HOMOMORPHISMS OF (0, 1)-LATTICES WITH A GIVEN SUBLATTICE AND QUOTIENT

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(Received 14 November, 1989)

1. Introduction. Let us recall the notions of full embedding and universality of categories we will be using throughout.

A *full embedding* is a functor F taking the objects of a source category A injectively to objects of a target category B and the hom-sets $\mathrm{Hom}_A(a,b)$ bijectively to the hom-sets $\mathrm{Hom}_B(F(a),F(b))$. If A is a subcategory of B and the corresponding inclusion functor is a full embedding then A is said to be a *full subcategory* of B. In this case we have $\mathrm{Hom}_A(a,b)=\mathrm{Hom}_B(a,b)$ for any a, b in A; that is to say, a full subcategory is completely determined, within a given category, by specifying the class of its objects. A category U is termed *universal* if an arbitrary category of algebras can be fully embedded in U.

If the one-object categories (monoids) can be fully embedded into a category V (that is, if an arbitrary monoid M is isomorphic to the endomorphism monoid of a suitable object of V) then V is said to be *monoid universal*.

Let us note in passing that in a universal (concrete) category it is impossible to draw any conclusions about the structure of an object from its monoid of endomorphisms. From this point of view universality is rather a negative property.

The reader wishing to learn more about full embeddings and universal categories is referred to the monograph [**15**].

We are going to study, from the universality point of view, some natural full subcategories of the category $\mathbf{B}$ of (0, 1)-lattices and (0, 1)-homomorphisms (the latter are supposed to preserve the universal bounds 0 and 1 of (0, 1)-lattices).

Universality of $\mathbf{B}$ was established in [**7**]. Recently the universal subvarieties of $\mathbf{B}$ were characterized as follows.

THEOREM 1.1 [**5**]. *For an arbitrary variety* $\mathbf{V}$ *of* (0, 1)*-lattices the following are equivalent*:

(1) $\mathbf{V}$ *is universal*,

(2) $\mathbf{V}$ *is monoid universal*,

(3) $\mathbf{V}$ *contains a non-trivial* (0, 1)*-lattice with no prime ideal*,

(4) $\mathbf{V}$ *contains a simple* (0, 1)*-lattice* L *with* $\mathrm{card}(L)>2$.

Consequently, a variety $\mathbf{V}$ of (0, 1)-lattices is not universal if and only if every non-trivial (0, 1)-lattice L in $\mathbf{V}$ has a prime ideal. ■

Besides varieties, the classes ${}_S\mathbf{B}$ consisting of all (0, 1)-lattices L containing as a (0, 1)-sublattice a fixed (0, 1)-lattice S, as well as the classes $\mathbf{B}_Q$ of all (0, 1)-lattices with a fixed quotient (0, 1)-lattice Q, seem to be fairly natural and worth studying. The following results have been obtained.

THEOREM 1.2 [**1**]. *For every non-trivial lattice S the category* ${}_S\mathbf{B}$ *is universal.* ■

* The support of NSERC is gratefully acknowledged.

THEOREM 1.3 [13]. *For a finitely generated lattice Q, the category $\mathbf{B}_Q$ is universal if and only if Q has no prime ideal.* ■

In this paper we extend this line of study to the classes ${}_S\mathbf{B}_Q = {}_S\mathbf{B} \cap \mathbf{B}_Q$ of all $(0, 1)$-lattices with a given $(0, 1)$-sublattice S and quotient lattice Q. Our goal is to establish the following result.

THEOREM 1.4. *If S is finite or has a prime idea or if Q is finitely generated then ${}_S\mathbf{B}_Q$ is universal if and only if there exists a $(0, 1)$-homorphism from S to Q and no $(0, 1)$-homomorphism from Q to S.*

As is the case with Theorem 1.2 and 1.3, the necessity of the condition in Theorem 1.4 is nearly obvious: the existence of a $(0, 1)$-homomorphism $f: Q \to S$ implies that there is at most one object representing the trivial monoid (a *rigid* object).

The proof of Theorem 1.4 makes use of certain categories of hypergraphs and categories of partial lattices. All the necessary data on these structures will be given in the next section, preliminary to the actual proof carried out in the concluding section. The proof will be accomplished by a sequence of full embeddings yielding a composite full embedding of a special universal category of hypergraphs into ${}_S\mathbf{B}_Q$.

The resulting full embedding will pass through an intermediate category of the form $\mathrm{Biext}(S, Q, \varphi)$ where $\varphi: S \to Q$ is a $(0, 1)$-homomorphism, whose objects are all triples (ι, L, σ) where L is a $(0, 1)$-lattice, $\iota: S \to L$ is an injective $(0, 1)$-homomorphism, $\sigma: L \to Q$ is a surjective $(0, 1)$-homomorphism with $\varphi = \sigma \circ \iota$, and morphisms from (ι, L, σ) to (ι', L', σ') are all $(0, 1)$-homomorphisms $f: L \to L'$ satisfying $\iota' = f \circ \iota$ and $\sigma = \sigma' \circ f$. In such a way, as a by-product of our proof we obtain a sufficient condition for the universality of $\mathrm{Biext}(S, Q, \varphi)$.

2. Preliminaries. A *partial lattice* is a triple $\mathbf{P} = (P, J, M)$ where P is a poset, and J, M are sets of finite subsets of P such that

if $A \in J$ then A has a join in P,

if $A \in M$ then A has a meet in P,

if $x \leqslant y$ in P then $\{x, y\} \in J \cap M$.

A mapping $f: P \to P'$ is a *partial lattice homomorphism* from $\mathbf{P} = (P, J, M)$ into $\mathbf{P}' = (P', J', M')$ if for every $A \in J$ (or $A \in M$) we have $f(A) \in J'$ (or $f(A) \in M'$) and $f(\bigvee A) = \bigvee f(A)$ (or $f(\bigwedge A) = \bigwedge f(A)$), respectively. Every lattice L can be considered as a partial lattice for which $J = M$ is the set of all finite non-empty subsets of L.

A lattice L together with an inclusion $f: P \to L$ is called a *free completion* of a partial lattice $\mathbf{P}$ if f is a partial lattice homomorphism and for every partial lattice homomorphism $g: \mathbf{P} \to L'$ into a lattice L' there exists exactly one lattice homomorphism $g^{\#}: L \to L'$ with $g = g^{\#} \circ f$. (Then $g^{\#}$ is called the *free extension* of g.) The lattice L is denoted by $\mathbf{P}^{\#}$.

R. A. Dean has proved the following theorem.

THEOREM 2.1[4]. *Every partial lattice has a free completion which is uniquely determined up to an isomorphism.*

Let $\mathbf{P} = (P, J, M)$ be a partial lattice. A subset $A \subseteq P$ is called *J-ideal* (or *M-filter*) if the following hold:

if $x \in A$ and $y \in P$ with $y \leqslant x$ (or $y \geqslant x$) then $y \in A$,

if $B \subseteq A$ and $B \in J$ (or $B \in M$) then $\bigvee B \in A$ (or $\bigwedge B \in A$).

