THE INJECTIVE ENVELOPE OF S-SETS

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Introduction. If S is a semigroup, then an S-set A_S is a set A together with a representation of S by mappings of A into itself. In this article, the theory of injective envelopes is carried from R-modules to S-sets. These results are known to hold in every Grothendieck category, but the category Ens_S of (right) S-sets is not even additive.

In the first section we show that Ens_S has enough injectives; in the second we proceed to construct the injective envelope as a maximal essential extension. These results are applied in the last section to show that, for instance, the extended system of reals is the injective envelope of the rationals in Ens_S.

This article is essentially a generalization and simplification of part of my doctoral dissertation written under the direction of Professor J. Lambek.

1. Injectivity.

DEFINITION 1. A right set (A, f) over a semigroup S, or a right S-set A_S, consists of a set A, and a mapping f from $A \times S$ into A, written $f(a, s) = as$, such that for any a in A, and s, s' in S, we have:

$$a(ss') = (as)s' . \tag{1}$$

A homomorphism ϕ from A_S to B_S, both right S-sets, is a mapping from A to B such that for any a in A and s in S, $\phi(as) = (\phi(a))s$.

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The category of right S-sets (henceforth called S-sets) and homomorphisms will be denoted by Ens_S. Left S-sets are defined dually (c.f. also [6], [3]).

There are two subcategories of Ens_S which deserve attention. The first is Ens_M, where M is a monoid with identity element 1, satisfying (1) and

(2) $$a \cdot 1 = a \text{ for all } a \text{ in } A$$

with same homomorphisms as above. The second is $Ens^*_{M_o}$ in which A is a pointed set with distinguished element $*$, M_o is a monoid with a zero element 0, satisfying (1), (2) and:

(3) $$* \cdot m = * \text{ for all } m \text{ in } M_o ,$$

(4) $$a \cdot 0 = * \text{ for all } a \text{ in } A ,$$

and every homomorphism maps distinguished element on same.

All the results and definitions in this paper are stated for Ens_S, but they can be trivially extended to the latter two categories.

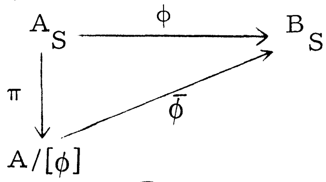
There are numerous examples of S-sets, a semigroup over itself and the set of all maps on a set being the two most obvious ones. A more interesting one is treated in greater detail at the end.

Now the notions of injection, surjection, isomorphism, sub S-set, congruence relation, as well as the relationship with semigroup representations, are all immediate. The direct product of two S-sets is their cartesian product with operations defined component-wise, while their coproduct is their disjoint union: the two are obviously not isomorphic, which in itself shows that Ens_S is not additive.

The proofs of the next three theorems are easy and will be omitted.

THEOREM 1. If ϕ is a homomorphism from A_S to

B_S then $[\phi]$, defined on A by $x[\phi]y$ if and only if $\phi(x) = \phi(y)$, for all x and y in A, is a congruence relation on A_S and the following diagram:



commutes, where π and $\bar{\phi}$ are the usual surjection and injection respectively.

THEOREM 2. If θ is a congruence relation on B_S, θb the congruence class of b for any b in B, then every sub S-set of B/θ is of the form $A/\theta = \{\theta a \mid a \in A\}$, where A_S is a sub S-set of B_S.

THEOREM 3. If $\theta \subset \theta'$ are two congruence relations on A_S, then the relation θ'/θ, defined on A/θ by $\theta x(\theta'/\theta)\theta y \Longleftrightarrow x\theta' y$ for all x and y in A, is a congruence relation on A/θ, and every congruence relation on A/θ is of that form.

We now proceed to show that Ens_S has enough injectives, but first:

DEFINITION 2. An S-set I_S is injective if and only if for any injection $K: A_S \rightarrow B_S$ and homomorphism $\phi: A_S \rightarrow I_S$ there is a homomorphism $\bar{\phi}: B_S \rightarrow I_S$ such that $\bar{\phi}K = \phi$.

The next theorem is obvious.

THEOREM 4. A retract of an injective S-set is injective.

For the next two theorems we need the following concept. We let S' denote the monoid obtained by adjoining an element 1 to S, with $s \cdot 1 = 1 \cdot s = s$ for all s in S, with the additional convention that it be equal to S if S is already a monoid (i.e. $S' = S$). S'_S is then an S-set extending S_S.

