{O}_X\otimes_{\mathbb{C}_X}\mathbb{C}_X[t]}\mathrm{Im}[V_j(\mathcal{D}_X\otimes_{\mathbb{C}_X}\mathbb{C}_X[t,\partial_t])\longrightarrow \mathcal{M}_0]\\ \simeq&(\mathcal{O}_{X\times\mathbb{C}}|_{\{t=0\}})\otimes_{\mathcal{O}_X\otimes_{\mathbb{C}_X}\mathbb{C}_X[t]}V_j(\mathcal{D}_X\otimes_{\mathbb{C}_X}\mathbb{C}_X[t,\partial_t])\sigma_m \qquad (j=0,-1).\end{aligned}$$

Then the assertion immediately follows. ∎

§3. Proof of Theorem 1.4

First of all, we shall recall the classical theory of Kashiwara–Malgrange filtrations. For more precise explanations on them, we refer to [19] and [25]. We assume first that $\mathcal{M}$ is a general regular holonomic $\mathcal{D}_{X\times\mathbb{C}}$-module on the product of a complex manifold X and $\mathbb{C}_t$. Set $\theta=t\partial_t\in\mathcal{D}_{X\times\mathbb{C}}$ and for a section $\sigma\in\mathcal{M}$ of $\mathcal{M}$ denote by $p_\sigma(s)\in\mathbb{C}[s]$ the minimal polynomial such that

$$p_\sigma(\theta)\sigma\in V_{-1}(\mathcal{D}_{X\times\mathbb{C}})\sigma. \tag{3.1}$$

Furthermore, we set

$$\mathrm{ord}_{\{t=0\}}(\sigma):=p_\sigma^{-1}(0)\subset\mathbb{C}. \tag{3.2}$$

On the set $\mathbb{C}$ of complex numbers, let us consider the lexicographic order $\geq$ defined by

$$z\geq w \iff \mathrm{Re}z>\mathrm{Re}w \quad\text{or}\quad \mathrm{Re}z=\mathrm{Re}w,\ \mathrm{Im}z\geq\mathrm{Im}w. \tag{3.3}$$

Then, for $\alpha\in\mathbb{C}$, we define a $V_0(\mathcal{D}_{X\times\mathbb{C}})$-submodule $V_\alpha(\mathcal{M})$ of $\mathcal{M}$ by

$$V_\alpha(\mathcal{M})=\{\sigma\in\mathcal{M}\mid \mathrm{ord}_{\{t=0\}}(\sigma)\geq-\alpha-1\}. \tag{3.4}$$

We can easily see that there exists a finite subset $A\subset\{z\in\mathbb{C}\mid -1\leq z<0\}$ such that, for any section $\sigma\in\mathcal{M}$ of $\mathcal{M}$, we have

$$\mathrm{ord}_{\{t=0\}}(\sigma)\subset A+\mathbb{Z}. \tag{3.5}$$

Moreover, for each element $\alpha\in A$ of such A, the filtration $\{V_{\alpha+j}(\mathcal{M})\}_{j\in\mathbb{Z}}$ of $\mathcal{M}$ is a good V-filtration. For $\alpha\in A+\mathbb{Z}$, we set

$$V_{<\alpha}(\mathcal{M})=\bigcup_{\beta<\alpha}V_\beta(\mathcal{M})=\{\sigma\in\mathcal{M}\mid \mathrm{ord}_{\{t=0\}}(\sigma)>-\alpha-1\} \tag{3.6}$$

and

$$\mathrm{gr}^V_\alpha(\mathcal{M})=V_\alpha(\mathcal{M})/V_{<\alpha}(\mathcal{M}). \tag{3.7}$$

Then $\mathrm{gr}^V_\alpha(\mathcal{M})$ is a regular holonomic $\mathcal{D}_X$-module and we can easily show that there exists $N\gg0$ such that

$$(\theta+\alpha+1)^N\mathrm{gr}^V_\alpha(\mathcal{M})=0. \tag{3.8}$$

The following lemma is well known to the specialists.

LEMMA 3.1. *Let $\sigma\in\mathcal{M}$ be a section of $\mathcal{M}$ such that $\mathcal{D}_{X\times\mathbb{C}}\sigma=\mathcal{M}$. Then:*

(i) *For any section $\tau\in\mathcal{M}$ of $\mathcal{M}$, we have*

$$\mathrm{ord}_{\{t=0\}}(\tau)\subset\mathrm{ord}_{\{t=0\}}(\sigma)+\mathbb{Z}. \tag{3.9}$$

(ii) *For any* $\lambda \in \mathrm{ord}_{\{t=0\}}(\sigma)$, *we have*

$$\mathrm{gr}^V_{-\lambda-1}(\mathcal{M}) \neq 0. \tag{3.10}$$

(iii) *Conversely, if* $\mathrm{gr}^V_\alpha(\mathcal{M}) \neq 0$, *then we have*

$$-\alpha - 1 \in \mathrm{ord}_{\{t=0\}}(\sigma) + \mathbb{Z}. \tag{3.11}$$

Now, we return to the situation in the proof of Theorem 1.1. Namely, for the meromorphic function $f = \frac{F}{G}$, we have

$$\mathcal{M} \simeq \frac{\mathcal{O}_{X\times\mathbb{C}}[\frac{1}{(tG-F)G}]}{\mathcal{O}_{X\times\mathbb{C}}[\frac{1}{G}]} \tag{3.12}$$

and

$$\sigma_m = \frac{1}{G^m}\delta(t - f(x)) = \left[\frac{G}{(tG-F)G^m}\right] \in \mathcal{M} \qquad (m \geq 0). \tag{3.13}$$

Then we have the following result, whose proof is inspired from Sabbah's exposition [35].

LEMMA 3.2. *Assume that* $m \geq 2\dim X$. *Then* $\mathcal{M}$ *is generated by the section* $\sigma_m \in \mathcal{M}$ *over* $\mathcal{D}_{X\times\mathbb{C}}$, *that is,* $\mathcal{M} = \mathcal{D}_{X\times\mathbb{C}}\sigma_m$.

