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STRICHARTZ ESTIMATES FOR THE WAVE EQUATION INSIDE CYLINDRICAL CONVEX DOMAINS

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Abstract

We establish local-in-time Strichartz estimates for solutions of the model case Dirichlet wave equation inside cylindrical convex domains $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega \neq \emptyset$. The key ingredients to prove Strichartz estimates are dispersive estimates, energy estimates, interpolation and TT^* arguments. Strichartz estimates for waves inside an arbitrary domain Ω have been proved by Blair, Smith and Sogge ['Strichartz estimates for the wave equation on manifolds with boundary', *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009), 1817–1829]. We provide a detailed proof of the usual Strichartz estimates from dispersive estimates inside cylindrical convex domains for a certain range of the wave admissibility.

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1. Introduction

1.1. The cylindrical model problem. Let $\Omega = \{x \geq 0, (y,z) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega = \{x = 0\}$ and let $\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2$. We consider solutions of the linear Dirichlet wave equation inside Ω:

$$(\partial_t^2 - \Delta)u = 0, \quad u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \quad u_{|x=0} = 0. \tag{1.1}$$

The Riemannian manifold (Ω, Δ) with $\Delta = \partial_x^2 + (1+x)\partial_y^2 + \partial_z^2$ can be locally seen as a cylindrical domain in $\mathbb{R}^3$ by taking cylindrical coordinates (r, θ, z), where we set $r = 1 - x/2, \theta = y$ and $z = z$. The main goal of this work is to prove the Strichartz estimates inside cylindrical convex domains for the solution u to (1.1).

1.2. Some known results. Let us recall a few results about Strichartz estimates (see [10, Section 1]). Let (Ω, g) be a Riemannian manifold without boundary of dimension $d \geq 2$. Local-in-time Strichartz estimates state that

$$\|u\|_{L^q((-T,T);L^r(\Omega))} \leq C_T(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)}), \tag{1.2}$$

where $\dot{H}^\beta$ denotes the homogeneous Sobolev space over Ω of order β, $2 \le q, r \le \infty$ and

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta, \quad \frac{1}{q} \le \frac{d-1}{2}\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

Here $u = u(t, x)$ is a solution to the wave equation

$$(\partial_t^2 - \Delta_g)u = 0 \text{ in } (-T, T) \times \Omega, \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x),$$

where Δ_g denotes the Laplace–Beltrami operator on (Ω, g). The estimates (1.2) hold on $\Omega = \mathbb{R}^d$ and $g_{ij} = \delta_{ij}$.

Blair *et al.* [4] proved the Strichartz estimates for the wave equation on a (compact or noncompact) Riemannian manifold with boundary. They proved that the Strichartz estimates (1.2) hold if Ω is a compact manifold with boundary and (q, r, β) is a triple satisfying

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta \quad \text{for} \begin{cases} \dfrac{3}{q} + \dfrac{d-1}{r} \le \dfrac{d-1}{2}, & d \le 4, \\ \dfrac{1}{q} + \dfrac{1}{r} \le \dfrac{1}{2}, & d \ge 4. \end{cases}$$

Recently in [10], Ivanovici *et al.* deduced local-in-time Strichartz estimates (1.2) from the optimal dispersive estimates inside strictly convex domains of dimension $d \ge 2$ for a triple (d, q, β) satisfying

$$\frac{1}{q} \le \Big(\frac{d-1}{2} - \frac{1}{4}\Big)\Big(\frac{1}{2} - \frac{1}{r}\Big) \quad \text{and} \quad \beta = d\Big(\frac{1}{2} - \frac{1}{r}\Big) - \frac{1}{q}.$$

For $d \ge 3$, this improves the range of indices for which sharp Strichartz estimates hold compared to the result by Blair *et al.* [4]. However, the results in [4] apply to any domains or manifolds with boundary. The latest results in [11] on Strichartz estimates inside the Friedlander model domain have been obtained for pairs (q, r) such that

$$\frac{1}{q} \le \Big(\frac{1}{2} - \frac{1}{9}\Big)\Big(\frac{1}{2} - \frac{1}{r}\Big).$$

This result improves on the known results for strictly convex domains for $d = 2$, while [10] only gives a loss of $\frac{1}{4}$.