DEFINITION 3. An S-set A_S is weakly injective if and only if for any right ideal K of S (i.e. $KS \subset K$) and homomorphism $\phi: K_S \rightarrow A_S$ there exists an element a in A such that

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for all s in K, $\phi(s) = as$.

This is a transcription of the well known criterion of injectivity for R-modules, but all we can prove here is

THEOREM 5. If A_S is injective, then it is weakly injective.

Proof. Let ϕ be as in the above definition, and extend it to $\bar{\phi}: S'_S \to A_S$ by the injectivity of A_S. If $\bar{\phi}(1) = a$ then for any s in K, $\phi(s) = \bar{\phi}(1s) = (\bar{\phi}(1))s = as$. Q.E.D.

It will be shown in the last section that the converse is not true. Meanwhile, we prove the key result of this section.

THEOREM 6. If $A^{S'}$ denotes the set of all mappings from S' to A, then it is an injective S-set $A^{S'}_S$ extending to A_S.

Proof. Defining for any mapping $f: S' \to A$ and s in S, fs by $(fs)(t) = f(st)$ for all t in S' turns $A^{S'}$ into an S-set $A^{S'}_S$ which extends A_S for the canonical embedding ψ defined for each a in A by $(\psi(a))(s) = as$ for all s in S and $(\psi(a))(1) = a$ is an injection. If ϕ is a homomorphism from B_S to $A^{S'}_S$ and $K: B_S \to C_S$ an injection, then $\bar{\phi}$ is defined by $(\bar{\phi}(c))(t) = (\phi(K^{-1}(ct)))(1)$ if ct is in $K(B)$, and to any fixed element a of A otherwise (in particular when $t = 1$). $\bar{\phi}$ is the required extension of ϕ since $(\bar{\phi}(cs))(t) = (\phi(K^{-1}(cst)))(1) = (\bar{\phi}(c))(st) = ((\bar{\phi}(c))s)(t)$ if cst is in $K(B)$ with c in C and s and t in S.

COROLLARY 1. Every S-set can be embedded into an injective S-set.

COROLLARY 2. A_S is injective if and only if it is a retract of every extension.

COROLLARY 3. A_S is injective if and only if it is a retract of $B^{S'}_S$ for some set B.

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These results could also have been obtained by using a "transfer" theorem, due to Maranda ([2] and [4]), or by embedding A_S into a product of S^*'s, where S^* is the right S-set of all mappings of S into some fixed set X containing at least two elements, since that product is injective.

2. The Injective Envelope.

DEFINITION 4. A sub S-set A_S of B_S is large in B_S, written $A_S \leq B_S$, if and only if any homomorphism ϕ from B_S to C_S, for any S-set C_S, with restriction to A_S an injection is itself an injection. If $A_S \leq B_S$, then B_S is also said to be an essential extension of A_S.

LEMMA 1. If a_1 and a_2 are in A_S, then the relation $[a_1, a_2]$ defined on A_S by:

$$x[a_1, a_2]y \iff x = y \text{ ; or } x = a_i \text{ , } y = a_j \text{ , with } i \text{ and } j \text{ in } \{1, 2\} \text{ ;}$$

or there exists a finite sequence $s_1, s_2, \ldots, s_n$ of elements of S such that $x = a_{j_1} s_1$ and $a_{j'_1} s_1 = a_{j_2} s_2$ and $a_{j'_2} s_2 = \ldots$ and $a_{j'_{i-1}} s_{i-1} = a_{j_i} s_i$ and $a_{j'_i} s_i = \ldots = a_{j_n} s_n$ and $a_{j'_n} s_n = y$, with j_i and j'_i in $\{1, 2\}$, $i = 1, 2, \ldots, n$, is the smallest congruence relation on A_S relating a_1 to a_2.

<u>Proof.</u> It is easy to verify that it is a congruence relation relating a_1 to a_2. Now assume θ is a congruence relation on A_S such that $a_1 \theta a_2$, and suppose that $x[a_1, a_2]y$, where x and y are in A_S. With the above notation, we have $a_{j_i} s_i \theta a_{j'_i} s_i$, and ultimately $x \theta y$ in the only non trivial case.

THEOREM 7. If A_S is a sub S-set of B_S, and C_S is a sub S-set of B_S extending A_S, then the following are all equivalent:

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(1) A_S is large in B_S.