Proof. Set $g := (tG(x) - F(x)) \cdot G(x) \in \mathcal{O}_{X\times\mathbb{C}}$, and let $b_g(s) \in \mathbb{C}[s]$ be its Bernstein–Sato polynomial. Then by [36, Th. 0.4], for any root $\alpha \in \mathbb{Q}$ of $b_g(s)$, we have

$$-\dim(X \times \mathbb{C}) = -\dim X - 1 < \alpha < 0. \tag{3.14}$$

Moreover, for any $k \geq 1$, there exists $P_k(s) \in \mathcal{D}_{X\times\mathbb{C}}[s]$ such that

$$b_g(s-k)\cdot\dots\cdot b_g(s-2)b_g(s-1)g^{s-k} = P_k(s)g^s. \tag{3.15}$$

Set $n := \dim X$. Then, by substituting s in the above formula by $-n$, we see that for any $k \geq 1$ the meromorphic function g^{-n-k} is a nonzero constant multiple of $P_k(-n)g^{-n}$. This implies that

$$\mathcal{M} \simeq \frac{\mathcal{O}_{X\times\mathbb{C}}[\frac{1}{(tG-F)G}]}{\mathcal{O}_{X\times\mathbb{C}}[\frac{1}{G}]} \tag{3.16}$$

is generated by its section

$$\left[\frac{1}{g^n}\right] = \left[\frac{1}{(tG-F)^n G^n}\right] \in \mathcal{M} \tag{3.17}$$

over $\mathcal{D}_{X\times\mathbb{C}}$. On the other hand, the section $\partial_t^{n-1}\sigma_m \in \mathcal{M}$ of $\mathcal{M}$ is a nonzero constant multiple of

$$\left[\frac{1}{(tG-F)^n G^{m-n}}\right] \in \mathcal{M}. \tag{3.18}$$

Therefore, if $m \geq 2n = 2\dim X$, it generates $\mathcal{M}$ over $\mathcal{D}_{X\times\mathbb{C}}$. ∎

Now, let us prove Theorem 1.4. By the proof of Theorem 1.1 and the correspondence $s \longleftrightarrow -\partial_t t = -\theta - 1$, the Bernstein–Sato polynomial $b^{\mathrm{mero}}_{f,m}(s)$ of f coincides with $p_{\sigma_m}(-s-1)$. This, in particular, implies that we have

$$(b^{\mathrm{mero}}_{f,m})^{-1}(0) = \{-\lambda - 1 \mid \lambda \in \mathrm{ord}_{\{t=0\}}(\sigma_m)\}. \tag{3.19}$$

Note also that for the $\mathcal{D}_X$-module $\mathcal{M}$ in the proof of Theorem 1.1, we have an isomorphism

$$\mathrm{DR}_{X\times\mathbb{C}}(\mathcal{M}) \simeq R\Gamma_{(X\setminus G^{-1}(0))\times\mathbb{C}}(\mathbb{C}_Z)[n] \tag{3.20}$$

and the nearby cycle sheaf $\psi_t(\mathrm{DR}_{X\times\mathbb{C}}(\mathcal{M}))$ coincides with the meromorphic nearby cycle $\psi_f^{\mathrm{mero}}(\mathbb{C}_X)$ introduced in [29] up to some shift. Assume first that $m \geq 2\dim X$. Then, by Lemma 3.2, the section $\sigma_m \in \mathcal{M}$ generates $\mathcal{M}$ over $\mathcal{D}_{X\times\mathbb{C}}$ and the second assertion of Theorem 1.4 follows from Lemma 3.1, Kashiwara's isomorphism

$$\bigoplus_{-1\leq\alpha<0} \mathrm{DR}_X(\mathrm{gr}_\alpha^V(\mathcal{M})) \simeq \psi_t(\mathrm{DR}_{X\times\mathbb{C}}(\mathcal{M})) \simeq \bigoplus_{-1\leq\alpha<0} \psi_{t,\exp(2\pi i\alpha)}(\mathrm{DR}_{X\times\mathbb{C}}(\mathcal{M})) \tag{3.21}$$

and [29, Lem. 2.1 (iii)]. If we do not have the condition $m \geq 2\dim X$, by considering the $\mathcal{D}_{X\times\mathbb{C}}$-submodule $\mathcal{D}_{X\times\mathbb{C}}\sigma_m \subset \mathcal{M}$ instead of $\mathcal{M}$ itself, we obtain the first assertion of Theorem 1.4. This completes the proof. □

By the proofs of Theorems 1.1 and 1.4, we obtain the following weak relation among the roots of the b-functions $b_{f,m}^{\mathrm{mero}}(s)$ for various $m \geq 0$.

Lemma 3.3. *Let $m, m' \geq 0$ be two nonnegative integers such that $m \geq m'$. Then, for some $l \gg 0$, we have an inclusion*

$$(b_{f,m'}^{\mathrm{mero}})^{-1}(0) \subset \bigcup_{i=0}^{l} \{(b_{f,m}^{\mathrm{mero}})^{-1}(0) - i\}. \tag{3.22}$$

Proof. By the proofs of Theorems 1.1 and 1.4, we have $b_{f,m}^{\mathrm{mero}}(s) = p_{\sigma_m}(-s-1)$ and $b_{f,m'}^{\mathrm{mero}}(s) = p_{\sigma_{m'}}(-s-1)$. Moreover, by our assumption $m \geq m'$, we have

$$\sigma_{m'} = G^{m-m'} \cdot \sigma_m \in \mathcal{O}_X\sigma_m \subset V_0(\mathcal{D}_{X\times\mathbb{C}})\sigma_m. \tag{3.23}$$

Set $\mathcal{N} := \mathcal{D}_{X\times\mathbb{C}}\sigma_m$ and $\mathcal{N}' := \mathcal{D}_{X\times\mathbb{C}}\sigma_{m'}$. Then the V-filtration $\{V_j(\mathcal{D}_{X\times\mathbb{C}})\sigma_m\}_{j\in\mathbb{Z}}$ (resp. $\{V_j(\mathcal{D}_{X\times\mathbb{C}})\sigma_{m'}\}_{j\in\mathbb{Z}}$) of $\mathcal{N}$ (resp. $\mathcal{N}'$) is good. By Artin-Rees's lemma, the V-filtration $\{U_j(\mathcal{N}')\}_{j\in\mathbb{Z}}$ of $\mathcal{N}'$ defined by

$$U_j(\mathcal{N}') := \mathcal{N}' \cap (V_j(\mathcal{D}_{X\times\mathbb{C}})\sigma_m) \qquad (j \in \mathbb{Z}) \tag{3.24}$$

is also good and satisfies the condition $\sigma_{m'} \in U_0(\mathcal{N}')$. Then there exists $l \gg 0$ such that

$$U_{-l-1}(\mathcal{N}') \subset V_{-1}(\mathcal{D}_{X\times\mathbb{C}})\sigma_{m'} \subset \mathcal{N}'. \tag{3.25}$$