Denote by $I(\mathbf{P})$ the set of all J-ideals, $F(\mathbf{P})$ the set of all M-filters. Obviously, for every $x \in P$, $\{y \in P; y \leqslant x\}$ is a J-ideal, $\{y \in P; y \geqslant x\}$ is a M-filter; they are called *principal*. Ordered by inclusion, $I(\mathbf{P})$ and $F(\mathbf{P})$ are complete lattices. Denote by $I_0(\mathbf{P})$ (or $F_0(\mathbf{P})$) the sublattice of $I(\mathbf{P})$ (or $F(\mathbf{P})$) generated by all principal J-ideals (or M-filters). Define $r_{\mathbf{P}}: \mathbf{P}^{\#} \to I_0(\mathbf{P})$, $q_{\mathbf{P}}: \mathbf{P}^{\#} \to F_0(\mathbf{P})$ such that for $w \in \mathbf{P}^{\#}$, $r_{\mathbf{P}}(w) = \{x \in P; x \leqslant w \text{ in } \mathbf{P}^{\#}\}$, $q_{\mathbf{P}}(w) = \{x \in P; x \geqslant w \text{ in } \mathbf{P}^{\#}\}$.

Proposition 2.2 [**4**]. *The mappings $r_{\mathbf{P}}$, $q_{\mathbf{P}}$ are lattice homomorphisms.* ∎

In what follows we shall omit the index $\mathbf{P}$ in $r_{\mathbf{P}}$ and $q_{\mathbf{P}}$ if a misunderstanding cannot occur. The following theorem says, essentially, that the order in $\mathbf{P}^{\#}$ can be recursively reduced to that of $\mathbf{P}$.

Theorem 2.3 [**4**]. *If $a, b \in \mathbf{P}^{\#}$ for some partial lattice $\mathbf{P}$ then $a \leqslant b$ in $\mathbf{P}^{\#}$ if and only if one of the following conditions holds*:

(1) *$a = a_0 \vee a_1$ and $a_i \leqslant b$ for every $i \in 2 = \{0, 1\}$,*
(2) *$a = a_0 \wedge a_1$ and $a_i \leqslant b$ for some $i \in 2$,*
(3) *$b = b_0 \vee b_1$ and $b_i \geqslant a$ for some $i \in 2$,*
(4) *$b = b_0 \wedge b_1$ and $b_i \geqslant a$ for every $i \in 2$,*
(5) *$q_{\mathbf{P}}(a)$ and $r_{\mathbf{P}}(b)$ have an element of P in common.*

Consequently, every element $a \in \mathbf{P}^{\#} \setminus P$ is either meet or join irreducible. ∎

For a partial lattice $\mathbf{P} = (P, J, M)$ define *polynomials* over $\mathbf{P}$ and the *rank of a polynomial* as usual (see e.g. [**3**] or [**6**]):

(1) each element $x \in P$ is a polynomial over $\mathbf{P}$ and $\operatorname{rank}(x) = 1$,
(2) if p_1, p_2 are polynomials over $\mathbf{P}$ then $p_1 \vee p_2$, $p_1 \wedge p_2$ are polynomials over $\mathbf{P}$ and $\operatorname{rank}(p_1 \vee p_2) = \operatorname{rank}(p_1 \wedge p_2) = \operatorname{rank}(p_1) + \operatorname{rank}(p_2) + 1$.

For a polynomial p over $\mathbf{P}$ denote by $v(p)$ the element of $\mathbf{P}^{\#}$ corresponding to p. We say that p is a *minimal polynomial* if for every polynomial p' with $v(p') = v(p)$ we have $\operatorname{rank}(p) \leqslant \operatorname{rank}(p')$. Following H. Lakser [**14**] we obtain:

Lemma 2.4. *If $x \in \mathbf{P}^{\#}$ is meet-irreducible then there exist a minimal polynomial p over $\mathbf{P}$ with $v(p) = x$ and a finite family $\{p_i; i \in I\}$ of minimal polynomials over $\mathbf{P}$ such that*:

(a) *$v(p) = \bigvee\{v(p_i); i \in I\}$ in $\mathbf{P}^{\#}$;*
(b) *for every $i \in I$ either $v(p_i)$ is join-irreducible or* $\operatorname{rank}(p_i) = 1$;
(c) *for every $i \in I$, $v(p_i) \not\leqslant \bigvee\{v(p_j); j \in I \setminus \{i\}\}$ in $\mathbf{P}^{\#}$;*
(d) *for every $i \in I$, if* $\operatorname{rank}(p_i) > 1$ *then there exists no $y \in P$ with $v(p_i) \leqslant y \leqslant v(p)$ in $\mathbf{P}^{\#}$ and if $p_i = p_{i,1} \wedge p_{i,2}$ then $v(p_{i,j}) \not\leqslant v(p)$ for $j \in \{1, 2\}$.*

Proof. If x is meet-irreducible in $\mathbf{P}^{\#}$ and p is a minimal polynomial with $v(p) = x$ then $p = p_1 \wedge p_2$ implies that either $v(p_1) = x$ or $v(p_2) = x$; this contradicts the minimality of p; thus $p = p_1 \vee p_2$ or $\operatorname{rank}(p) = 1$. Hence we easily obtain that there exists a finite family of minimal polynomials $\{p_i; i \in I\}$ satisfying (a) and (b). We prove (c) and (d). Assume that (c) or (d) does not hold. We construct a polynomial p' such that $\operatorname{rank}(p') < \operatorname{rank}(p)$ and $v(p') = v(p)$; it contradicts the minimality of rank of p. If (c) or (d) does not hold, then there exists $i \in I$ such that either $\bigvee\{v(p_j); j \in I \setminus \{i\}\} \geqslant v(p_i)$ or $\operatorname{rank}(p_i) > 1$ and there exists $y \in P$ with $v(p_i) \leqslant y \leqslant v(p)$ or $p_i = p_{i,1} \wedge p_{i,2}$ and $v(p_{i,j}) \leqslant v(p)$ for $j \in \{1, 2\}$. Let r be a "join" of $\{p_j; j \in I \setminus \{i\}\}$. In the first case we set $p' = r$, in the second case $p' = r \vee y$, in the last case $p' = r \vee p_{i,j}$. Clearly, $\operatorname{rank}(p') < \operatorname{rank}(p)$, $v(p') = v(p)$. ∎

The family $\{p_i; i \in I\}$ from Lemma 2.4 is called a *normal decomposition of* x. A dual statement holds for join-irreducible elements. The following theorem generalizes Whitman's result on semidistributivity of free lattices.

THEOREM 2.5. *If* $u = x \vee y = x \vee z > x \vee (y \wedge z)$ *in* $\mathbf{P}^{\#}$ *for a partial lattice* $\mathbf{P}$ *and if* $\{p_i; i \in I\}$ *is a normal decomposition of* u *then there exists* $i \in I$ *such that* $\operatorname{rank}(p_i) = 1$ *and* $v(p_i) \not\leqslant x \vee (y \wedge z)$.

Proof. Assume that $\{p_i; i \in I\}$ is a normal decomposition of u. We first consider the case that $\operatorname{rank}(p_i) > 1$ and $v(p_i) \leqslant x \vee y$. According to Theorem 2.3. either $v(p_i) \leqslant x$ or $v(p_i) \leqslant y$, because an application of (2) or (5) in Theorem 2.3 contradicts (d) of Lemma 2.4. Analogously we obtain that either $v(p_i) \leqslant x$ or $v(p_i) \leqslant z$ and so $v(p_i) \leqslant x \vee (y \wedge z)$. If for every p_i, $i \in I$ we have $v(p_i) \leqslant x \vee (y \wedge z)$ then $u = \bigvee\{v(p_i); i \in I\} \leqslant x \vee (y \wedge z) \leqslant u$. The proof is complete. ■

For a lattice L, we say a set $A \subseteq L$ *has property* (n), where $n \geqslant 2$ is a natural number, if the following holds: there exist $x, y \in L$ such that for every $B \subseteq A$ with $|B| = n$ we have $\bigvee B = x$, $\bigwedge B = y$. If A has the property (n) then denote $\sup(A)$ by x, $\inf(A)$ by y.