(2) If θ is a congruence relation on B_S which is not the identity, then its restriction to A_S is not the identity relation.

(3) For any two distinct elements b_1 and b_2 in B, there are two distinct elements a_1 and a_2 in A such that $a_1[b_1, b_2]a_2$.

(4) (2) holds for any congruence relation with domain C_S.

(5) Definition (4) holds for any homomorphism with domain C_S.

Proof. If θ is not the identity on B_S, then the canonical surjection of B_S onto B/θ is not an injection, nor is its restriction to A_S by (1), which shows (2). If $b_1 \neq b_2$ in B, the restriction of $[b_1, b_2]$ to A is not the identity, and thus (2) implies (3). If θ is a congruence relation on C_S, with $A_S \subseteq C_S \subseteq B_S$, not the identity, then there exist two distinct b_1 and b_2 in C with $b_1 \theta b_2$, and by (3) there are two distinct a_1 and a_2 in A such that $a_1[b_1, b_2]a_2$, which implies that $a_1 \theta a_2$ by Lemma 1. If ϕ is a homomorphism with domain C_S, not an injection, then the congruence relation $[\phi]$ of Theorem 1 is not the identity on C_S and the result follows by (4). Finally, (1) is a special case of (5).

DEFINITION 5. A_S is strictly large in an extension B_S if and only if for any $b_1 \neq b_2$ in B there is an s in S such that $b_1 s \neq b_2 s$ and both are in A.

It is obvious that strictly large implies large, but the converse is not true as will be seen in the last section.

COROLLARY. (1) $\leq$ is a transitive relation.

(2) If $A_S \leq B_S$ and $A_S \subseteq C_S \subseteq B_S$,

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then $A_S \leq C_S \leq B_S$ and A_S is not a proper retract of C_S.

(3) Every essential extension of A_S is contained in every injective extension up to isomorphism over A_S.

THEOREM 8. If A_S is a sub S-set of B_S and θ is a congruence relation on B_S maximal in the set of all congruence relations on B_S with restriction to A_S the identity, then B/θ contains a large sub S-set A/θ isomorphic to A_S.

Proof. Using theorems 2 and 3, we see that A_S is isomorphic to A/θ, and if η/θ is a congruence relation on B/θ with restriction to A/θ the identity, then η is a congruence relation on B_S, containing θ, and with restriction to A_S the identity; for if a_1 and a_2 are in A, then

$$a_1 \; \eta \; a_2 \Longleftrightarrow \theta\, a_1(\eta/\theta)\theta a_2 \Longleftrightarrow \theta a_1 = \theta a_2 \Longleftrightarrow a_1 = a_2 .$$

It follows from the maximality of θ that $\theta = \eta$ and thus η/θ is the identity relation.

THEOREM 9. An S-set A_S is injective if and only if it has no proper essential extension.

Proof. Let us assume that the condition holds and that B_S is a proper extension of A_S. By Corollary 2 of Theorem 6 it suffices to show that A_S is a retract of B_S. B_S is not an essential extension, by the hypothesis, and thus there is a congruence relation θ on B_S with restriction to A_S the identity. Since the union of any chain of such congruence relations still has that property, θ may be assumed to be maximal by Zorn's Lemma. By Theorem 8, B/θ is an essential extension of A/θ, the latter being isomorphic to A_S. The hypothesis implies that $B/\theta = A/\theta$, or that for each b in B there is a unique a in A such that $\theta b = \theta a$; this in turn implies the existence of a homomorphism from B_S to A_S with restriction to A_S the identity. The converse is immediate.

THEOREM 10. Every S-set A_S has a maximal essential

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extension which is injective and unique up to isomorphism over A_S.

Proof. Let I_S be an injective extension of A_S, as guaranteed by Corollary 1 of Theorem 6. Then the union of any chain of essential extensions of A_S contained in I_S is an essential extension of A_S, and thus by Zorn's Lemma, A_S has maximal essential extensions in I_S. If B_S is any such maximal essential extension in I_S and C_S any other essential extension of B_S then it easily follows that $B_S = C_S$, and thus B_S is a maximal essential extension of A_S and is thus injective by the preceding proposition. If B_S and C_S are two maximal essential extensions of A_S, then the embedding of A_S in C_S can be extended to an injection of B_S into C_S, the latter being onto by the maximality of B_S.