This implies that we have

$$p_{\sigma_m}(\theta - l)\cdots p_{\sigma_m}(\theta-1)p_{\sigma_m}(\theta)\sigma_{m'} \in V_{-1}(\mathcal{D}_{X\times\mathbb{C}})\sigma_{m'}. \tag{3.26}$$

Then the assertion immediately follows. □

§4. Upper and lower bounds for the roots of b-functions

Recall that in [18], Kashiwara proved that if f is holomorphic, the roots of the Bernstein–Sato polynomial $b_f(s)$ are negative rational numbers. In this section, we prove an analogous result for the meromorphic function $f = \frac{F}{G}$. We can easily prove that the roots of our b-function $b_{f,m}^{\mathrm{mero}}(s)$ are rational numbers, but their negativity does not follow from our proof. For this reason, here we only give an upper bound

$$(b_{f,m}^{\mathrm{mero}})^{-1}(0) \subset B_{f,m}^{\pi} \subset \mathbb{Q} \qquad (m \geq 0) \tag{4.1}$$

https://doi.org/10.1017/nmj.2023.10 Published online by Cambridge University Press

for the set $(b^{\mathrm{mero}}_{f,m})^{-1}(0)$ in terms of resolutions of singularities π of $D \subset X$. The precise statement is as follows. Let $\pi : Y \longrightarrow X$ be a resolution of singularities of the divisor $D = F^{-1}(0) \cup G^{-1}(0) \subset X$, which means that $\pi : Y \longrightarrow X$ is a proper morphism of n-dimensional complex manifolds such that $\pi^{-1}(D) \subset Y$ is normal crossing and $\pi|_{Y \setminus \pi^{-1}(D)} : Y \setminus \pi^{-1}(D) \longrightarrow X \setminus D$ is an isomorphism. Then we define a meromorphic function g on Y by

$$g := f \circ \pi = \frac{F \circ \pi}{G \circ \pi}. \tag{4.2}$$

From now on, we fix a nonnegative integer $m \geq 0$ and consider the (local) Bernstein–Sato polynomials of g of order m. At each point $q \in \pi^{-1}(D)$ of the normal crossing divisor $\pi^{-1}(D)$, there exists a local coordinate system $y = (y_1, y_2, \ldots, y_n)$ such that $q = (0, 0, \ldots, 0)$ and

$$(F \circ \pi)(y) = \prod_{i=1}^{n} y_i^{a_i} \quad (a_i \geq 0), \qquad (G \circ \pi)(y) = \prod_{i=1}^{n} y_i^{b_i} \quad (b_i \geq 0). \tag{4.3}$$

Then we have

$$\left(\frac{1}{G^m} f^s \right)(y) = \prod_{i=1}^{n} y_i^{(a_i - b_i)s - m b_i}. \tag{4.4}$$

It follows that the set $K_q \subset \mathbb{Q}$ of the roots of the (local) Bernstein–Sato polynomial of g at q is explicitly given by

$$K_q = \bigcup_{i:\ a_i > b_i} \left\{ \frac{m b_i}{a_i - b_i} - \frac{k}{a_i - b_i} \mid 1 \leq k \leq a_i - b_i \right\} \subset \mathbb{Q} \tag{4.5}$$

(see, e.g., [20, Lem. 6.10]). It is clear that this set K_q does not depend on the choice of the local coordinates. For the point $x_0 \in D$, its inverse image $\pi^{-1}(x_0) \subset \pi^{-1}(D)$ being compact, we obtain a finite subset

$$K := \bigcup_{q \in \pi^{-1}(x_0)} K_q \subset \mathbb{Q}. \tag{4.6}$$

THEOREM 4.1. *For any $m \geq 0$, the roots of the (local) Bernstein–Sato polynomial $b^{\mathrm{mero}}_{f,m}(s)$ of f at $x_0 \in D$ are contained in the set*

$$B^{\pi}_{f,m} := \bigcup_{l = 0,1,2,\ldots} (K - l) = \{ r - l \mid r \in K,\ l = 0, 1, 2, \ldots \} \subset \mathbb{Q}. \tag{4.7}$$

In particular, for $m = 0$, the roots of $b^{\mathrm{mero}}_{f,0}(s)$ are negative rational numbers.

Proof. Our proof is similar to the one in [18] for the case where f is holomorphic. But we also need some new ideas to treat the meromorphic case. Recall that for the section $\sigma := \sigma_m \in \mathcal{M}$ (see (2.5)) of the regular holonomic $\mathcal{D}_{X \times \mathbb{C}}$-module $\mathcal{M}$, we denote by $p_\sigma(s) \in \mathbb{C}[s]$, the minimal polynomial $p(s) \neq 0$ such that

$$p(\theta)\sigma \in V_{-1}(\mathcal{D}_{X \times \mathbb{C}})\sigma \tag{4.8}$$

and we have $b^{\mathrm{mero}}_{f,m}(s) = p_\sigma(-s-1)$. Let $i : Y \longrightarrow Y \times X$ $(y \longmapsto (y, \pi(y)))$ be the graph embedding by π and $p : Y \times X \to X$ $((y, x) \longmapsto x)$, the projection such that $\pi = p \circ i$.