COROLLARY 2.6. *Let* $\mathbf{P}$ *be a partial lattice. If* $A \subseteq \mathbf{P}^{\#}$ *is infinite with property* (n), *then for a normal decomposition* $\{p_i; i \in I\}$ *of* $\sup(A)$ *there exists* $i \in I$ *with* $\operatorname{rank}(p_i) = 1$ *and* $a \not\geqslant v(p_i)$ *for every* $a \in A$. *On the other hand for every* $i \in I$ *with* $\operatorname{rank}(p_i) > 1$ *we have* $v(p_i) \leqslant a$ *for every* $a \in A$.

Proof. Let $\{p_i; i \in I\}$ be a normal decomposition of $\sup(A)$ and assume that for every $i \in I$ either $\operatorname{rank}(p_i) > 1$ or $v(p_i) \geqslant a$ for every $a \in A$. Let $a_0, a_1, \ldots, a_n$ be distinct elements of A. For every $j, k \in n+1$, $j \neq k$ set $b = \bigvee\{a_i; i \in n+1, i \neq j, k\}$, and by property (n) and Theorem 2.5 we obtain $b = (b \vee a_j) \wedge (b \vee a_k) = \sup(A)$. Now by induction we prove that $a_i = \sup(A)$ for every $i \in n+1$—a contradiction. ■

The dual statements of Theorem 2.5 and Corollary 2.6 are also true.

Denote by $H(n)$ for $n \geqslant 2$ the category of all *n-hypergraphs* and their *compatible mappings*: objects are pairs (X, E) where X is a set and E is a set of subsets of X with cardinality n; a mapping $f: X \to Y$ is a compatible mapping from (X, E) into (Y, F) if for every $e \in E$ we have $f(e) \in F$. For two cardinals $m \geqslant n \geqslant 2$, n finite, denote by $H(n, m)$ the full subcategory of $H(n)$ formed by all n-hypergraphs (X, E) without *isolated elements* (that is, for every $x \in X$ there exists $e \in E$ with $x \in e$) such that for every $e \in E$ there exists a set B with $e \subseteq B \subseteq X$, $|B| = m$ and every subset $C \subseteq B$ with $|C| = n$ belongs to E.

We first strengthen the results from **[13]** and **[9]**, **[10]**.

PROPOSITION 2.7. *For all cardinals* $m \geqslant n \geqslant 2$, n *finite, the category* $H(n, m)$ *is universal.*

Proof. For an arbitrary cardinal $m > 2$, the universality of $H(2, m)$ has been proved in **[9]** and **[10]**. For arbitrary cardinals $m \geqslant n$, n finite, $n > 2$, define a functor $\Omega: H(2, m) \to H(n, m)$ as follows: For $(X, E) \in H(2, m)$ define $\Omega(X, E) = (X, E')$ where

$$E' = \{B \subseteq X;\ |B| = n, \text{ there exists a set } C \text{ with } B \subseteq C \subseteq X,\ |C| = m \text{ and } \{c, b\} \in E \text{ for every distinct } c, b \in C\};$$

for a compatible mapping f we set $\Omega(f) = f$. It is easy to verify that Ω is a full embedding. ■

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If we combine Proposition 2.7 with results of **[8]** (see also **[15]**), we immediately obtain

COROLLARY 2.8. *Let $m \geqslant n \geqslant 2$ be cardinals, n finite. For every set I there exist a system $R(I) = \{(X_i, E_i) : i \in I\}$ of rigid hypergraphs in $H(n,m)$ and a universal full subcategory U of $H(n,m)$ such that there are no compatible mappings between distinct members of $R(I)$ or between members of $R(I)$ and objects of U.* ■

For a fixed system $\mathbf{R} = \{(X_i, E_i); i \in I\}$ of hypergraphs in $H(n,m)$ and for every n-hypergraph $(X, E) \in H(n,m)$, denote by $\mathbf{R}(X,E)$ the n-hypergraph $(X_{\mathbf{R}}, E_{\mathbf{R}})$, where $X_{\mathbf{R}}$ is a disjoint union of X and all X_i with $i \in I$, and $E_{\mathbf{R}}$ is a disjoint union of corresponding sets E and all E_i with $i \in I$.

For an n-hypergraph (X, E) define a partial lattice $\mathrm{PL}(X,E) = (P, J, M)$ where $P = X \cup \{0, 1\}$ (we assume that $0, 1 \notin X$), where 0 is the smallest element of P, 1 is the largest element of P, the elements of X are incomparable and

$$J = M = E \cup \{\{p, i\}; p \in P, i \in \{0,1\}\} \cup \{0, 1\}.$$

Let $\Lambda(X,E)$ be a free completion of $\mathrm{PL}(X,E)$. For a compatible mapping $f:(X,E) \to (X', E')$ denote by $\mathrm{PL}(f)$ the partial lattice homomorphism from $\mathrm{PL}(X,E)$ into $\Lambda(X',E')$ such that $\mathrm{PL}(f)(x) = f(x)$ for every $x \in X$, $\mathrm{PL}(f)(0) = 0$, $\mathrm{PL}(f)(1) = 1$. It is easy to verify that $\mathrm{PL}(f)$ is a partial lattice homomorphism; denote by $\Lambda(f)$ the free extension of $\mathrm{PL}(f)$. Then Λ is a functor and the following holds.

PROPOSITION 2.9 **[13]**. *If n is finite and $m \geqslant 3n - 3$ then Λ is an embedding of $H(n,m)$ into $\mathbf{L}$ such that any non-constant lattice homomorphism $f: \Lambda(X,E) \to \Lambda(X',E')$ with (X,E), $(X',E') \in H(n,m)$ has the form $f = \Lambda g$ for a compatible mapping $g:(X,E) \to (X',E')$ and moreover, f preserves 0 and 1.* ■

We state several properties of Λ. For an independent set Z (that is, no $A \in E$ satisfies $A \subseteq Z$) in $(X,E) \in H(n,m)$ write

$$F_Z = \{u \in \Lambda(X,E); \text{ there exist finite sets } Z' \subseteq Z,\ X' \subseteq \Lambda(X,E) \text{ such that for every } x \in X' \text{ there exists } y \in X \setminus Z \text{ with } x > y \text{ and } u \geqslant \bigwedge(Z' \cup X')\}.$$

Since X is a set of doubly irreducible elements, $\Lambda(X,E) \setminus (X \setminus Z)$ is a (0, 1)-sublattice of $\Lambda(X,E)$.

PROPOSITION 2.10 **[13]**. *If n is finite and $m \geqslant n \geqslant 2$ then for every $(X,E) \in H(n,m)$ we have*

(a) *if $x \vee y = x \vee z < 1$ (or $x \wedge y = x \wedge z > 0$) in $\Lambda(X,E)$ then $x \vee y = x \vee (y \wedge z)$ (or $x \wedge y = x \wedge (y \vee z)$);*

(b) *for every independent set Z of (X,E), F_Z is a prime filter of $\Lambda(X,E) \setminus (X \setminus Z)$;*

(c) *if $A \subseteq \Lambda(X,E)$ is finite and $\bigvee A = 1$ (or $\bigwedge A = 0$) then there exists $B \in E$ such that for every $b \in B$ there exists $a \in A$ with $a \geqslant b$ (or $b \geqslant a$).* ■

3. Universality of biextensions. Let $\varphi = S \to Q$ be a (0, 1)-homorphism under certain conditions on (0, 1)-lattices S and Q, for any finite $n > 1$ and any regular cardinal $m > \max\{\aleph_0, |S|, |Q|\}$ we construct a full embedding of $H(n,m)$ into ${}_S\mathbf{B}_Q$ or $\mathrm{Biext}(S, Q, \varphi)$.