Let us also recall that dispersive estimates for the wave equation in $\mathbb{R}^d$ follow from the representation of the solution as a sum of Fourier integral operators (see [1, 5, 8]). They read as follows:

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \le Ch^{-d} \min\Big\{1, \Big(\frac{h}{|t|}\Big)^{(d-1)/2}\Big\}, \tag{1.3}$$

where $\Delta_{\mathbb{R}^d}$ is the Laplace operator in $\mathbb{R}^d$. Here and in the following, the function χ belongs to $C_0^\infty(]0, \infty[)$ and is equal to 1 on $[1, 2]$ and $D_t = (1/i)\partial_t$. Inside strictly convex domains Ω_D of dimensions $d \ge 2$, the optimal (local-in-time) dispersive estimates for the wave equation have been established by Ivanovici *et al.* [10]. More precisely, they

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have proved that

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_D}}\|_{L^1(\Omega_D)\to L^\infty(\Omega_D)} \leq Ch^{-d}\min\Big\{1,\Big(\frac{h}{|t|}\Big)^{(d-1)/2-1/4}\Big\}, \tag{1.4}$$

where Δ_D is the Laplace operator on Ω_D. Due to the formation of caustics in arbitrarily small times, (1.4) induces a loss of $\frac{1}{4}$ powers of the $(h/|t|)$ factor compared to (1.3). The local-in-time dispersive estimates for the wave equation inside cylindrical convex domains in dimension 3 have been derived in [13, 14] as follows:

$$\|\chi(hD_t)\mathcal{G}_a(t,x,y,z)\|_{L^1(\Omega)\to L^\infty(\Omega)} \leq Ch^{-3}\min\Big\{1,\Big(\frac{h}{|t|}\Big)^{3/4}\Big\},$$

where $\mathcal{G}_a$ is the Green function for (1.1).

2. Main result

We now state our main result concerning the Strichartz estimates inside cylindrical convex domains in dimension 3.

THEOREM 2.1. *Let (Ω, Δ) be defined as before. Let u be a solution of the wave equation on Ω:*

$$(\partial_t^2 - \Delta)u = 0 \text{ in } \Omega, \quad u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \quad u_{|x=0} = 0.$$

Then for all T, there exists C_T such that

$$\|u\|_{L^q((0,T);L^r(\Omega))} \leq C_T(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)}),$$

with

$$\frac{1}{q} \leq \frac{3}{4}\Big(\frac{1}{2}-\frac{1}{r}\Big) \quad \text{and} \quad \beta = 3\Big(\frac{1}{2}-\frac{1}{r}\Big)-\frac{1}{q}.$$

To prove Theorem 2.1, we first prove the frequency-localised Strichartz estimates by utilising the frequency-localised dispersive estimates, interpolation and TT^* arguments. We then apply the Littlewood–Paley square function estimates (see [2, 3, 12]) to get the Strichartz estimates (Theorem 2.1) in the context of cylindrical domains. For $d = 3$, Theorem 2.1 improves the range of indices for which the sharp Strichartz estimates hold. However, our result is restricted to cylindrical domains, while [4] applies to any domain.

3. Strichartz estimates for the model problem

Let us recall some notation. For any $I \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, we define the mixed space-time norms

$$\|u\|_{L^q(I;L^r(\Omega))} := \Big(\int_I \|u(t,.)\|^q_{L^r(\Omega)}\,dt\Big)^{1/q} \quad \text{if } 1 \leq q < \infty,$$

$$\|u\|_{L^\infty(I;L^r(\Omega))} := \operatorname*{ess\,sup}_{t\in I} \|u(t,.)\|_{L^r(\Omega)}.$$

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3.1. Frequency-localised Strichartz estimates. In this section, we prove Theorem 3.1. The classical strategy is as follows. We begin by interpolating between the energy estimates and dispersive estimates. This yields a new estimate, which we further manipulate via a classical L^p inequality to establish (3.8). This last step imposes conditions on the space-time exponent pair (q, r); these are precisely the wave admissibility criteria. The classical inequalities used are the Young, Hölder and Hardy–Littlewood–Sobolev inequalities.