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We set also

$$\tilde{i} := i \times \mathrm{id}_{\mathbb{C}} : Y \times \mathbb{C} \longrightarrow (Y \times X) \times \mathbb{C},$$
$$\tilde{p} := p \times \mathrm{id}_{\mathbb{C}} : (Y \times X) \times \mathbb{C} \longrightarrow X \times \mathbb{C},$$

so that we have $\tilde{\pi} := \pi \times \mathrm{id}_{\mathbb{C}} = \tilde{p} \circ \tilde{i}$. As in the case of the meromorphic function $f = \frac{F}{G}$, we define a regular holonomic $\mathcal{D}_{Y\times\mathbb{C}}$-module $\mathcal{N}$ associated with $g = \frac{F\circ\pi}{G\circ\pi}$ and its section

$$\tau := \frac{1}{\{(G\circ\pi)(y)\}^m}\delta(t - g(y)) \in \mathcal{N}. \tag{4.9}$$

Then the roots of its minimal polynomial $p_\tau(s) \in \mathbb{C}[s]$ such that

$$p_\tau(\theta)\tau \in V_{-1}(\mathcal{D}_{Y\times\mathbb{C}})\tau \tag{4.10}$$

is contained in the set $\{-r-1 \mid r \in K\} \subset \mathbb{Q}$. Since $\tilde{\pi} : Y \times \mathbb{C} \longrightarrow X \times \mathbb{C}$ is an isomorphism over $(Y \setminus \pi^{-1}(D)) \times \mathbb{C} \simeq (X \setminus D) \times \mathbb{C}$, the section $\tau \in \mathcal{N}$ is naturally identified with $\sigma \in \mathcal{M}$ there. Let

$$\mathbf{D}\tilde{i}_*\mathcal{N} \simeq H^0\mathbf{D}\tilde{i}_*\mathcal{N} = \tilde{i}_*\left(\mathcal{D}_{(Y\times X)\times\mathbb{C}\leftarrow Y\times\mathbb{C}} \otimes_{\mathcal{D}_{Y\times\mathbb{C}}} \mathcal{N}\right) \tag{4.11}$$

be the direct image of $\mathcal{N}$ by $\tilde{i}$ and

$$\widetilde{\tau} := 1_{(Y\times X)\times\mathbb{C}\leftarrow Y\times\mathbb{C}} \otimes \tau \in \mathbf{D}\tilde{i}_*\mathcal{N}, \tag{4.12}$$

its section defined by $\tau \in \mathcal{N}$. Then it is easy to see that the minimal polynomial $p_{\widetilde{\tau}}(s) \in \mathbb{C}[s]$ such that

$$p_{\widetilde{\tau}}(\theta)\widetilde{\tau} \in V_{-1}(\mathcal{D}_{(Y\times X)\times\mathbb{C}})\widetilde{\tau} \tag{4.13}$$

is equal to $p_\tau(s)$. Let us consider the $\mathcal{D}_{Y\times\mathbb{C}}$-submodule $\mathcal{N}_0 := \mathcal{D}_{Y\times\mathbb{C}}\tau \subset \mathcal{N}$ of $\mathcal{N}$ generated by $\tau \in \mathcal{N}$. Then we have $\widetilde{\tau} \in \mathbf{D}\tilde{i}_*\mathcal{N}_0 \subset \mathbf{D}\tilde{i}_*\mathcal{N}$ and $\mathcal{D}_{(Y\times X)\times\mathbb{C}}\widetilde{\tau} = \mathbf{D}\tilde{i}_*\mathcal{N}_0$. Hence, we can define a good V-filtration $\{U_j(\mathbf{D}\tilde{i}_*\mathcal{N}_0)\}_{j\in\mathbb{Z}}$ of $\mathbf{D}\tilde{i}_*\mathcal{N}_0$ by

$$U_j(\mathbf{D}\tilde{i}_*\mathcal{N}_0) := V_j(\mathcal{D}_{(Y\times X)\times\mathbb{C}})\widetilde{\tau} \qquad (j \in \mathbb{Z}). \tag{4.14}$$

Then it is easy to see that, for any $j \in \mathbb{Z}$, we have

$$p_\tau(\theta + j)U_j(\mathbf{D}\tilde{i}_*\mathcal{N}_0) \subset U_{j-1}(\mathbf{D}\tilde{i}_*\mathcal{N}_0). \tag{4.15}$$

Let us consider the relationship between $p_{\widetilde{\tau}}(s) = p_\tau(s)$ and $p_\sigma(s) = b^{\mathrm{mero}}_{f,m}(-s-1)$. For this purpose, let

$$\mathcal{M}' := H^0\mathbf{D}\tilde{\pi}_*\mathcal{N}_0 \simeq H^0\mathbf{D}\tilde{p}_*(\mathbf{D}\tilde{i}_*\mathcal{N}_0) \tag{4.16}$$

be the zeroth direct image of $\mathbf{D}\tilde{i}_*\mathcal{N}_0$ by $\tilde{p}$ and as in Gyoja [16, §4.2] define its section $\sigma' \in \mathcal{M}'$ to be the image of a section $1_{X\leftarrow Y} \otimes \tau \in \tilde{p}_*[\Omega^n_{Y\times X/X} \otimes_{\mathcal{O}_{Y\times X}} \mathbf{D}\tilde{i}_*\mathcal{N}_0]$ by the morphism

$$\tilde{p}_*\left[\Omega^n_{Y\times X/X} \otimes_{\mathcal{O}_{Y\times X}} \mathbf{D}\tilde{i}_*\mathcal{N}_0\right] \longrightarrow H^0\mathbf{D}\tilde{p}_*(\mathbf{D}\tilde{i}_*\mathcal{N}_0) = \mathcal{M}'.$$

For the construction of $1_{X\leftarrow Y} \otimes \tau$, see [16, §4.2] for details. Note that on the open subset $(X \setminus D) \times \mathbb{C}$ of $X \times \mathbb{C}$, we have $\mathcal{M}' = \mathcal{D}_{X\times\mathbb{C}}\sigma = \mathcal{M}$ and σ' coincides with σ. For $j \in \mathbb{Z}$, we denote by $U_j(\mathcal{M}') \subset \mathcal{M}' = H^0\mathbf{D}\tilde{p}_*(\mathbf{D}\tilde{i}_*\mathcal{N}_0)$ the image of the natural morphism

$$H^0R\tilde{p}_*\left\{\mathrm{DR}_{Y\times X/X}\left(U_j(\mathbf{D}\tilde{i}_*\mathcal{N}_0)\right)\right\} \longrightarrow \mathcal{M}'. \tag{4.17}$$