Write $I = (S \setminus \{0,1\}) \cup (Q \setminus \mathrm{Im}(\varphi)) \cup \{0,1\}$. Select $l_0 \in I$ and set $I' = I \setminus \{l_0\}$. For $i \in I$ define $l(i), u(i)$ as follows: if $i \in S \setminus \{0,1\}$ then $l(i) = (\varphi(i), 0)$, $u(i) = i$; otherwise $l(i) =$

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$(i, 0)$, $u(i) = (i, 1)$. By Corollary 2.8 there exists a family of rigid n-hypergraphs $\mathbf{R} = \{(X_i, E_i); i \in I'\}$ in $H(n, m)$ and a universal full subcategory U of $H(n, m)$ such that there exist no compatible mappings between distinct members of $\mathbf{R}$ or between members of $\mathbf{R}$ and objects in U. For $(X, E) \in H(n, m)$ define two partial lattices $\mathbf{P}(X, E) = (P(X, E), J_0, M_0)$ and $\mathbf{Q}(X, E) = (Q(X, E), J_1, M_1)$ (for simplicity we set $(X_{i_0}, E_{i_0}) = (X, E)$) such that: $P(X, E)$ is a poset on the set $(Q \times \{0, 1\}) \cup (S \setminus \{0, 1\}) \cup X_{\mathbf{R}}$ where the union is disjoint and the ordering is the smallest one satisfying

(a) $x \leqslant y$ for $x, y \in S$ whenever $x \leqslant y$ in S;

(b) $(x, i) \leqslant (y, j)$ for $x, y \in Q$, $i, j \in \{0, 1\}$ whenever $x \leqslant y$ in Q and $i \leqslant j$;

(c) $(\varphi(x), 0) \leqslant x \leqslant (\varphi(x), 1)$ for every $x \in S \setminus \{0, 1\}$;

(d) for every $i \in I$, $y \in X_i$ we have $l(i) \leqslant y \leqslant u(i)$.

Note that $l(i)$, $u(i) \in P(X, E)$ for every $i \in I$, and $Q(X, E)$ is the subposet of $P(X, E)$ on $\{(0, 0), (1, 1)\} \cup (S \setminus \{0, 1\}) \cup X_{\mathbf{R}}$.

In addition to all comparable pairs of $P(X, E)$ let J_0 consist of all members of $E_{\mathbf{R}}$ and all pairs $\{x, y\} \in P(X, E)$ such that either both x and y are in $Q \times \{0\}$, or both x and y are in S, or at least one of x, y is in $Q \times \{1\}$. Let M_0 consist of all comparable pairs of $P(X, E)$, of all members of $E_{\mathbf{R}}$, and all pairs $\{x, y\} \in P(X, E)$ such that either both x and y are in $Q \times \{1\}$, or both x and y are in S, or at least one of x, y is in $Q \times \{0\}$. Set

$$J_1 = \{A \in J_0; A \subseteq Q(X, E)\},\ M_1 = \{Ax \in M_0;\ A \subseteq Q(X, E)\}.$$

Finally, define mappings:

$\nu_{(X,E)}: P(X, E) \to Q(X, E)$ by $\nu_{(X,E)}(x) = x$ for every $x \in Q(X, E)$, $\nu_{(X,E)}(x, i) = (i, i)$ for every $x \in Q$, $i \in \{0, 1\}$.

$\iota_{(X,E)}: S \to P(X, E)$ by $\iota_{(X,E)}(x) = x$ for every $x \in S \setminus \{0, 1\}$, $\iota_{(X,E)}(0) = (0, 0)$, $\iota_{(X,E)}(1) = (1, 1)$,

$\sigma_{(X,E)}: P(X, E) \to Q$ by $\sigma_{(X,E)}(x, i) = x$ for every $x \in Q$, $i \in \{0, 1\}$, $\sigma_{(X,E)}(x) = \varphi(x)$ for every $x \in S \setminus \{0, 1\}$, $\sigma_{(X,E)}(x) = y$ for every $x \in X_i$, $i \in I$ where $y \in Q$ is determined by the equation $l(i) = (y, 0)$.

Then we have:

LEMMA 3.1. *For every $(X, E) \in H(n, m)$, $\mathbf{P}(X, E)$ and $\mathbf{Q}(X, E)$ are partial lattices and $\nu: \mathbf{P}(X, E) \to \mathbf{Q}(X, E)$, $\iota: S \to \mathbf{P}(X, E)$, $\sigma: \mathbf{P}(X, E) \to Q$ are partial lattice homomorphisms.*

Proof. By a direct inspection. ■

Write $\Phi(X, E) = \mathbf{P}(X, E)^{\#}$, $\Psi(X, E) = \mathbf{Q}(X, E)^{\#}$, and let $\nu^{\#}_{(X,E)}: \Phi(X, E) \to \Psi(X, E)$, $\iota^{\#}_{(X,E)}: S \to \Phi(X, E)$, $\sigma^{\#}_{(X,E)}: \Phi(X, E) \to Q$ be the free extensions of $\nu_{(X,E)}$, $\iota_{(X,E)}$, and $\sigma_{(X,E)}$. In the following, if misunderstanding cannot occur, we will omit the index (X, E). We have:

LEMMA 3.2.(a) *For every $x \in Q$, $(\sigma^{\#})^{-1}(x) = \{y \in \Phi(X, E);\ (x, 0) \leqslant y \leqslant (x, 1)\}$.*

(b) $(\nu^{\#})^{-1}(0, 0) = Q \times \{0\}$ *and* $(\nu^{\#})^{-1}(1, 1) = Q \times \{1\}$.

(c) *For every $x \in X_{\mathbf{R}}$, $(\nu^{\#})^{-1}(x) = \{x\}$.*

Proof. We prove by induction over the rank of polynomials that for every polynomial p over $\mathbf{P}(X, E)$ there exists $x(p) \in Q$ with $(x(p), 0) \leqslant \nu(p) \leqslant (x(p), 1)$, thus obtaining (a). If $\operatorname{rank}(p) = 1$ then by a direct inspection we obtain the required statement.

Assume that $\mathrm{rank}(p) > 1$. If $p = p_1 \wedge p_2$ then $x(p) = x(p_1) \wedge x(p_2)$; if $p = p_1 \vee p_2$ then $x(p) = x(p_1) \vee x(p_2)$; because $\mathrm{rank}(p_1)$, $\mathrm{rank}(p_2) < \mathrm{rank}(p)$ the proof of (a) is complete.

