The follow-up question is then: does adding this final extra rule change the answer on what happens if one plays forever? The answer is 'no' the expected gain is still infinity for a player willing to play forever. Students are led to argue by estimation again. Namely in the worst case scenario, the player gets all straight 1s and in the last roll a whopping 6. In this (unfortunate to the player) scenario, his or her share would be $\frac{1}{n^6}1.01^n$, which still goes to infinity as $n \to \infty$.

Other variants of the game can be designed. For instance, a competitive game between two players and a fair coin. At each turn, a player flips a coin and, say, if he/she gets a head a 1% is increased, but if he/she gets tail, the pool's value decreases. We often ask students to come up with different games by themselves, try to model their games and make further questions about what to expect.

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Two quick direct proofs of the irrationality of tan15°

Figure 1 depicts a square *ACDE* of side length $2p$ on one side of which is constructed an equilateral triangle *ABC*. Line *BHGF* is an axis of symmetry with $BF = q$, and triangle *CBG* is isosceles with $BG = BC = 2p$. A short angle-chase shows that the two angles marked α are both 15°.

In triangle *BFE* we have

$$\tan 15^\circ = \frac{p}{q}. \quad (1)$$

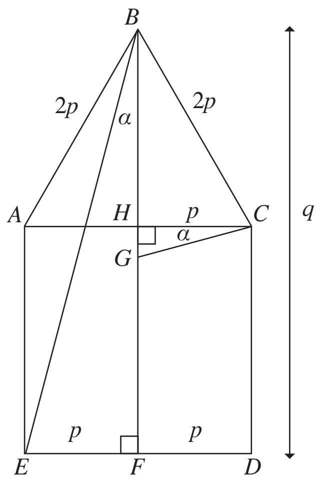

FIGURE 1: The height, BF, is q; angles marked α are both 15°.

In triangle CHG we have

$$\tan 15^\circ = \frac{HG}{p} = \frac{HF - GF}{p} = \frac{2p - (q - 2p)}{p} = \frac{4p - q}{p}. \qquad (2)$$

For contradiction, suppose that $\tan 15^\circ$ is rational. Then, in Figure 1, p, q may be taken as positive integers with $p < q$ so that, in (1), $\tan 15^\circ = \frac{p}{q}$. The quickest way of completing the proof is to assume that, in the fraction $\frac{p}{q}$, q is minimal. Invoking (2) then gives a rational representation of $\tan 15^\circ = \frac{4p-q}{p}$ with denominator smaller than the posited minimum, q.

Alternatively, we can combine (1) and (2) to deduce that $\frac{p}{q} + \frac{q}{p} = 4$. (This is equivalent to a geometric proof that $\tan 15^\circ + \cot 15^\circ = 4$.) Then $p^2 + q^2 = 4pq$, which forces $p = 2P$ and $q = 2Q$ both to be even. This leads to $(2P)^2 + (2Q)^2 = 4(2P)(2Q)$, so that $P^2 + Q^2 = 4PQ$. But we can repeat this argument to produce an unending sequence of smaller and smaller positive integer solutions – a contradiction. (This type of proof by descent always provokes a lively reaction from classes!)

Exactly the same argument shows that $\tan\theta$ is irrational if $\tan\theta + \cot\theta = 4k$, for any positive integer k.

We can also read off from Figure 1 that

$$\tan 15^\circ = \frac{EF}{FH + HB} = \frac{p}{2p + \sqrt{3}p} = \frac{1}{2 + \sqrt{3}}$$

or, if you prefer,

$$\tan 15^\circ = \frac{GH}{HC} = \frac{GB - HB}{HC} = \frac{2p - \sqrt{3}p}{p} = 2 - \sqrt{3}.$$

Either way, irrationality of $\tan 15^\circ$ is equivalent to that of $\sqrt{3}$.

In the case that $\tan\theta + \cot = 4k$, the quadratic formula shows that the irrationality of $\tan\theta$ is equivalent to that of $\sqrt{4k^2 - 1}$.

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This prompts a query. If positive integers k, l, M satisfy $4k^2 - Ml^2 = 1$, so that M and l are necessarily odd, then $M \equiv 3 \pmod 4$, and the irrationality of $\sqrt{4k^2 - 1}$ entails that of $l\sqrt{M}$ and hence $\sqrt{M}$. For many values of $M \equiv 3 \pmod 4$, the fundamental solution of the Pell equation $x^2 - My^2 = 1$ has x even and y odd, which supplies the $k = \frac{1}{2}x$ and $l = y$ required. But what explains the exceptional values of such M, the first few examples of which are 39, 55, 95 and 111?

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Infinite sums of reciprocal quadratics

Summing series such as $\sum_1^\infty \frac{1}{(n+p)(n+q)}$ with $0 \leqslant p < q$ integers is a familiar topic for many students. Partial fractions rewrite the terms of the series as

$$\frac{1}{q-p}\sum_1^\infty \left(\frac{1}{n+p} - \frac{1}{n+q}\right)$$

which telescopes to give

$$\sum_1^\infty \frac{1}{(n+p)(n+q)} = \frac{1}{q-p}\sum_1^{q-p}\frac{1}{n+p}.$$

After this, a natural question to ask is 'Can we evaluate the sum

$$\sum_1^\infty \frac{1}{n^2 + An + B}$$

with integer coefficients A, B when the quadratic $n^2 + An + B$ does not factorise?'

A clue that something more sophisticated is needed comes from the case $p = q$ above where we need to invoke the sum

$$\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

to deduce that

$$\sum_1^\infty \frac{1}{(n+p)^2} = \frac{\pi^2}{6} - \sum_1^p \frac{1}{n^2}.$$

It is convenient to split cases according to the parity of A.

- *Case* 1: $A = 2a$ is even

  Then $n^2 + An + B = (n + a)^2 - \Delta$, where $4\Delta = A^2 - 4B$ is the discriminant of $n^2 + An + B$, assumed not to be the square of a non-negative integer. We also assume that $\Delta \neq 0$, since the case $\Delta = 0$ has already been dealt with.