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Then, by the proof of [25, Th. 4.8.1(1)], $\{U_j(\mathcal{M}')\}_{j\in\mathbb{Z}}$ is a good V-filtration of $\mathcal{M}'$; and for any $j\in\mathbb{Z}$, we have

$$p_\tau(\theta+j)U_j(\mathcal{M}')\subset U_{j-1}(\mathcal{M}'). \tag{4.18}$$

Moreover, by our construction, the section $\sigma'\in\mathcal{M}'$ is contained in $U_0(\mathcal{M}')$. Let

$$\mathcal{M}'':=\mathcal{D}_{X\times\mathbb{C}}\sigma'\subset\mathcal{M}' \tag{4.19}$$

be the $\mathcal{D}_{X\times\mathbb{C}}$-submodule of $\mathcal{M}'$ generated by σ'. Then, by Artin–Rees's lemma, the V-filtration $\{U_j(\mathcal{M}'')\}_{j\in\mathbb{Z}}$ of $\mathcal{M}''$ defined by

$$U_j(\mathcal{M}''):=\mathcal{M}''\cap U_j(\mathcal{M}')\qquad (j\in\mathbb{Z}) \tag{4.20}$$

is also good, and hence there exists $l\gg 0$ such that

$$U_{-l}(\mathcal{M}'')=\mathcal{M}''\cap U_{-l}(\mathcal{M}')\subset V_{-1}(\mathcal{D}_{X\times\mathbb{C}})\sigma'. \tag{4.21}$$

Combining these results together, we get

$$p_\tau(\theta-(l-1))\cdots p_\tau(\theta-1)p_\tau(\theta)\sigma'\in\mathcal{M}''\cap U_{-l}(\mathcal{M}')\subset V_{-1}(\mathcal{D}_{X\times\mathbb{C}})\sigma'. \tag{4.22}$$

This implies that the minimal polynomial $p_{\sigma'}(s)\in\mathbb{C}[s]$ for the section $\sigma'\in\mathcal{M}'$ divides the product

$$p_\tau(s-(l-1))\cdots p_\tau(s-1)p_\tau(s)\in\mathbb{C}[s]. \tag{4.23}$$

Now, according to Kashiwara [20], there exists an adjunction morphism

$$\mathbf{D}\widetilde{\pi}_*(\mathbf{D}\widetilde{\pi}^*\mathcal{M})\longrightarrow\mathcal{M} \tag{4.24}$$

of $\mathcal{D}_{X\times\mathbb{C}}$-modules. Since $\mathbf{D}\widetilde{\pi}^*\mathcal{M}$ is isomorphic to $\mathcal{N}$ (use, e.g., the Riemann–Hilbert correspondence) and $\mathcal{N}_0\subset\mathcal{N}$, we obtain a morphism

$$\Psi:\mathcal{M}'=H^0\mathbf{D}\widetilde{\pi}_*\mathcal{N}_0\longrightarrow\mathcal{M} \tag{4.25}$$

of $\mathcal{D}_{X\times\mathbb{C}}$-modules. Then the section $\Psi(\sigma')\in\mathcal{M}$ of $\mathcal{M}$ coincides with $\sigma\in\mathcal{M}$ on the open subset $(X\setminus D)\times\mathbb{C}\subset X\times\mathbb{C}$. Moreover, by the isomorphism $\mathcal{M}\simeq\mathcal{D}_X[\partial_t]$ on the open subset $(X\setminus G^{-1}(0))\times\mathbb{C}\subset X\times\mathbb{C}$, this coincidence can be extended to $(X\setminus G^{-1}(0))\times\mathbb{C}$. Here, we used the classical theorem on the unique continuation of holomorphic functions. Since we have $\mathcal{M}\simeq\mathcal{M}[\frac{1}{G}]$, by Hilbert's nullstellensatz, we get $\Psi(\sigma')=\sigma$ on the whole $X\times\mathbb{C}$. This implies that the minimal polynomial $p_\sigma(s)=b^{\mathrm{mero}}_{f,m}(-s-1)$ divides the one $p_{\sigma'}(s)$. Now the assertion is clear. This completes the proof. ∎

We have seen that the roots of our b-functions $b^{\mathrm{mero}}_{f,m}(s)$ are rational numbers. Let $\rho:\mathbb{Q}\longrightarrow\mathbb{Q}/\mathbb{Z}$ be the quotient map. Then Lemma 3.3 means that the subset $A_m:=\rho\,\{(b^{\mathrm{mero}}_{f,m})^{-1}(0)\}\subset\mathbb{Q}/\mathbb{Z}$ increases with respect to $m\geq 0$. By Theorem 1.4, this sequence is stationary for $m\geq 2\dim X$.

Next, we shall give a lower bound for the subsets $(b^{\mathrm{mero}}_{f,m})^{-1}(0)\subset\mathbb{Q}$. In the proof of Theorem 1.1, we have seen that the minimal polynomial of s acting on the $\mathcal{D}_X$-module

$$\mathcal{K}\simeq\frac{\sum_{k=0}^{+\infty}\mathcal{D}_X[s](\frac{1}{G^m}f^{s+k})}{\sum_{k=1}^{+\infty}\mathcal{D}_X[s](\frac{1}{G^m}f^{s+k})} \tag{4.26}$$

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is equal to our b-function $b^{\mathrm{mero}}_{f,m}(s)$. Localizing it along the hypersurface $G^{-1}(0) \subset X$, we obtain a new $\mathcal{D}_X$-module

$$\mathcal{K}\left[\frac{1}{G}\right] \simeq \frac{\{\mathcal{D}_X[s](\frac{1}{G^m}f^s)\}[\frac{1}{G}]}{\{\mathcal{D}_X[s](\frac{1}{G^m}f^{s+1})\}[\frac{1}{G}]} \tag{4.27}$$

on which s still acts. Obviously, we have $b^{\mathrm{mero}}_{f,m}(s) = 0$ on $\mathcal{K}[\frac{1}{G}]$. By this observation, we obtain the following result. We denote the localized ring $\mathcal{D}_X[\frac{1}{G}]$ simply by $\widetilde{\mathcal{D}}_X$.

THEOREM 4.2. *Let $m \geq 0$ be a nonnegative integer. Then there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ satisfying the equation*

$$b(s)\left(\frac{1}{G^m}f^s\right) = \tilde{P}(s)\left(\frac{1}{G^m}f^{s+1}\right) \tag{4.28}$$

for some $\tilde{P}(s) \in \widetilde{\mathcal{D}}_X[s]$.

DEFINITION 4.3. For $m \geq 0$, we denote by $\tilde{b}^{\mathrm{mero}}_{f,m}(s) \in \mathbb{C}[s]$ the minimal polynomial satisfying the equation in Theorem 4.2 and call it the reduced Bernstein–Sato polynomial or the reduced b-function of f of order m.