We prove (b). Write $r' = r_{\mathbf{Q}(X,E)}$. First we prove that for every polynomial p over $\mathbf{P}(X, E)$ and for every $x \in Q(X, E) \setminus \{1, 1)\}$ we have $x \in r' \circ v^{\#}(v(p))$ if and only if $v(p) \geq x$. If $v(p) \geq x$ then obviously, $x \in r'v^{\#}(x) \subseteq r' \circ v^{\#}(v(p))$. We show the converse implication by induction over the rank of polynomials over $\mathbf{P}(X, E)$. If $\mathrm{rank}(p) = 1$ then $r' \circ v^{\#}(v(p)) = \{y \in Q(X, E); y \leq v(p)\}$ and the statement is true. Assume that $\mathrm{rank}(p) > 1$. If $p = p_1 \wedge p_2$ then $\mathrm{rank}(p_1)$, $\mathrm{rank}(p_2) < \mathrm{rank}(p)$ and by the induction hypothesis we obtain the statement. Assume that $p = p_1 \vee p_2$. Then by Proposition 2.2, x belongs to the smallest J_1-ideal containing $r' \circ v^{\#}(v(p_1)) \cup r' \circ v^{\#}(v(p_2))$. Since every $y \in Q(X, E)$ is either join-irreducible or lies in S, direct inspection of J_1 shows that there exist $A_i \subseteq Q(X, E)$ with $A_i \subseteq r' \circ v^{\#}(v(p_i))$ for $i \in \{1, 2\}$ such that $x \leq \bigvee(A_1 \cup A_2)$, $A_1 \cup A_2 \in J_1$. Since $\mathrm{rank}(p_i) < \mathrm{rank}(p)$ for $i \in \{1, 2\}$ we obtain by the induction hypothesis that $a_i \leq v(p_i)$ for all $a_i \in A_i$, $i \in \{1, 2\}$ and hence $x \leq v(p)$; thus the statement is true. Secondly we show that $r' \circ v^{\#}(v(p)) = Q(X, E)$ if and only if $v(p) \in Q \times \{1\}$. Since $(r')^{-1}(Q(X, E)) = \{1, 1\}$ we obtain that $(v^{\#})^{-1}(1, 1) = Q \times \{1\}$. Obviously, if $v(p) \in Q \times \{1\}$ then $r' \circ v^{\#}(v(p)) = Q(X, E)$. We show the converse implication by induction over the rank of polynomials. If $\mathrm{rank}(p) = 1$ then obviously $v(p) \in Q \times \{1\}$. Assume that $\mathrm{rank}(p) > 1$. If $p = p_1 \wedge p_2$ then $\mathrm{rank}(p_1)$, $\mathrm{rank}(p_2), < \mathrm{rank}(p)$ and $r' \circ v^{\#}(v(p_1)) = r' \circ v^{\#}(v(p_2)) = Q(X, E)$. By the induction hypothesis $v(p_1)$, $v(p_2) \in Q \times \{1\}$; hence also $v(p) \in Q \times \{1\}$. If $p = p_1 \vee p_2$ then by Proposition 2.2 $Q(X, E)$ is the smallest J_1-ideal containing $r' \circ v^{\#}(v(p_1)) \cup r' \circ v^{\#}(v(p_2))$. Hence there exist $A_i \subseteq r' \circ v^{\#}(v(p_i))$, $i \in \{1, 2\}$ such that $A_1 \cup A_2 \in J_1$ and $\bigvee(A_1 \cup A_2) = (1, 1)$ in $\mathbf{Q}(X, E)$. By direct inspection either $A_1 \cup A_2 \in J_0$ and $\bigvee(A_1 \cup A_2) \in Q \times \{1\}$ in $\mathbf{P}(X, E)$ or $(1, 1) \in A_1 \cup A_2$. Thus, by the induction hypothesis and by the first part of the proof, we obtain $v(p) \geq x \in Q \times \{1\}$.

Now by (a) we obtain the required statement. The proof that $(v^{\#})^{-1}(0, 0) = Q \times \{0\}$ is dual.

Finally, we prove (c). Assume the contrary; then there exists y comparable with x such that $v^{\#}(x) = v^{\#}(y)$. Further there exists $A \in E_{\mathbf{R}}$ with $x \in A$. Set $A' = (A \setminus \{x\}) \cup \{y\}$. Assume for example that $y > x$. By Proposition 2.10 we obtain $\bigwedge q_{P(X,E)}(A') \notin Q \times \{0\}$ but $\bigwedge v^{\#}(A') = \bigwedge v^{\#}(A) = (0, 0)$—a contradiction to (b). If $y < x$ then $\bigvee A \in S \cup (Q \times \{1\})$ but $\bigvee A'$ does not have this property by the first part of the proof of (b) and by Proposition 2.10. ■

LEMMA 3.3. *A J_0-ideal A belongs to $I_0(\mathbf{P}(X, E))$ if and only if one of the following conditions holds*:

(a) *A is the principal ideal generated by $(x, 1)$ for some $x \in Q$;*

(b) *$A = A_1 \cup A_2$ where A_1 is the principal ideal generated by $(x, 0)$ for some $x \in Q$ and $A_2 \subseteq X_{\mathbf{R}}$ is a finite set such that for every $i \in I$, if $X_i \cap A \neq \emptyset$ then $l(i) \leq (x, 0)$;*

(c) *$A = A_1 \cup A_2 \cup A_3$ where A_1 is the principal ideal generated by some $x \in S \setminus \{0, 1\}$, A_2 is the principal ideal generated by $(y, 0)$ for some $y \in Q$ with $\varphi(x) \leq y$, $A_3 \subseteq X_{\mathbf{R}}$ is a finite set such that for every $i \in I$ if $A_3 \cap X_i \neq \emptyset$ then $l(i) \leq (y, 0)$ and $u(i) \nleq x$.*

Proof. Clearly, A is always an ideal and since it is a finite join of principal ideals it belongs to $I_0(\mathbf{P}(X, E))$. By a direct inspection we obtain that J-ideals satisfying (a), (b), and (c) are closed under meets and joins in $I(\mathbf{P}(X, E))$; thus, they form $I_0(\mathbf{P}(X, E))$. ■

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LEMMA 3.4. *Let $A \subseteq \Phi(X, E)$ be a set of cardinality m with property (n) and* $\inf(A) = (x, 0)$ *for some $x \in Q$. Then $A \subseteq X_i$, $i \in I$ with $l(i) = (x, 0)$.*

Let $A \subseteq \Phi(X, E)$ be a set of cardinality m with property (n) and $\sup(A) = (1, 1)$. *Then $A \subseteq X_1$.*

Proof. Assume that $|A| = m$, A has property (n) and $\inf(A) = (x, 0)$ for some $x \in Q$. Write $r = r_{\mathbf{P}(X,E)}$, $q = q_{\mathbf{P}(X,E)}$. By Lemma 3.3, for every $a \in A$ we have $x_a \in Q$, so that $r(a) \cap Q \times \{0\}$ is the principal ideal generated by $(x_a, 0)$. Obviously, $x_a \geqslant x$. Since m is a regular infinite cardinal with $m > |Q|$ we obtain that there exists $A' \subseteq A$ with $|A'| = m$ such that $x_a = x_b$ for every pair of $a, b \in A'$. Since $\bigwedge r(B)$ is the principal ideal generated by x for every $B \subseteq A'$ with $|B| = n$, we have $x_a = x$ for every $a \in A'$. Denote by $Y_x = \bigcup \{X_i;\ l(i) = (x, 0)\}$. For $a \in A$, if $r(a)$ contains an element $z \in P(X, E) \setminus (Y_x \cup (Q \times \{0\})$ then $q(a)$ is contained in a principal filter generated by some $y \in S$ with $\varphi(y) = x$, because for every $u \in r(a)$, $v \in q(a)$ we have $v \geqslant u$. Since $\bigwedge q(B)$ is the principal filter generated by $(x, 0)$, this is possible for at most $n - 1$ elements of A, since A has property (n). Thus for all but $n - 1$ elements $r(a)$ is a principal ideal generated by some $z_a \in X_i$, where $\sup(A) = u(i)$ and $(x, 0) = l(i)$ (by Corollary 2.6 $r(\sup(A)) \neq r(\inf(A))$). If $q(a)$ is a proper subset of the principal filter generated by z_a then, moreover, it is contained in the principal filter generated by $u(i)$; thus, at most $n - 1$ elements of A have this property. Hence, for all but finitely many $a \in A$, the ideal $r(a)$ is the principal ideal generated by z_a and similarly $q(a)$ is the principal filter generated by z_a; thus $a = z_a$. Hence $\sup(A) = u(i)$ and, by Proposition 2.10, we conclude that there exists $i \in I$ with $l(i) = (x, 0)$ and $A \subseteq X_i$. The second statement is proved dually. ■