DEFINITION 6. Any maximal essential extension of an S-set A_S is called an injective envelope of A_S. It is unique up to isomorphism over A_S.

THEOREM 11. I_S is the injective envelope of A_S if and only if I_S is a minimal injective extension of A_S.

Proof. If I_S is the injective envelope of A_S and I'_S is injective between A_S and I_S then the identity map of I'_S can be extended to an injection of I_S into I'_S since $A_S \leq I'_S \leq I_S$ by the second part of the corollary of Theorem 7, and thus $I'_S = I_S$. If conversely I'_S is the injective envelope of A_S and I_S a minimal injective extension of A_S then it follows that $I_S = I'_S$.

COROLLARY. The following are all equivalent:

(1) I_S is the injective envelope of A_S,

(2) I_S is both an injective and an essential extension of A_S,

(3) I_S is a minimal injective extension of A_S.

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3. Example. Let S be a lower semi-lattice, i.e., a partially ordered set $(S, \leq)$ in which any two elements have an inf. Then S can be regarded as a commutative idempotent semigroup if the product of any two of its elements is defined to be their inf, and it is well known that S can be embedded in a complete lattice D in which the order relation of S is preserved together with all sups and infs already existing in S, and such that moreover for any d in D, $\sup\{s \in S : s \leq d\} = d = \inf\{s \in S : d \leq s\}$. This lattice D will be referred to as the Dedekind-MacNeille or DM completion of S, and it is of course an S-set D_S extending S_S.

In the sequel, semi-lattice will mean lower semi-lattice, but everything is dually true for upper semi-lattices.

We now show that if S is a chain then its DM completion, itself a chain, is its injective envelope, but we first recall

THEOREM 12. If D is the DM completion of the chain S and d_1 and d_2 are any two distinct elements of D with at most one of them in S, then there exists an infinite sequence $s_1, s_2, s_3, \ldots$ of elements of S such that $d_1 < s_1 < s_2 < s_3 < \ldots < d_2$ (strict inequalities).

Proof. Let us assume that $d_1 < d_2$. If d_2 is not in S then $d_1 < d_2$ implies that there exists an s_1 in S such that $d_1 < s_1 < d_2$. Repeating this argument with s_1, s_2 etc. yields the required sequence. The argument is the same if d_1 is not in S. Q.E.D.

This theorem is used to prove the following important

THEOREM 13. The DM completion D_S of a chain S_S is an essential extension of it.

Proof. Let θ be a congruence relation on D_S not the identity. Then there exist two distinct elements d_1 and d_2 in D, say $d_1 < d_2$, such that $d_1 \,\theta\, d_2$. If both are in S, there is nothing to prove. If one of them is not in S, then by the preceding theorem there exist two distinct elements s_1 and s_2 in S such that $d_1 < s_1 < s_2 < d_2$ and $d_1 \,\theta\, d_2$ implies that

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$d_1 = d_1 s_1 \; \theta \; d_2 s_1 = s_1$ and $d_1 = d_1 s_2 \; \theta \; d_2 s_2 = s_2$. Thus $s_1 \; \theta \; s_2$ by symmetry and transitivity of θ. Q.E.D.

The result also follows from the fact that every congruence relation on D_S is a partition of D into disjoint intervals and that the smallest congruence relation linking d_1 to d_2 (that of Lemma 1) is of the form $x[d_1, d_2]y$ if and only if $x = y$ or x and y are both in the closed interval $[d_1, d_2]$ for any x and y in D.

It now suffices to show that the DM completion D of a chain S is injective, and in the sequel 0 and 1 will denote the smallest and largest elements of D respectively.

DEFINITION 7. If f is a mapping from S' to the chain S then $\ell_f = \sup\{x \in S' : f$ restricted to $<0, x]$ (half closed) is the identity$\}$, and 0 if no such x exists.

The following lemma is obvious.

LEMMA 2. If f is as above and s in S then

(1) $f\,s = f$ on $<0, s]$ and $fs = f(s)$, a constant map on $[s, 1]$.

(2) If fs is the identity mapping on $<0, x]$ then so is f for any x in S'.

THEOREM 14. ϕ defined for any f in $S^{S'}$ by $\phi(f) = \ell_f$ is a homomorphism from $S_S^{S'}$ to D_S.