Since $\tilde{P}(s) \in \widetilde{\mathcal{D}}_X[s]$ in the equation (4.28) can be rewritten as

$$\tilde{P}(s) = \frac{1}{G^m} \circ \tilde{Q}(s) \circ G^m \qquad (\tilde{Q}(s) \in \widetilde{\mathcal{D}}_X[s]), \tag{4.29}$$

in fact, the condition on $b(s)$ in Theorem 4.2 is equivalent to the existence of some $\tilde{Q}(s) \in \widetilde{\mathcal{D}}_X[s]$ satisfying the simpler equation

$$b(s)f^s = \tilde{Q}(s)f^{s+1} \tag{4.30}$$

independent of $m \geq 0$. This shows that we have

$$\tilde{b}^{\mathrm{mero}}_{f,0}(s) = \tilde{b}^{\mathrm{mero}}_{f,1}(s) = \tilde{b}^{\mathrm{mero}}_{f,2}(s) = \cdots\cdots\cdots. \tag{4.31}$$

Therefore, we denote $\tilde{b}^{\mathrm{mero}}_{f,m}(s)$ simply by $\tilde{b}^{\mathrm{mero}}_{f}(s)$. Then, by our construction, for any $m \geq 0$, our b-function $b^{\mathrm{mero}}_{f,m}(s)$ is divided by the reduced one $\tilde{b}^{\mathrm{mero}}_{f}(s)$. We thus obtain a lower bound

$$(\tilde{b}^{\mathrm{mero}}_{f})^{-1}(0) \subset (b^{\mathrm{mero}}_{f,m})^{-1}(0) \subset \mathbb{Q} \tag{4.32}$$

for the subset $(b^{\mathrm{mero}}_{f,m})^{-1}(0) \subset \mathbb{Q}$. Several authors studied (global) b-functions on algebraic varieties. In particular, the result in [24] ensures the existence of b-functions on smooth affine varieties (see [2] for a review on this subject). Since for algebraic X and $f = \frac{F}{G}$, the variety $X \setminus G^{-1}(0)$ is affine, our Theorem 4.2 could be considered as an analytic counterpart of their result in a very special case.

REMARK 4.4. It looks that the $\mathcal{D}_X$-modules $\mathcal{K}$ and $\mathcal{K}[\frac{1}{G}]$ above are regular holonomic, but we could not prove it. We conjecture that they are regular holonomic.

From now on, we consider the special case where the meromorphic function $f = \frac{F}{G}$ is quasi-homogeneous. More precisely, for a local coordinate system $x = (x_1, x_2, \ldots, x_n)$ of X such that $x_0 = \{x = 0\}$, we assume that there exist a vector field $v = \sum_{i=1}^{n} w_i x_i \partial_{x_i} \in \mathcal{D}_X$

https://doi.org/10.1017/nmj.2023.10 Published online by Cambridge University Press

($w=(w_1,w_2,\dots,w_n)\in\mathbb{Z}^n_{\geq 0}\setminus\{0\}$ is a weight vector) and $d_1,d_2\in\mathbb{Z}_{>0}$ with $d:=d_1-d_2\neq 0$ such that

$$vF=d_1\cdot F,\quad vG=d_2\cdot G,\quad vf=d\cdot f\neq 0. \tag{4.33}$$

Let us calculate $\tilde{b}_f^{\mathrm{mero}}(s)$ of such f following the arguments in [20, §6.4]. First, by the condition $vf=d\cdot f$ $(d\neq 0)$, we have isomorphisms

$$\widetilde{\mathcal{D}}_X[s]f^s\simeq\widetilde{\mathcal{D}}_Xf^s,\quad \mathcal{K}\left[\frac{1}{G}\right]\simeq\frac{\widetilde{\mathcal{D}}_Xf^s}{\widetilde{\mathcal{D}}_Xf^{s+1}}, \tag{4.34}$$

and for our reduced b-function $\tilde{b}_f^{\mathrm{mero}}(s)$, there exists $\tilde{P}\in\widetilde{\mathcal{D}}_X$ such that

$$\tilde{b}_f^{\mathrm{mero}}(s)f^s=\tilde{P}f^{s+1}. \tag{4.35}$$

In this situation, by the proof of [20, Lem. 6.6], we see that $\mathcal{K}[\frac{1}{G}]$ is a holonomic $\mathcal{D}_X$-module and Theorem 4.2 can be proved also by using the trick in the proof of [20, Th. 6.7]. If we set $s=-1$ in (4.35), we obtain

$$\tilde{b}_f^{\mathrm{mero}}(-1)=f\tilde{P}(1). \tag{4.36}$$

Restricting this equality to the subset $F^{-1}(0)\setminus G^{-1}(0)\subset X\setminus G^{-1}(0)$, we see that $\tilde{b}_f^{\mathrm{mero}}(-1)=0$. Namely, for a nonzero polynomial $\tilde{\beta}_f^{\mathrm{mero}}(s)\in\mathbb{C}[s]$, we have

$$\tilde{b}_f^{\mathrm{mero}}(s)=(s+1)\cdot\tilde{\beta}_f^{\mathrm{mero}}(s). \tag{4.37}$$

On the other hand, by (4.36), we have $\tilde{P}(1)=0$, and hence $\tilde{P}\in\sum_{i=1}^n\widetilde{\mathcal{D}}_X\partial_{x_i}$. Namely, there exist $\tilde{Q}_i\in\widetilde{\mathcal{D}}_X$ $(1\leq i\leq n)$ such that $\tilde{P}=\sum_{i=1}^n\tilde{Q}_i\partial_{x_i}$. Moreover, if we set

$$f_i:=f_{x_i}=\frac{\partial f}{\partial x_i}=\frac{F_{x_i}G-FG_{x_i}}{G^2}\quad (1\leq i\leq n), \tag{4.38}$$

then we have

$$\partial_{x_i}f^{s+1}=(s+1)f_if^s\quad (1\leq i\leq n). \tag{4.39}$$

Therefore, we obtain

$$\tilde{\beta}_f^{\mathrm{mero}}(s)f^s=\sum_{i=1}^n\tilde{Q}_if_if^s. \tag{4.40}$$