LEMMA 3.5. *Let $Z \subseteq X$ be an independent set in (X, E). Then there exists a surjective* $(0, 1)$*-homomorphism from $\Psi(X, E) \setminus (X \setminus Z)$ into S.*

Proof. Set $X_Z = X_{\mathbf{R}} \setminus Z$. For every $i \in I$, the (ordinary) sublattice $\langle X_i \rangle$ of $\Psi(X, E)$ generated by X_i is isomorphic to $\Lambda(X_i, E_i)$. Denote by $\mathbf{M} = (\mathrm{MP}, J, M)$ the partial lattice where MP is the subposet of $\Psi(X, E)$ on the set $\cup \{\langle X_i \rangle; i \in I\}$. Clearly $S \setminus \{0, 1\}$, X_i, $\{(0, 0), (1, 1)\} \subseteq \mathrm{MP}$. Set

$$J = M = \{\{(x, y\}; \text{ either } x, y \in \langle X_i \rangle \text{ for some } i \in I, \text{ or } x, y \in S, \text{ or } x \text{ and } y \text{ are comparable in MP}\}.$$

By a direct inspection we obtain that $\theta : \mathbf{M} \to \Psi(X, E)$ is a free completion, where θ is the inclusion. Write

$$P = \{p; p \text{ is a polynomial over } \mathbf{M} \text{ with } v(p) \notin X_Z\}.$$

By Proposition 2.10, for every $i \in I$ we have the prime filter $F_i = F_{Z \cap X_i}$ in $\Lambda(X_i, E_i) \setminus X_Z$. Define a function $f : P \to S$ by induction, as follows.

If $p \in P$ with rank 1 then for $v(p) \in S$ set $f(v(p)) = v(p)$, for $v(p) \in \Lambda(X_i, E_i)$, $i \in I$ set $f(v(p)) = 1$ whenever $v(p) \in F_i$ and $u(i) = (x, 1)$ for some $x \in Q$, $f(v(p)) = u(i)$ whenever $v(p) \in F_i$ and $u(i) \in S$, $f(v(p)) = 0$ whenever $v(p) \notin F_i$.

If $p = p_1 \vee p_2 \in P$ then for $j \in \{1, 2\}$, $i \in I$ such that $v(p_j) \in X_i \cap X_Z$ define $v(p_j) = 1$ whenever $u(i) \in Q \times \{1\}$, $v(p_j) = u(i)$ whenever $u(i) \in S$ and set $f(v(p)) = f(v(p_1)) \vee f(v(p_2))$.

If $p = p_1 \wedge p_2 \in P$ then $f(v(p)) = f(v(p_1)) \wedge f(v(p_2))$ whenever $p_1, p_2 \in P$, $f(v(p)) = 0$ whenever $p_1 \notin P$ or $p_2 \notin P$.

We shall prove that for p_1, $p_2 \in P$ with $v(p_1) = v(p_2)$ we have $f(v(p_1)) = f(v(p_2))$; this shows that $f : \Psi(X, E) \backslash X_Z \to S$ is a surjective (0, 1)-homomorphism. Define a quasi-ordering on the set of all pairs of polynomials from P such that $\{p_1, p_2\} \leqslant \{r_1, r_2\}$ if

$$\max\{\mathrm{rank}(p_1), \mathrm{rank}(p_2)\} < \max\{\mathrm{rank}(r_1), \mathrm{rank}(r_2)\}$$

or if

$$\max\{\mathrm{rank}(p_1), \mathrm{rank}(p_2)\} = \max\{\mathrm{rank}(r_1), \mathrm{rank}(r_2)\}$$

and

$$\min\{\mathrm{rank}(p_1), \mathrm{rank}(p_2)\} \leqslant \min\{\mathrm{rank}(r_1), \mathrm{rank}(r_2)\}.$$

We prove that $f(v(p_1)) \leqslant f(v(p_2))$ whenever $v(p_1) \leqslant v(p_2)$ in $\mathbf{M}^{\#}$. Assume the contrary and let $\{p_1, p_2\}$ be a least pair of lattice polynomials in P such that $v(p_1) \leqslant v(p_2)$ and $f(v(p_1)) \nleqslant f(v(p_2))$. By Theorem 2.3, $v(p_1) \leqslant s \leqslant v(p_2)$ for some $s \in \Psi(X, E)$. If $s \in X_Z$ then $f(v(p_1)) = 0 \leqslant f(v(p_2))$. If s is a lattice polynomial from P then by the induction hypothesis either $\mathrm{rank}(p_1) = 1$ or $\mathrm{rank}(p_2) = 1$ or $f(v(p_1)) \leqslant f(v(s)) \leqslant f(v(p_2))$. If $\mathrm{rank}(p_1) = 1$ then $f(v(p_1)) \nleqslant f(v(p_2))$ implies $\mathrm{rank}(p_2) > 1$. If $p_2 = p_{2,1} \wedge p_{2,2}$ then $\mathrm{rank}(p_{2,1})$, $\mathrm{rank}(p_{2,2}) < \mathrm{rank}(p_2)$ and $v(p_1) \leqslant v(p_{2,1})$, $v(p_{2,2})$, and by induction we have $f(v(p_1)) \leqslant f(v(p_2))$. If $p_2 = p_{2,1} \vee p_{2,2}$, then by Proposition 2.2 and the special form of $\mathbf{M}$ either $v(p_1) < v(p_{2,i})$ for some $i \in \{1, 2\}$ or there exist $A_i \subseteq \mathrm{MP}$ with $v(a_i) \leqslant v(p_{2,i})$, for every $a_i \in A_i$, $i \in \{1, 2\}$ and such that $A_1 \cup A_2 \in J$ with $\bigvee(A_1 \cup A_2) \geqslant v(p_1)$. Since $\mathrm{rank}(p_{2,i}) < \mathrm{rank}(p_2)$ we conclude by the induction hypothesis that $f(v(p_1)) \leqslant f(v(p_2))$. Therefore $f(v(p_1)) \leqslant f(v(p_2))$ whenever $v(p_1) \leqslant v(p_2)$. If $\mathrm{rank}(p_2) = 1$ the proof is analogous; hence $f(v(p_1)) = f(v(p_2))$ if $v(p_1) = v(p_2)$. ■.