We first recall the Littlewood–Paley decomposition and some links with Sobolev spaces [1]. Let $\chi \in C_0^\infty(\mathbb{R}^*)$ and equal to 1 on $[\frac{1}{2}, 2]$ such that

$$\sum_{j\in\mathbb{Z}} \chi(2^{-j}\lambda) = 1, \quad \lambda > 0.$$

We define the associated Littlewood–Paley frequency cutoffs $\chi(2^{-j}\sqrt{-\Delta})$ using the spectral theorem for Δ and we have

$$\sum_{j\in\mathbb{Z}} \chi(2^{-j}\sqrt{-\Delta}) = \mathrm{Id} : L^2(\Omega) \longrightarrow L^2(\Omega).$$

This decomposition takes a single function and writes it as a superposition of a countably infinite family of functions χ each one having a frequency of magnitude $\sim 2^j$ for $j \geq 1$. A norm of the homogeneous Sobolev space $\dot{H}^\beta$ is defined as follows: for all $\beta \geq 0$,

$$\|u\|_{\dot{H}^\beta} := \Big(\sum_{j\in\mathbb{Z}} 2^{2j\beta}\|\chi(2^{-j}D_t)u\|_{L^2}^2\Big)^{1/2}.$$

With this decomposition, the Littlewood–Paley square function estimate (see [2, 3, 12]) reads as follows: for $f \in L^r(\Omega)$ and for all $r \in [2, \infty[$,

$$\|f\|_{L^r(\Omega)} \leq C_r \Big\|\Big(\sum_{j\in\mathbb{Z}} |\chi(2^{-j}\sqrt{-\Delta})f|^2\Big)^{1/2}\Big\|_{L^r(\Omega)}. \tag{3.1}$$

The proof follows from the classical Stein argument involving Rademacher functions and an appropriate Mikhlin–Hörmander multiplier theorem.

We define the frequency localisation v_j of u by $v_j = \chi(2^{-j}\sqrt{-\Delta})u$. Hence, $u = \sum_{j\geq 0} v_j$. Let $h = 2^{-j}$. We deduce from the dispersive estimates inside cylindrical convex domains established in [13, 14] the frequency-localised dispersive estimates for the solution $v_j = \chi(hD_t)u$ of the (frequency-localised) wave equation

$$(\partial_t^2 - \Delta)v_j = 0 \text{ in } \Omega, \quad v_{j|t=0} = \chi(hD_t)u_0, \quad \partial_t v_{j|t=0} = \chi(hD_t)u_1, \quad v_{j|\partial\Omega} = 0, \tag{3.2}$$

which read as follows:

$$\|\dot{\mathcal{U}}(t)\chi(hD_t)u_0\|_{L^\infty} \lesssim h^{-3} \min\Big\{1, \Big(\frac{h}{t}\Big)^{3/4}\Big\}\|\chi(hD_t)\, u_0\|_{L^1}, \tag{3.3}$$

$$\|\mathcal{U}(t)\chi(hD_t)u_1\|_{L^\infty} \lesssim h^{-2} \min\Big\{1, \Big(\frac{h}{t}\Big)^{3/4}\Big\}\|\chi(hD_t)\, u_1\|_{L^1},$$

where we use the notation

$$\mathcal{U}(t) := \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \quad \text{and} \quad \dot{\mathcal{U}}(t) := \cos(t\sqrt{-\Delta}).$$

These estimates yield the following Strichartz estimates.

