SOME REMARKS ON A PAPER OF WONG

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1. The aim of this note is to point out a mistake in the proof of Theorem 2 of Wong's paper [1]. We first give an example to show that the theorem as stated is not true.

Example 1. Consider the cartesian plane and the graph H with equation $y = \sqrt{x^2 + 1}$. For each n let f_n denote a graph which is such that *

(1) $|f_n(x) - H(x)| < 1/n$ for all x, $n = 1, 2, \ldots$

(2) $f_n(x)$ intersects $f(x) = x$ exactly once and in a point with abscissa greater than n.

(3) $f_n(x)$ is continuous.

Then each f_n is a continuous function of reals into itself. The metric d defined by

$$d(x, y) = \begin{cases} |x - y| & \text{if } |x - y| \leq 1 \\ 1 & \text{if } |x - y| > 1 \end{cases}$$

is a bounded complete metric for the reals equivalent to the usual metric. It is easy to see that $\{f_n : n = 1, 2 \ldots\}$ is a Cauchy sequence, converging to H under the sup. norm topology. Each f_n has a fixed point but H does not. This completes the example.

As a special case in the above example, we may consider

$$f_n(x) = \begin{cases} H(x) & x \leq n \\ H(n) & n \leq x \leq n + \frac{1}{2} \\ g_n(x) & n + \frac{1}{2} \leq x \leq n + 1 \\ H(x) & x \geq n + 1 \end{cases}$$

where $g_n(x)$ is the line segment connection the points $(n + \frac{1}{2}, H(n))$ and $(n+1, H(n + 1))$.

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2. Let X be a compact metric space with metric ρ. Let X^* be the set of all continuous functions from X into itself, and define for any $f, g \in X^*$,

$$d(f, g) = \sup_{x \in X} \rho(f(x), g(x))$$

Then d is a complete metric on X^*. Let

$$F = \{f \in X^* : f(x) = x \text{ for some } x \in X\} .$$

THEOREM 1. F *is a closed subset of* X^*.

Proof. Let $\{f_n\}$ be a Cauchy sequence of functions in F. Since X^* is complete, $\{f_n\}$ converges to a function $f \in X^*$. We show that $f \in F$.

Let x_n be any fixed point of f_n, $n = 1, 2, \ldots$. Since X is compact, $\{f(x_n) : n = 1, 2, \ldots\}$ has a convergent subsequence $\{f(x_{n_k}) : k = 1, 2, \ldots\}$ converging, say, to $x \in X$. Since $\{f_n\}$ is a Cauchy sequence it is easy to check that $\{f_{n_k}(x_{n_k}) : k = 1, 2, \ldots\}$ also converges to x. That is, $\{x_{n_k} : k = 1, 2, \ldots\}$ converges to x. Hence $f(x) = x$ and $f \in F$. This completes the proof.

3. Let X be a topological space, and X^* be the space of all continuous functions of X into itself with the compact open topology [2]. Let $F = \{f \in X^* : f(x) = x \text{ for some } x \in X\}$.

THEOREM 2. *If* X *is compact and Hausdorff, then* F *is closed*.

Proof. We show that the complement of F is open. Suppose $f \in X^* - F$. Then $f(x) \neq x$ for any $x \in X$. Hence for any $x \in X$ there exist open sets U_x and V_x containing x and $f(x)$ respectively such that $\overline{U}_x$ is compact, $\overline{U}_x \cap V_x = \emptyset$ and $f(\overline{U}_x) \subset V_x$. Then

$$f \in M(\overline{U}_x, V_x) = \{g \in X^* : g(\overline{U}_x) \subset V_x\} .$$

Choosing such pairs for each point $x \in X$, we get an open covering $\{U_x : x \in X\}$ of X, which has a finite subcover $\{U_{x_i} : i = 1, 2, \ldots, n\}$.

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Let $\{V_{x_i} : i = 1, 2, \ldots, n\}$ be the corresponding members of $\{V_x : x \in X\}$. Then

$$V = \bigcap_{i=1}^{n} M(\bar{U}_{x_i}, V_{x_i})$$

is an open set containing f. Furthermore if $g \in V$ then for any $x \in X$, $g(x) \neq x$. Hence $f \in V \subset X^* - F$ and $X^* - F$ is an open set. This completes the proof.

REFERENCES

1. J.S.W. Wong, Some remarks on transformations in a metric space. Canad. Math. Bull. 8 (1965) 659-666.

2. S. T. Hu, Elements of general topology (Holden-Day, 1964).

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