Set $\Phi'(X, E) = (\iota^{\#}, \Phi(X, E), \sigma^{\#})$. For a compatible mapping $f : (X, E) \to (X', E')$ denote by f' the mapping $f' : \mathbf{P}(X, E) \to \mathbf{P}(X', E')$ which is the extension of f by the identity. Obviously f' is a partial lattice homomorphism. Denote by $\Phi' f$ the free extension of f' and set $\Phi f = \Phi' f$. Then we obtain:

Proposition 3.6. *Φ' is an embedding from $H(n, m)$ into* Biext(S, Q, φ) *and thus Φ is an embedding from $H(n, m)$ into ${}_S\mathbf{B}_Q$.*

Proof. Obviously, for every $(X, E) \in H(n, m)$, $\iota^{\#} : S \to \Phi(X, E)$ is an injective homomorphism and $\sigma^{\#} : \Phi(X, E) \to Q$ is a surjective homomorphism with $\sigma^{\#} \circ \iota^{\#} = \varphi$; thus $(\iota^{\#}, \Phi(X, E), \sigma^{\#})$ is an object of Biext(S, Q, φ). For every compatible mapping $f : (X, E) \to (X', E')$ the following diagram commutes:

$$\begin{tikzcd}
& \mathbf{P}(X, E) \arrow[dd, "f'"] \arrow[dr, "\sigma"] & \\
S \arrow[ur, "\iota"] \arrow[dr, "\iota'"'] & & Q \\
& \mathbf{P}(X', E') \arrow[ur, "\sigma'"'] &
\end{tikzcd}$$

Consequently, $\Phi' f : (\iota^{\#}, \Phi(X, E), \sigma^{\#}) \to ((\iota')^{\#}, \Phi(X', E'), (\sigma')^{\#})$ is a morphism of Biext(S, Q, φ). ■

LEMMA 3.7. *Let M be a lattice, and let $(X, E) \in H(n, m)$. Then every homomorphism $f: \Phi(X, E) \to M$ with $f(x, 0) = f(x, 1)$ for some $x \in Q$ satisfies $f(y, 0) = f(y, 1)$ for every $y \in Q$, and thus $f(x) = f(y)$ for every $x, y \in \Phi(X, E)$ with $\sigma(x) = \sigma(y)$.*

Proof. Observe that in $\Phi(X, E)$ for every $x \in Q$ we have $(x, 1) \vee (1, 0) = (1, 1)$, $(x, 0) \wedge (0, 1) = (0, 0)$, $(x, 0) \vee (0, 1) = (x, 1)$, $(x, 1) \wedge (1, 0) = (x, 0)$. ∎

LEMMA 3.8. *If $(X, E), (X', E') \in U$ and $f: \Phi(X, E) \to \Phi(X', E')$ is a homomorphism with $f(x, 0) \neq f(x, 1)$ for every $x \in L$ then there exists a compatible mapping $g: (X, E) \to (X', E')$ with $\Phi g = f$.*

Proof. For every $x \in X_0$ there exists $D_x \subseteq X_0$ with $|D_x| = m$ such that $B \in E_0$ for every $B \subseteq D_x$ with $|B| = n$. Then D_x has property (n) in $\Phi(X, E)$ and $\inf(D_x) = (0, 0)$, $\sup(D_x) = (0, 1)$. Hence $f(D_x)$ has property (n), and $\inf(f(D_x)) = f(0, 0) = (0, 0)$. By Lemma 3.4 we conclude that $f(D_x) \subseteq X_0'$ and $\sup(f(D_x)) = (0, 1)$. Thus $f(X_0) \subseteq X_0'$. Analogously we obtain that $f(X_1) \subseteq X_1'$ and $f(1, 0) = (1, 0)$. Thus $f(Q \times \{0\}) \subseteq Q \times \{0\}$ and, if we apply Lemma 3.4 for every (X_i, E_i), $i \in I$, we conclude that $f(X_i) \subseteq X_j'$ where $u(j) = f(u(i))$. By the properties of $\mathbf{R}$ (see Corollary 2.8 and Proposition 2.9), we obtain that $f(x, i) = (x, i)$ for every $x \in Q$ and $i \in I$, $f(y) = y$ for every $y \in S$, $f(z) = z$ for every $z \in X_i$ with $i \in I'$ and there exists a compatible mapping $g: (X_{i_0}, E_{i_0}) \to (X_{i_0}', E_{i_0}')$ defined by $g(z) = f(z)$ for every $z \in X_{i_0}$. Since $f(x) = \Phi g(x)$ for every $x \in P(X, E)$ and $P(X, E)$ generates $\Phi(X, E)$ we have $\Phi g = f$. ∎

If we summarize results in Lemmas 3.7 and 3.8 we obtain:

COROLLARY 3.9. *If there is no $(0, 1)$-homomorphism of Q into $\Phi(X, E)$ for any $(X, E) \in U$, then ${}_S\mathbf{B}_Q$ is universal. If for every $(X, E) \in U$ and for every $(0, 1)$-homomorphism $f: Q \to \Phi(X, E)$ we have either $\iota^\# \neq f \circ \sigma^\# \circ \iota^\#$ or $\sigma^\# \neq \sigma^\# \circ f \circ \sigma^\#$ then* Biext(S, Q, φ) *is universal.*

LEMMA 3.10. *If there exists a $(0, 1)$-sublattice Q_0 of Q such that every quotient of Q_0 has at most $n - 1$ doubly irreducible elements and there exists no $(0, 1)$-homomorphism from Q_0 to S then for every $(X, E) \in U$ there exists no $(0, 1)$-homomorphism from Q to $\Phi(X, E)$.*

Proof. Assume that there exists a $(0, 1)$-homomorphism $f: Q \to \Phi(X, E)$. Consider $g = \nu^\# \circ f \circ \theta: Q_0 \to \mathbf{Q}(X, E)^\#$ where $\theta: Q_0 \to Q$ is the inclusion. By the assumption, $\mathrm{Im}(g)$ has at most $n - 1$ doubly irreducible elements, and thus $\mathrm{Im}(g) \cap X_{\mathbf{R}}$ is an independent set and according to Lemma 3.5 there exists a $(0, 1)$-homomorphism from Q_0 to S—a contradiction. ∎

LEMMA 3.11. *If there exists a simple $(0, 1)$-sublattice Q_0 of Q which is not a $(0, 1)$-sublattice of S, then for every $(X, E) \in U$ there exists no $(0, 1)$-homomorphism from Q to $\Phi(X, E)$.*

Proof. Assume that there exists a $(0, 1)$-homomorphism $f: Q \to \Phi(X, E)$. Consider $g = \nu^\# \circ f \circ \theta: Q_0 \to \mathbf{Q}(X, E)^\#$ where $\theta: Q_0 \to Q$ is the inclusion. If $\mathrm{Im}(g) \cap X_{\mathbf{R}}$ is an independent set then there exists a $(0, 1)$-homomorphism from Q_0 to S—a contradiction. Thus there exists $A \subseteq \mathrm{Im}(g) \cap X_{\mathbf{R}}$ with $A \in E_{\mathbf{R}}$. By Proposition 3.2(c) $A \subseteq \mathrm{Im}(f \circ \theta)$ and hence $\bigwedge A \in \mathrm{Im}(f \circ \theta) \cap Q \times \{0\}$. Since Q_0 is simple and $\nu^\#(Q \times \{0\}) = (0, 0)$ we

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conclude that $\bigwedge A = (0,0)$ and $A \in E_0$. Then $\bigvee(A) = (0,1) \in \mathrm{Im}(f \circ \theta)$ but $\nu^{\#}(1,1) = \nu^{\#}(0,1)$—a contradiction to the simplicity of Q_0. ■