THEOREM 3.1 (Frequency-localised Strichartz estimates). *Let* (Ω, Δ) *be defined as before. Let* v_j *be a solution of the (frequency-localised) wave equation (3.2). Then for all T, there exists* C_T *such that*

$$h^{\beta}\|\dot{\mathcal{U}}(t)\chi(hD_t)u_0\|_{L^q_t(L^r_x)} \lesssim \|\chi(hD_t)u_0\|_{L^2}, \tag{3.4}$$

$$h^{\beta-1}\|\mathcal{U}(t)\chi(hD_t)u_1\|_{L^q_t(L^r_x)} \lesssim \|\chi(hD_t)u_1\|_{L^2}, \tag{3.5}$$

with

$$q \in]2,\infty], \quad r \in [2,\infty], \quad \frac{1}{q} \le \alpha_3\Big(\frac{1}{2}-\frac{1}{r}\Big), \quad \alpha_3 = \frac{3}{4}, \quad \beta = 3\Big(\frac{1}{2}-\frac{1}{r}\Big)-\frac{1}{q}.$$

REMARK 3.2. If $1/q = \alpha_3(1/2 - 1/r)$, then $\beta = (3-\alpha_3)(1/2-1/r)$.

PROOF OF THEOREM 3.1. We prove only (3.4) since (3.5) follows analogously. We have the frequency-localised dispersive estimates in Ω in (3.3) for $|t| \ge h$,

$$\|\dot{\mathcal{U}}(t)\chi(hD_t)u_0\|_{L^\infty} \lesssim h^{-3}\Big(\frac{h}{t}\Big)^{\alpha_3}\|\chi(hD_t)u_0\|_{L^1}, \tag{3.6}$$

and the energy estimates,

$$\|\dot{\mathcal{U}}(t)\chi(hD_t)u_0\|_{L^2} \lesssim \|\chi(hD_t)u_0\|_{L^2}. \tag{3.7}$$

We apply the Riesz–Thorin interpolation theorem [9] to the operator $\dot{\mathcal{U}}(t)\chi(hD_t)$ for fixed time $t \in \mathbb{R}$. Interpolating between (3.6) and (3.7) with $\theta = 1 - 2/r$ yields

$$\|\dot{\mathcal{U}}(t)\chi(hD_t)u_0\|_{L^r} \lesssim h^{(-3+\alpha_3)(1-2/r)}t^{-\alpha_3(1-2/r)}\|\chi(hD_t)u_0\|_{L^{r'}}, \tag{3.8}$$

for $2 \le r \le \infty$, where r' denotes the exponent conjugate to r (that is, $1/r + 1/r' = 1$). Let T be the operator solution defined by

$$T : \phi_0 \in L^2 \longmapsto T\phi_0 = \dot{\mathcal{U}}(t)\chi(hD_t)\phi_0 \in L^q_tL^r_x.$$

Its adjoint is given by

$$T^* : \psi \in L^{q'}_tL^{r'}_x \longmapsto T^*\psi = \int \dot{\mathcal{U}}(t)\chi^*(hD_t)\psi(t)\,dt \in L^2.$$

Moreover,

$$T^*T : \psi \in L^{q'}_tL^{r'}_x \longmapsto T^*T\psi = \int \dot{\mathcal{U}}(t-s)\chi^*(hD_t)\chi(hD_t)\psi(s)\,ds \in L^q_tL^r_x.$$

By the TT^* argument in [7], it is sufficient to prove

$$\|T^*T\psi\|_{L^q_tL^r_x} \lesssim h^{-2\beta}\|\psi\|_{L^{q'}_tL^{r'}_x}.$$

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We have

$$\|T^*T\psi\|_{L^q_tL^r_x} = \Big\| \int \dot{\mathcal{U}}(t-s)\chi^*(hD_t)\chi(hD_t)\psi(s)\,ds\Big\|_{L^q_tL^r_x},$$
$$\lesssim h^{-2(3-\alpha_3)(1/2-1/r)}\Big\| \int |t-s|^{-2\alpha_3(1/2-1/r)}\|\psi\|_{L^{r'}_x}\,ds\Big\|_{L^q_t}. \qquad (3.9)$$