<u>Proof.</u> ϕ is obviously a mapping and it suffices to show that $\ell_{fs} = \ell_f s$ for any s in S. Now it follows from the second part of the above lemma that $\ell_{fs} \leq \ell_f$ and thus the theorem is true for $\ell_f = 0$. If $\ell_f \neq 0$ there are two possibilities:

(1) $s < \ell_f$. Then $fs = f =$ identity on $<0, s]$ and thus $s \leq \ell_{fs}$. If $s < \ell_{fs}$, then there is an x in S such that $s < x \leq \ell_{fs}$ and fs restricted to $<0, x]$ is the identity which implies that $x = fs(x) = f(sx) = f(s) = s$ a contradiction. Thus

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$\ell_{fs} = s = \ell_f s$.

(2) $\ell_f \leq s$. If $\ell_{fs} < \ell_f$, then there is an x in S such that $\ell_{fs} < x \leq \ell_f$ and f restricted to $< 0, x]$ is the identity, which means that $fs = f =$ identity on $< 0, x]$, again a contradiction, and thus $\ell_{fs} = \ell_f = \ell_f s$.

THEOREM 15. If D is the DM completion of a chain S , then D_S is the injective envelope of S_S .

Proof. It suffices to show that D_S is a retract of $S_S^{S'}$ and to this effect we embed D_S in $S_S^{S'}$ as follows. We define a mapping K from D to $S^{S'}$ by $(K(d))(x) = dx$ if dx is in S , and to a fixed element k of S otherwise, for all x in S' . K is obviously a homomorphism and it is also an injection. For, let $d_1 < d_2$ in D . If both are in S , then $(K(d_2))(d_2) = d_2$ and $(K(d_1))(d_2) = d_1$. If only one is in S , then by Theorem 12 there is at least one element x of S distinct from k and such that $d_1 < x < d_2$; then $(K(d_1))(x) = d_1$ if d_1 is in S and to k otherwise, while $(K(d_2))(x) = x$. In every case, $K(d_1) \neq K(d_2)$. Finally it follows from theorem 14 that $\phi \circ K(d) = d$ for all d in D .

COROLLARY. The chain of extended reals is the injective envelope of the chain of rationals.

The rest of this section is devoted to proving the statements made after Definition 5 and Theorem 5.

Firstly, we remark that the DM completion D_S of a non complete chain S_S is not a strictly essential extension. For if $d_1 < d_2$ are not in S , then s in S must be between d_1 and d_2 or larger than d_2 in order to have $d_1 s \neq d_2 s$; but then $d_1 s = d_1$ is not in S .

To prove the second assertion, we recall that the ideals, or the sub S-sets, of a semi-lattice S are the semi-filters, i.e., subsets K of S such that for any s in S and k in K ,

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$s \leq k$ implies that s is in K. We now have

THEOREM 16. Every partial endomorphism f of a semi-lattice S_S with domain an ideal K is an inf preserving idempotent contraction with $f(K) \subset K$.

Proof. For any k in K, $f^2(k) = f^2(k^2) = f(f(kk)) = f(kf(k)) = (f(k))^2 = f(k)$. If k' is also in K, then $f(k) \cdot f(k') = f(kf(k')) = f^2(kk') = f(kk')$. Finally, $f(k) = f(k^2) = (f(k))k$ and thus $f(k) \leq k$, which implies that $f(K) \subset K$.

THEOREM 17. If f is a partial endomorphism on the semi-lattice S with domain an ideal K, then there is an element d in the DM completion D of S such that for any x in K, $f(x) = dx$.

Proof. Let $d = \sup\{k \in K : f(k) = k\}$. This set is not empty since for any x in K, $f(f(x)) = f(x)$ by the last theorem. Thus $f(x) \leq d$ for all x in K and $f(x) = (f(x))x \leq dx$. If now k in K is such that $f(k) = k$ then for any x in K, $kx \leq f(x)$ and in K which implies that $d \leq f(x)$ and $dx \leq f(x)$.

THEOREM 18. Every chain is weakly injective.

Proof. Let f be a partial endomorphism on the chain S with domain an ideal K, and let d be the element of the DM completion D of S whose existence is guaranteed by the preceding theorem. If x is in K, then dx must be in K by Theorem 16, and thus, if d is not in S, x must be less than d. But by Theorem 12, there is an s in S between d and 1 with $f(x) = dx = x = sx$, and this for all x in K. Q.E.D.

And so, any non complete chain is weakly injective but not injective, since the DM completion is a minimal injective extension.

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