THEOREM 3.12. *Let S,Q be* $(0,1)$*-lattices and let $\varphi: S \to Q$ be a* $(0,1)$*-homomorphism. If there exists a* $(0,1)$*-sublattice Q_0 of Q which has no* $(0,1)$*-homomorphisms into S and either there exists finite n such that every quotient of Q_0 has at most n doubly irreducible elements or Q_0 is simple, then ${}_S\mathbf{B}_Q$ and* $\mathrm{Biext}(S, Q, \varphi)$ *are universal. In particular, if Q has a finitely generated* $(0,1)$*-sublattice without any* $(0,1)$*-homomorphisms into S then ${}_S\mathbf{B}_Q$ and* $\mathrm{Biext}(S, Q, \varphi)$ *are universal.* $\mathrm{Biext}(S, Q, \varphi)$ *is also universal whenever φ is not one-to-one.*

Proof. The first statement follows from Proposition 3.6, Corollary 3.9 and Lemmas 3.10 and 3.11. Since a doubly irreducible element has to belong to every set of generators, we obtain that if Q_0 has n generators then it has at most n doubly irreducible elements. Moreover, if Q_0 has n generators then every quotient of Q_0 has also n generators; thus we obtain the second statement. The last statement immediately follows from Corollary 3.9 because if $f: Q \to \Phi(X, E)$ is a $(0,1)$-homomorphism then $f \circ \sigma^{\#} \circ \iota^{\#} = f \circ \varphi$ and therefore $f \circ \sigma^{\#} \circ \iota^{\#}$ is not one-to-one, so that no morphism f of $\mathrm{Biext}(S, Q, \varphi)$ can factorize through $\sigma^{\#}$. ■

COROLLARY 3.13. *Let S,Q be* $(0,1)$*-lattices, and let $\varphi: S \to Q$ be a* $(0,1)$*-homomorphism. If S is finite or has a prime ideal, or if Q is finitely generated, then the following conditions are equivalent*:

(a) *there exists no* $(0,1)$*-homomorphism from Q to S*;

(b) *${}_S\mathbf{B}_Q$ contains a rigid lattice and S and Q are non-isomorphic*;

(c) *${}_S\mathbf{B}_Q$ contains an arbitrarily large rigid lattice*;

(d) *for every monoid M the class ${}_S\mathbf{B}_Q$ contains a proper class C of non-isomorphic lattices such that* $\mathrm{End}(N)$ *is isomorphic to M for every $N \in C$*;

(e) *${}_S\mathbf{B}_Q$ is universal.*

Proof. If S is finite and there exists no $(0,1)$-homomorphism from Q to S then from the equational compactness **[16]** of S there exists a finitely generated $(0,1)$-sublattice Q_0 of Q without any $(0,1)$-homomorphisms from Q_0 to S. If S has a prime ideal then Q has a $(0,1)$-homomorphism to S if and only if Q has a prime ideal. If Q has no prime ideal then by the equational compactness of the two-element lattice **[16]** there exists a finitely generated $(0,1)$-sublattice Q_0 of Q without any prime ideals. Thus if one of the conditions holds then by Theorem 3.12 ${}_S\mathbf{B}_Q$ is universal. Thus $(a) \Rightarrow (e)$. The implications (e)$\Rightarrow$(d)$\Rightarrow$(c)$\Rightarrow$(b) are proved in **[15]** (note that ${}_S\mathbf{B}_Q$ for $S = Q$ fails to satisfy (c)). We prove $(b) \Rightarrow (a)$. If there exists a $(0,1)$-homomorphism $f: Q \to S$, then every lattice M in ${}_S\mathbf{B}_Q$ has an endomorphism $\iota^{\#} \circ f \circ \sigma^{\#}$, so only Q can be a rigid lattice in ${}_S\mathbf{B}_Q$. In this case S and Q are isomorphic since $\iota^{\#}$ is one-to-one. ■.

Thus Theorem 1.4 is proved. The following corollary generalizes Theorem 1.3 and thus it gives a solution of the problem given in **[13]**.

COROLLARY 3.14. *For a lattice Q, the following conditions are equivalent*:

(a) *Q has no* $(0,1)$*-homomorphism into a free* $(0,1)$*-lattice*;

(b) *Q has no prime ideal*;

(c) *$\mathbf{B}_Q$ contains a rigid lattice*;

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(d) $\mathbf{B}_Q$ *contains an arbitrarily large rigid lattice*;

(e) *for every monoid* M *the class* $\mathbf{B}_Q$ *contains a proper class* C *of non-isomorphic lattices such that* $\mathrm{End}(N)$ *is isomorphic to* M *for every* $N \in C$;

(f) $\mathbf{B}_Q$ *is universal.*

Proof. Obviously (a) and (b) are equivalent (a two-element lattice is a free $(0,1)$-lattice over the empty set). The equivalence of the other conditions follows from Corollary 3.13 applied in the case of a two-element lattice S. ■

It is well known that any $(0,1)$-lattice S is a $(0,1)$-sublattice of a simple lattice Q of a cardinality greater than that of S. Thus Theorem 3.12, when applied to S and Q, provides also an alternative proof of Theorem 1.2.

REFERENCES

1. M. E. Adams and J., Sichler, Homomorphisms of bounded lattices with a given sublattice, *Arch. Math. Basel* **30** (1978), 122–128.

2. M. E. Adams and J. Sichler, Cover set lattices, *Canad. J. Math.* **32** (1980), 1177–1205.

3. P. Crawley and R. P. Dilworth, *Algebraic theory of lattices* (Prentice-Hall, 1973).

4. R. A. Dean, Free lattices generated by partially ordered sets and preserving bounds, *Canad. J. Math.* **16** (1964), 136–148.

5. P. Goralčik, V. Koubek and J. Sichler, Universal varieties of $(0,1)$-lattices, *Canad. J. Math.* **42** (1990), 470–490.

6. G. Grätzer, *General lattice theory* (Academic Press, 1978).

7. G. Grätzer and J. Sichler, On the endomorphism semigroup (and category) of bounded lattices, *Pacific J. Math.* **35** (1970), 639–647.

8. Z. Hedrlin and J. Sichler, Any boundable binding category contains a proper class of mutually disjoint copies of itself, *Algebra Universalis* **1** (1971), 97–103.

9. V. Koubek, Graphs with given subgraphs represent all categories, *Comment. Math. Univ. Carolin.* **18** (1977), 115–127.

10. V. Koubek, Graphs with given subgraphs represent all categories II, *Comment. Math. Univ. Carolin.* **19** (1978), 249–263.

11. V. Koubek, Towards minimal binding varieties of lattices, *Canad. J. Math.* **36** (1984), 263–285.

12. V. Koubek and J. Sichler, Universality of small lattice varieties, *Proc. Amer. Math. Soc.* **91** (1984), 19–24.

13. V. Koubek and J. Sichler, Quotients of rigid $(0,1)$-lattices, *Arch. Math. Basel* **44** (1985), 403–412.

14. H. Lakser, Normal and canonical representations in free products of lattices, *Canad. J. Math.* **22** (1970), 394–402.

15. A. Pultr and V. Trnková, *Combinatorial, algebraic and topological representations of groups, semigroups and categories* (North Holland, 1980).

16. W. Taylor, Review of seventeen papers on equational compactness, *J. Symbolic Logic* **40** (1975), 88–92.

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