Conversely, for $\tilde{Q}_i\in\widetilde{\mathcal{D}}_X$ $(1\leq i\leq n)$ satisfying this equality, the differential operator $\tilde{P}=\sum_{i=1}^n\tilde{Q}_i\partial_{x_i}\in\widetilde{\mathcal{D}}_X$ satisfies the one (4.35). Consequently, our $\tilde{\beta}_f^{\mathrm{mero}}(s)\neq 0$ is the minimal polynomial $b(s)\in\mathbb{C}[s]$ satisfying the condition $b(s)f^s\in\sum_{i=1}^n\widetilde{\mathcal{D}}_Xf_if^s$. Since we have

$$v(f_if^s)=(d-w_i+d\cdot s)(f_if^s)\quad (1\leq i\leq n), \tag{4.41}$$

$\sum_{i=1}^n\widetilde{\mathcal{D}}_Xf_if^s$ is a $\widetilde{\mathcal{D}}_X[s]$-submodule of $\widetilde{\mathcal{D}}_Xf^s\simeq\widetilde{\mathcal{D}}_X[s]f^s$. Set $h_i:=F_{x_i}G-FG_{x_i}=G^2f_i\in\mathcal{O}_X$ $(1\leq i\leq n)$. Then the $\widetilde{\mathcal{D}}_X$-module

$$\widetilde{\mathcal{R}}:=\frac{\widetilde{\mathcal{D}}_Xf^s}{\sum_{i=1}^n\widetilde{\mathcal{D}}_Xf_if^s}\simeq\frac{\widetilde{\mathcal{D}}_Xf^s}{\sum_{i=1}^n\widetilde{\mathcal{D}}_Xh_if^s} \tag{4.42}$$

has an action of s and the minimal polynomial of s on it is equal to $\tilde{\beta}_f^{\mathrm{mero}}(s)$.

Proposition 4.5. *Let $f=\frac{F}{G}$ be as above and assume moreover that $f^{-1}(0)=F^{-1}(0)\setminus G^{-1}(0)\subset X\setminus G^{-1}(0)$ is smooth. Then $\widetilde{\mathcal{R}}=0$ and $\tilde{b}_f^{\mathrm{mero}}(s)=s+1$.*

Proof. Let us consider the coherent $\mathcal{D}_X$-module

$$\mathcal{S}:=\frac{\mathcal{D}_X}{\sum_{i=1}^n \mathcal{D}_X h_i} \tag{4.43}$$

and its localization

$$\widetilde{\mathcal{S}}:=\mathcal{S}\left[\frac{1}{G}\right]\simeq \frac{\widetilde{\mathcal{D}}_X}{\sum_{i=1}^n \widetilde{\mathcal{D}}_X h_i}. \tag{4.44}$$

Note that $\widetilde{\mathcal{R}}$ is a quotient of $\widetilde{\mathcal{S}}$. Since $f=\frac{F}{G}$ is quasi-homogeneous of degree $d=d_1-d_2\neq 0$, there is no singular point of f in $X\setminus(F^{-1}(0)\cup G^{-1}(0))$. Then by the smoothness of $f^{-1}(0)=F^{-1}(0)\setminus G^{-1}(0)\subset X\setminus G^{-1}(0)$, we have

$$\mathrm{Sing}\ f=\{x\in X\setminus G^{-1}(0)\mid h_1(x)=h_2(x)=\cdots=h_n(x)=0\}=\emptyset. \tag{4.45}$$

This implies that the support of the coherent $\mathcal{D}_X$-module $\mathcal{S}$ is contained in $G^{-1}(0)\subset X$. Then, by Hilbert's nullstellensatz, we get $\widetilde{\mathcal{S}}=0$; and hence, $\widetilde{\mathcal{R}}=0$. ∎

By using [29, Th. 3.3 and Cor. 3.4], we can construct many examples of $f=\frac{F}{G}$ satisfying the conditions in Proposition 4.5 and having a monodromy eigenvalue $\neq 1$ at the point $x_0\in X$. By Theorem 1.4, for such f, we thus obtain

$$b_{f,m}^{\mathrm{mero}}(s)\neq \tilde{b}_f^{\mathrm{mero}}(s)=s+1 \quad (m\geq 2\dim X). \tag{4.46}$$

Namely, in the situation of Proposition 4.5 the reduced b-function $\tilde{b}_f^{\mathrm{mero}}(s)$ captures only the tiny (trivial) part $s+1$ of $b_{f,m}^{\mathrm{mero}}(s)$ for $m\geq 2\dim X$.

§5. Multiplier ideals for meromorphic functions

In this section, we define multiplier ideal sheaves for the meromorphic function $f=\frac{F}{G}$ and study their basic properties. Recall that multiplier ideals for holomorphic functions were introduced by Nadel [28]. For their precise properties, we refer to the excellent book [22] by Lazarsfeld. For the meromorphic function $f=\frac{F}{G}$, we define them as follows. Denote by $\mathrm{L}^1_{\mathrm{loc}}$, the set of locally integrable functions on X.

Definition 5.1. For a positive real number $\alpha>0$, we define an ideal $\mathcal{J}(X,f)_\alpha\subset\mathcal{O}_X$ of $\mathcal{O}_X$ by

$$\mathcal{J}(X,f)_\alpha:=\left\{h\in\mathcal{O}_X \;\middle|\; \frac{|h|^2}{|f|^{2\alpha}}=\frac{|h|^2\cdot|G|^{2\alpha}}{|F|^{2\alpha}}\in \mathrm{L}^1_{\mathrm{loc}}\right\} \tag{5.1}$$

and call it the multiplier ideal of f of order $\alpha>0$.

Let $\pi:Y\longrightarrow X$ be a resolution of singularities of the divisor $D=F^{-1}(0)\cup G^{-1}(0)\subset X$ as in §4. Here, we assume, moreover, that the meromorphic function $g=\frac{F\circ\pi}{G\circ\pi}$ has no point of indeterminacy on the whole Y. Such a resolution $\pi:Y\longrightarrow X$ always exists. Let div g be the divisor on Y defined by g. Then there exist two effective divisors $(\mathrm{div}\ g)_+$ and $(\mathrm{div}\ g)_-$ such that

$$\mathrm{div}\ g=(\mathrm{div}\ g)_+-(\mathrm{div}\ g)_-. \tag{5.2}$$

By our assumption, their supports, which we denote by $g^{-1}(0)$ and $g^{-1}(\infty)$, respectively, are disjoint from each other. By using such a resolution $\pi : Y \longrightarrow X$, we can easily see that for $\alpha' > \alpha > 0$, we have $\mathcal{J}(X,f)_{\alpha'} \subset \mathcal{J}(X,f)_{\alpha}$. Then, as in the case where f is holomorphic, we can define the jumping numbers of the multiplier ideals $\{\mathcal{J}(X,f)_{\alpha}\}_{\alpha>0}$. In the situation as above, we have $g^{-1}(\infty) \subset (G\circ\pi)^{-1}(0)$ but $g : Y \setminus (G\circ\pi)^{-1}(0) \longrightarrow \mathbb{C}$ can be extended to a holomorphic function $\tilde{g} : Y \setminus g^{-1}(\infty) \longrightarrow \mathbb{C}$. Let