When $1/q < \alpha_3(1/2-1/r)$, we use Young's inequality which states that

$$\|K * u\|_{L^q} \le \|K\|_{L^{\tilde r}}\|u\|_{L^p} \quad \text{for } 1 \le p, q \le \infty, \qquad (3.10)$$

where $1 + 1/q = 1/\tilde r + 1/p$. We apply (3.10) with $\tilde r = q/2$, $p = q'$ and $1/q + 1/q' = 1$ to get the estimate

$$\Big\| \int_h^\infty |t-s|^{-2\alpha_3(1/2-1/r)}\|\psi\|_{L^{r'}_x}\,ds\Big\|_{L^q_t} \le \|\psi\|_{L^{q'}_tL^{r'}_x}\|t^{-2\alpha_3(1/2-1/r)}\|_{L^{q/2}_{|t|\ge h}}$$
$$\le h^{-2\alpha_3(1/2-1/r)+2/q}\|\psi\|_{L^{q'}_tL^{r'}_x}.$$

Since $1/q < \alpha_3(1/2-1/r)$,

$$\|t^{-2\alpha_3(1/2-2/r)}\|_{L^{q/2}_{|t|\ge h}} = \Big(\int_h^\infty t^{-2\alpha_3(1/2-2/r)q/2}\,dt\Big)^{2/q} \simeq h^{-2\alpha_3(1/2-1/r)+2/q}.$$

Then (3.9) becomes

$$\|T^*T\psi\|_{L^q_tL^r_x} \lesssim h^{-2(3-\alpha_3)(1/2-1/r)}\Big\| \int |t-s|^{-2\alpha_3(1/2-1/r)}\|\psi\|_{L^{r'}_x}\,ds\Big\|_{L^q_t},$$
$$\lesssim h^{-2[3(1/2-1/r)-\frac{1}{q}]}\|\psi\|_{L^{q'}_tL^{r'}_x} \lesssim h^{-2\beta}\|\psi\|_{L^{q'}_tL^{r'}_x}.$$

When $1/q = \alpha_3(1/2-1/r)$, we instead use the Hardy–Littlewood–Sobolev inequality (see [9, Theorem 4.5.3]) which says that for $K(t) = |t|^{-1/\gamma}$ and $1 < \gamma < \infty$,

$$\|K * u\|_{L^{\tilde r}(\mathbb{R})} \lesssim \|u\|_{L^{p'}(\mathbb{R})} \quad \text{for } \frac{1}{\gamma} = \frac{1}{p} + \frac{1}{\tilde r}. \qquad (3.11)$$

We apply (3.11) with $\tilde r = q$, $p = q$ and $1/\gamma = 2/q = 2\alpha_3(1/2-1/r)$ to show that $t^{-2/q}* : L^{q'} \to L^q$ is bounded for $q > 2$. Hence, from (3.9),

$$\|T^*T\psi\|_{L^q_tL^r_x} \lesssim h^{-2(3-\alpha_3)(1/2-1/r)}\|\psi\|_{L^{q'}_tL^{r'}_x} \lesssim h^{-2\beta}\|\psi\|_{L^{q'}_tL^{r'}_x}.$$

□

3.2. Homogeneous Strichartz estimates. We can restate Theorem 2.1 as follows.

THEOREM 3.3 (Theorem 2.1). *Let* (Ω, Δ) *be defined as before. Let u be a solution of the wave equation on* Ω*:*

$$(\partial_t^2 - \Delta)u = 0 \text{ in } \Omega, \quad u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \quad u_{|x=0} = 0. \qquad (3.12)$$

Then for all T, there exists C_T *such that*

$$\|u\|_{L^q((0,T);L^r(\Omega))} \le C_T(\|u_0\|_{\dot H^\beta(\Omega)} + \|u_1\|_{\dot H^{\beta-1}(\Omega)}),$$

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with

$$\frac{1}{q} \le \frac{3}{4}\Big(\frac{1}{2} - \frac{1}{r}\Big) \quad \textit{and} \quad \beta = 3\Big(\frac{1}{2} - \frac{1}{r}\Big) - \frac{1}{q}.$$