$$\iota_{\tilde{g}} : Y \setminus g^{-1}(\infty) \longrightarrow Y \times \mathbb{C} \qquad (y \longmapsto (y, \tilde{g}(y))) \tag{5.3}$$

be the graph embedding defined by $\tilde{g}$. From now, we shall use the terminologies of mixed Hodge modules. For example, regarding the holonomic $\mathcal{D}_{X\times\mathbb{C}}$-module $\mathcal{M}$ as a mixed Hodge module on $X \times \mathbb{C}$, for $\alpha \in \mathbb{Q}$ and $p \in \mathbb{Z}$, we set

$$F_p V_\alpha \mathcal{M} = F_p\mathcal{M} \cap V_\alpha\mathcal{M}. \tag{5.4}$$

We denote the normal crossing divisor $(G\circ\pi)^{-1}(0)$ in Y by E and consider the regular holonomic $\mathcal{D}_Y$-module $\mathcal{O}_Y(*E)$ as a mixed Hodge module. Then its Hodge filtration $\{F_p\mathcal{O}_Y(*E)\}_{p\in\mathbb{Z}}$ satisfies the condition

$$F_p\mathcal{O}_Y(*E) \simeq 0 \quad (p<0), \qquad F_0\mathcal{O}_Y(*E) \neq 0. \tag{5.5}$$

Moreover, $F_0\mathcal{O}_Y(*E) \subset \mathcal{O}_Y(*E)$ is the subsheaf of $\mathcal{O}_Y(*E)$ consisting of meromorphic functions on Y having poles of order ≤ 1 only along $E \subset Y$. See Mustata–Popa [27, Chap. D] for the details about the Hodge filtration of $\mathcal{O}_Y(*E)$. We denote the restriction of $F_0\mathcal{O}_Y(*E) \simeq \mathcal{O}_Y(E)$ to $Y \setminus g^{-1}(\infty)$ simply by $\mathcal{E}$. Then the following proposition can be proved just by following the arguments in Budur–Saito [9] (see also [7, §§3 and 4] for more precise explanations). We set $Y^\circ := Y \setminus g^{-1}(\infty)$ and

$$\varpi := (\pi \times \mathrm{id}_{\mathbb{C}}) \circ \iota_{\tilde{g}} : Y^\circ \longrightarrow X \times \mathbb{C} \qquad (y \longmapsto (\pi(y), \tilde{g}(y))). \tag{5.6}$$

Let $K_{Y/X}$ be the relative canonical divisor of $\pi : Y \longrightarrow X$.

Proposition 5.2. *Let $\alpha > 0$ be a positive real number. Then, for $0 < \varepsilon \ll 1$, there exists an isomorphism*

$$F_1 V_{-\alpha}\mathcal{M} \simeq \varpi_*\left\{\mathcal{O}_{Y^\circ}(K_{Y/X}) \otimes \mathcal{E} \cap \mathcal{O}_{Y^\circ}(-\lfloor(\alpha-\varepsilon)(\mathrm{div}\ g)_+\rfloor)\right\}. \tag{5.7}$$

Now, let $\mathrm{pr}_X : X \times \mathbb{C} \longrightarrow X$ be the projection. Then there exists an injective homomorphism of sheaves

$$\gamma : \mathcal{O}_X \longrightarrow (\mathrm{pr}_X)_*\mathcal{M} \qquad (h \longmapsto h \cdot \sigma_0), \tag{5.8}$$

by which we regard $\mathcal{O}_X$ as a subsheaf of $(\mathrm{pr}_X)_*\mathcal{M}$. By Proposition 5.2 and the local integrability condition in Definition 5.1, we obtain the following analog for meromorphic functions of Budur–Saito [9, Th. 0.1] (see also [7] for details). Note that by Proposition 5.2, for any $\alpha > 0$, we have

$$\mathcal{O}_X \cap (\mathrm{pr}_X)_* F_1 V_{-\alpha}\mathcal{M} = \mathcal{O}_X \cap (\mathrm{pr}_X)_* V_{-\alpha}\mathcal{M}. \tag{5.9}$$

Theorem 5.3. *Let $\alpha > 0$ be a positive real number. Then we have*

$$\mathcal{J}(X,f)_\alpha = \mathcal{O}_X \cap (\mathrm{pr}_X)_* V_{<-\alpha}\mathcal{M} = \{h \in \mathcal{O}_X \mid \mathrm{ord}_{\{t=0\}}(h\cdot\sigma_0) > \alpha - 1\}. \tag{5.10}$$

By this theorem and

$$\mathrm{ord}_{\{t=0\}}(h\cdot\sigma_0)\subset\bigcup_{i=0,1,2,\ldots}\{\mathrm{ord}_{\{t=0\}}(\sigma_0)+i\}\qquad(h\in\mathcal{O}_X)\tag{5.11}$$

(see the proof of Lemma 3.3), we immediately obtain the following generalization of the celebrated theorem of Ein–Lazarsfeld–Smith–Varolin [13] to meromorphic functions.

Corollary 5.4. *The jumping numbers of the multiplier ideals* $\{\mathcal{J}(X,f)_\alpha\}_{\alpha>0}$ *are contained in the set*

$$\bigcup_{i=0,1,2,\ldots}\left\{-(b^{\mathrm{mero}}_{f,0})^{-1}(0)+i\right\}\subset\mathbb{Q}_{>0}.\tag{5.12}$$

Moreover, the minimal jumping number $\alpha>0$ *is equal to the negative of the largest root of* $b^{\mathrm{mero}}_{f,0}(s)$.

Acknowledgments. The author thanks Professors Núñez-Betancourt, Kakehi, Oaku, Sabbah, and Schapira for some useful discussions during the preparation of this paper. He is also grateful to the anonymous referee whose comments and suggestions were very helpful.

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https://doi.org/10.1017/nmj.2023.10 Published online by Cambridge University Press