PROOF. Using the square function estimates (3.1),

$$\|u\|_{L_t^q L_x^r} \lesssim \Big(\sum_j \|v_j\|_{L_t^q L_x^r}^2\Big)^{1/2}.$$

Indeed,

$$\begin{aligned}\|u\|_{L^r(\Omega)} &\lesssim \Big\|\Big(\sum_{j\ge 0} |v_j|^2\Big)^{1/2}\Big\|_{L^r(\Omega)} = \Big\|\sum_{j\ge 0} |v_j|^2\Big\|_{L^{r/2}(\Omega)}^{1/2} \\ &\lesssim \Big\{\sum_{j\ge 0} \|v_j^2\|_{L^{r/2}(\Omega)}\Big\}^{1/2} = \Big\{\sum_{j\ge 0} \|v_j\|_{L^r(\Omega)}^2\Big\}^{1/2}.\end{aligned}$$

Hence,

$$\begin{aligned}\|u\|_{L_t^q L_x^r} &\lesssim \Big\|\Big\{\sum_{j\ge 0} \|v_j\|_{L_x^r}^2\Big\}^{1/2}\Big\|_{L_t^q} = \Big\{\Big\|\sum_{j\ge 0} \|v_j\|_{L_x^r}^2\Big\|_{L_t^{q/2}}\Big\}^{1/2}, \\ &\lesssim \Big\{\sum_{j\ge 0} \|\|\|v_j\|_{L_x^r}^2\|_{L_t^{q/2}}\Big\}^{1/2} = \Big\{\sum_{j\ge 0} \|v_j\|_{L_t^q L_x^r}^2\Big\}^{1/2}.\end{aligned}$$

The solution u to the wave equation (3.12) with localised initial data in frequency $1/h = 2^j$ is given by

$$u = \sum_j v_j \quad \text{where } v_j = \dot{\mathcal{U}}(t)\chi(2^{-j}D_t)u_0 + \mathcal{U}(t)\chi(2^{-j}D_t)u_1.$$

Therefore,

$$\begin{aligned}\|u\|_{L_t^q L_x^r} &\lesssim \Big(\sum_j \|\dot{\mathcal{U}}(t)\chi(2^{-j}D_t)u_0\|_{L_t^q L_x^r}^2 + \|\mathcal{U}(t)\chi(2^{-j}D_t)u_1\|_{L_t^q L_x^r}^2\Big)^{1/2}, \\ &\lesssim \Big(\sum_j 2^{2j\beta}\|\chi(2^{-j}D_t)u_0\|_{L^2}^2 + 2^{2j(\beta-1)}\|\chi(2^{-j}D_t)u_1\|_{L^2}^2\Big)^{1/2}, \\ &\lesssim \Big(\sum_j 2^{2j\beta}\|\chi(2^{-j}D_t)u_0\|_{L^2}^2\Big)^{1/2} + \Big(\sum_j 2^{2j(\beta-1)}\|\chi(2^{-j}D_t)u_1\|_{L^2}^2\Big)^{1/2}, \\ &\lesssim \|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)},\end{aligned}$$

where we used Minkowski's inequality in the third line. □

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4. Application

We can use the Strichartz estimates (Theorem 2.1) to obtain the well posedness of the following energy critical nonlinear wave equation in (Ω, Δ):

$$\begin{aligned} (\partial_t^2 - \Delta)u + u^5 = 0 \quad \text{in } \mathbb{R}_t \times \Omega, \\ u_{|t=0} = u_0, \quad \partial_t u_{|t=0} = u_1, \quad u_{|x=0} = 0. \end{aligned} \tag{4.1}$$

The solutions to (4.1) satisfy an energy conservation law:

$$E(u(t), \partial_t u(t)) = \int_\Omega \Big(\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6\Big)\, dx = E(u_0, u_1).$$

For initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, Theorem 2.1 allows the Strichartz triplet $q = 5, r = 10, \beta = 1$ and we get

$$\|u\|_{L^5((0,T);L^{10}(\Omega))} \le C_T(\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)}).$$

As a consequence, the critical nonlinear wave equation (4.1) is locally well posed in

$$X_T = C^0([0,T]; H_0^1(\Omega)) \cap L^5((0,T); L^{10}(\Omega)) \times C^0([0,T]; L^2(\Omega)).$$

Moreover, with the arguments in [6], we can extend local to global existence for arbitrary (finite energy) data.

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